WHAT PROOFS ARE FOR

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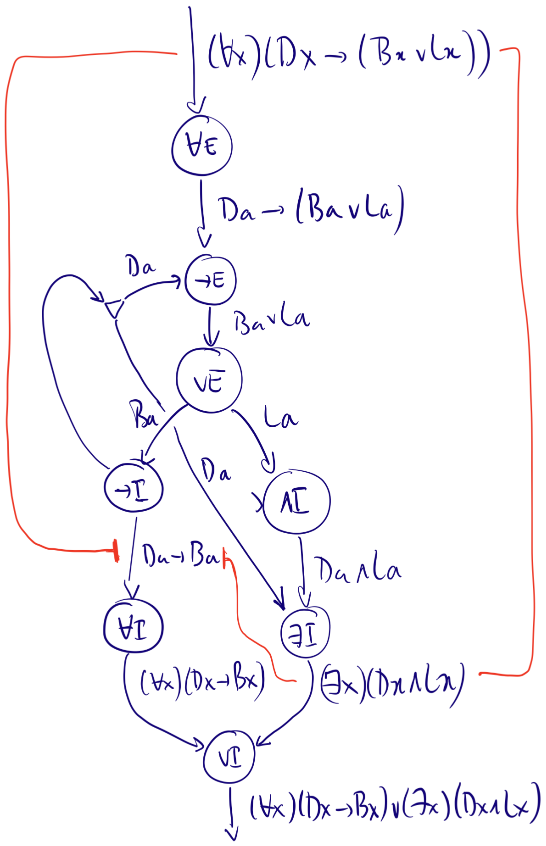
NEW YORK INSTITUTE OF PHILOSOPHY ∗ APRIL 6, 2018

My aim: To present a new account of the nature of proof, with the aim of explaining how proof could actually play the role in reasoning that it does, and answering some long-standing puzzles about the nature of proof. Along the way, I'll explain how Kreisel's *squeezing argument* helps us understand the connection between an informal notion of validity and the notions formalised in our accounts of proofs and models, and the relationship between proof-theoretic and model-theoretic analyses of logical consequence.

1 MOTIVATION

EXAMPLE PROOF: Every *drink* (in our fridge) is either a *beer* or a *lemonade* $(\forall x)(Dx \to (Bx \lor Lx))$. So either every *drink* is a *beer*, or some *drink* is a *lemonade* $(\forall x)(Dx \to Bx) \lor (\exists x)(Dx \land Lx)$. WHY? Take an arbitrary *drink*. If it's a *lemonade*, we have the conclusion that some *drink* is a *lemonade*. If we don't have that conclusion, then that arbitrary *drink* is a *beer*, and so, all the *drinks* are *beers*. ¶ The logical connections implicit in this reasoning can be formally represented in a *proof circuit* like this—



—and more traditionally (and with more redundancy, but equivalently) in Gentzen's sequent calculus, as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Da \succ Da \quad \dfrac{Ba \succ Ba \quad La \succ La}{Ba \lor La \succ Ba, La}\,{\lor L}}{Da \to (Ba \lor La), Da \succ Ba, La}\,{\to L}}{Da \to (Ba \lor La), Da \succ Ba, Da \land La}\,{\land R}}{Da \to (Ba \lor La) \succ Da \to Ba, Da \land La}\,{\to R}}{(\forall x)(Dx \to (Bx \lor Lx)) \succ Da \to Ba, Da \land La}\,{\forall L}}{(\forall x)(Dx \to (Bx \lor Lx)) \succ Da \to Ba, (\exists x)(Dx \land Lx)}\,{\exists R}}{(\forall x)(Dx \to (Bx \lor Lx)) \succ (\forall x)(Dx \to Bx), (\exists x)(Dx \land Lx)}\,{\forall R}}{(\forall x)(Dx \to (Bx \lor Lx)) \succ (\forall x)(Dx \to Bx) \lor (\exists x)(Dx \land Lx)}\,{\exists R}$$

My focus will be on categorical proofs, and in particular, my focus today will be on categorical proofs of a special kind—proofs in *first order predicate logic*, where the central concepts used are the logical notions of conjunction, disjunction, negation, the (material) conditional, the quantifiers. But what I say here can be extended to proof using other concepts.

PUZZLES ABOUT PROOF: How can proofs expand our knowledge, when the conclusion is already present (implicitly) in the premises? ¶ How can we be ignorant of a conclusion which actually already follows from what we already know? ¶ What grounds the necessity in the connection between premises and conclusion?

2 POSITIONS

POSITIONS collect together *assertions* and *denials* [X : Y]. ¶ *Assertions* and *denials* are moves in a communicative practice. I can *deny* what you *assert*. We can assert or deny the same thing. We can also *retract* assertions and denials. I can try on assertion or denial hypothetically (suppose p — then q...) ¶ Asserting or denying involves *taking a stand* on some matter. ¶ Assertion and denial *clash*.

The BOUNDS on positions — (1) IDENTITY: [A : A] is out of bounds. (2) WEAKENING: If [X : Y] is out of bounds, so are [X, A : Y] and [X : A, Y]. (3) CUT: If [X, A : Y] and [X : A, Y] are out of bounds, so is [X : Y]. ¶ A position that is *out of bounds* fails to successfully take a stand [7]. ¶ If a position is not out of bounds, we call it *available*.

DEFINITIONS: Definitions come in a number of flavours. One is obvious, and one is less so.

EXPLICIT DEFINITION: Define a concept by showing how you can compose this concept out of more primitive concepts ¶ (x is a square $=_{df}$ x is a rectangle $\land$ all sides of x are equal in length) ¶ Concepts given an explicit definition are sharply delimited (contingent on accepting the definition, of course). Logical concepts like conjunction, disjunction, negation, the (material) conditional, the quantifiers, and identity are similarly sharply delimited, but they cannot be given explicit definition. (They are *used*

*Thanks to Shawn Standefer for helpful conversations about this material. ¶ This research is supported by the Australian Research Council, through Grant DP150103801. ¶ Resources available at http://consequently.org/presentation/2018/what-proofs-are-for-nyu/.

in giving explicit definitions.)

DEFINITION THROUGH A RULE FOR USE: Define a concept by showing it could be added to one's vocabulary, giving rules for interpreting assertions and denials involving that concept. ¶ $[X, A \land B : Y]$ is out of bounds iff $[X, A, B : Y]$ is out of bounds.

$$\frac{X, A, B \succ Y}{X, A \land B \succ Y} \qquad \frac{X \succ A, B, Y}{X \succ A \lor B, Y} \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}$$

$$\frac{X \succ A, Y}{X, \neg A \succ Y} \qquad \frac{X \succ A|^x_n, Y}{X \succ (\forall x)A, Y} \qquad \frac{X, A|^x_n \succ Y}{X, (\exists x)A \succ Y} \qquad \frac{X, Fs \succ Ft, Y}{X \succ s = t, Y}$$

The concepts introduced in this way are *uniquely defined* (if you and I follow the same rule, our usages are intertranslatable) and they conservatively extend the original vocabulary (if a position was safe before we added the concept, it's still safe afterwards) [9]. ¶ Concepts defined in this way also play useful dialogical roles. They increase our expressive power. Once we have conjunction, for example, I can disagree with your assertion of A and B without disagreeing with A or disagreeing with B. They are also *subject-matter-neutral*. To use Brandom's terms, anthe new concepts *make explicit* some of what was previously merely implicit [3].

3 WHAT PROOFS ARE, AND WHAT THEY DO

Consider a tiny proof, consisting of a single step of *modus ponens*: *If it's Thursday, I'm in Melbourne. It's Thursday*. So, *I'm in Melbourne*. Here, we have two assertions (the premises), a connecting "*so*" and another assertion (the conclusion). This proof *crucially* uses the conditional. If we mean "→" by that "if," then since $A \to B \succ A \to B$, we have $A \to B, A \succ B$ (using the defining rule for "→"). ¶ Hence, a position in which I assert "*If it's Thursday, I'm in Melbourne*" and "*It's Thursday*" but I deny "*I'm in Melbourne*" is out of bounds. So, "*I'm in Melbourne*" is undeniable, and the assertion makes explicit what was previously implicit in granting the premises. ¶ We can show that the defining rules shown here give rise to the standard sequent calculus for classical logic [9].[1] The steps in a Gentzen-style derivation for $X \succ Y$ can be grounded in the defining rules as *definitions* of the concepts appealed to in that derivation. ¶ **A PROOF for the sequent $X \succ Y$ shows that the position $[X : Y]$ is out of bounds, by way of defining rules for the concepts used in X and Y.** ¶ Proofs in this sense are *analytic*. ¶ Proofs can contain a mix of assertions and denials. A proof of $A, B \succ C, D$, for example, can be understood as a proof of C from the position $[A, B : D]$, or a refutation of A from the position $[B : C, D]$.

4 COUNTEREXAMPLES & KREISEL'S SQUEEZE

If $X \succ Y$ is *not* derivable, then the position $[X : Y]$ can be enlarged into a *partition* $[X' : Y']$ of the original language, supplemented with a countable collection of new names [8]. (This is one way to understand Henkin's construction in the completeness proof for first order predicate logic.) ¶ Running the *Cut* rule in reverse, if $[X : Y]$ is available, then either $[X, A : Y]$ or $[X : A, Y]$ is available. Consider each sentence in the language in turn, and add it to the left or the right in your position and continue … ¶ At the limit of this process, we have a partition $[X' : Y']$ making a verdict on each sentence of the language, and we never have a derivation of $X \succ Y$ for any $X \subseteq X'$ and $Y \subseteq Y'$. ¶ Such a partition $[X' : Y']$ can be viewed as giving rise to a *model*, since it satisfies the truth conditions expected of Tarski's models for first order logic, according to which the formulas in X' are *true* and those in Y' are *false*.

- $A \in X'$ iff $\neg A \notin X'$ iff $\neg A \in Y'$,
- $A \land B \in X'$ iff $A \in X'$ and $B \in X'$.
- $A \lor B \in X'$ iff $A \in X'$ or $B \in X'$.
- $A \to B \in X'$ iff $A \in Y'$ or $B \in X'$.
- $(\forall x)A \in X'$ iff $A|^x_n \in X'$ for each name n.
- $(\exists x)A \in X'$ iff $A|^x_n \in X'$ for some name n.

We can think of a model, then, as the *limit* of a process of filling out a finite starting position. The completeness theorem states that if a sequent $X \succ Y$ is not derivable, then it may be extended by some limit position $[X' : Y']$—a model where each member of X is true and each member of Y is false.[2]

Now we have the resources to answer the following question: Given that the connectives and quantifiers are *defined* in the way given by these rules, is the logic determined by those rules *correct* and *comprehensive*? ¶ This is the question that Kreisel's squeezing argument addresses [5]. ¶ An argument from X to Y is *informally valid* if and only if there is a *clash* involved in asserting each member of X and denying each member of Y.[3] ¶ First, if $X \succ Y$ is *formally derivable*, then it is informally valid. Why? Because the *axiomatic* sequents are informally valid (there is always a clash involved in asserting A and denying A), and the *rules* show how assertions/denials involving complex vocabulary can be understood in terms of assertions/denials involving simpler vocabulary. We understand them to be definitions of those concepts, in the sense of being *rules* for their *use*. So, formal derivations underwrite informal validity for sequents using these concepts. ¶ If $X \succ Y$ is *underivable* then there is some *model* according to which all of X holds and all of Y fails. This model is uniquely determined by the *domain* (the family of names in the extended language), and the verdict it makes on each primitive sentence (of the form $Fn_1 \cdots n_m$). Provided that each primitive sentence is taken to be logically independent of any other (there is no clash involved in asserting Fab and denying Gcd, for example), the model shows *how* there is no clash involved in asserting each member of X and denying each member of Y. Given that you can take any position (assert/deny) on any primitive sentence, without any clash, the model gives you the reassurance that the position $[X : Y]$ is indeed clash-free. ¶ So, with this proviso, that the primitive non-logical vocabulary hides no clashes of its own, *informal* validity coincides exactly with *formal* validity. In other words, informal validity *in virtue of logical form* (*understood as first order logical form*) coincides with *formal validity*. The squeezing argument shows that formal logic is sound and complete for the informal notion.[4]

[1] And with a slight modification for the rules for the quantifiers, allowing for "non-denoting" singular terms, you get a sequent calculus for a standard negative free logic. The differences here are not important for the argument I am making.

[2] This is a little more complex if we include the identity predicate in the language. We need to add not only the stock of fresh names, but a stock of fresh *predicates*, and the domain of the model is not simply the collection of names, but equivalence classes under the relation of identity in the limit position. If $a = b \in X'$, then the names a and b denote the same object in the model, their equivalence class.

[3] So, a single-conclusion sequent $X \succ A$ is informally valid if asserting the premises X makes the conclusion A *undeniable*.

[4] This understanding of Kreisel's argument leaves open that other formal logics—intuitionistic logic, a paraconsistent logic, etc.,—may be sound and complete for *other* intuitive notions of logical validity. We had to fix the particular conception of validity (there is a clash between asserting the premises and denying the conclusion of the argument) to get the argument off the ground. One can be a pluralist about validity and have different formal notions corresponding to different informal validity concepts [1, 2].

5 CONSEQUENCES FOR HOW PROOFS WORK

Proofs make explicit the positions that are out of bounds. They help us navigate what claims (what assertions and denials) are open to us. Given this understanding of the role of proof, here are some consequences.

OBSERVATION 1: *Proofs preserve truth.* The definition of proof given here does not involve truth at all. ¶ However, given plausible (minimal, and minimal*ist*, amounting to T-intro and T-elim inferences) assumptions about the behaviour of the truth predicate, it follows that if there is a proof for $X \succ A, Y$ then if each member of X is *true* and each member of Y is not *true*, then A, indeed, is *true*. ¶ There are issues and complexities, of course, around the paradoxes. They are difficult for everyone, not just me.

OBSERVATION 2: What about proofs and the preservation of *warrant*? ¶ Our ability to *specify* consequence far outstrips our *ability* to recognise it. ¶ We have no idea if the position $[PA : GC]$ is out of bounds or not. This is not a bug—it is a *feature*, in particular, of the expressive power of the connectives and quantifiers—they give us the means to say things (think things, explore things) whose significance we continue to work out. It is straightforward to verify whether a putative proof is a proof. It is not always straightforward to find a proof of something that has a proof. ¶ Suppose $PA \succ GC$ holds and we know PA. Do we *know* GC? ∗ In a very weak sense, *yes*. It is a *logical consequence* of what we know. It is *implicitly present* in what we know. It holds in every possible world in which what we know is true. Denying GC is inconsistent (with PA). But this inconsistency is not transparent to us. ∗ In another sense, of course, the answer is *no*. Even if I believe GC (for inconclusive reasons), that may not count as knowledge if that belief is acquired in the wrong way. (By testimony, by misunderstanding, by inappropriate generalisation, by my mistaken proof.) ∗ Different accounts of knowledge will assess this case differently, but if the ground (or source) of the epistemic state plays some role in whether it counts as knowledge, then this provides a place where logical omniscience can break down. In this (hypothetical) case, there is evidence, in the sense of a proof from PA to GC, but if we do not possess it, and use it to ground our belief in GC, this proof is epistemically inert. ¶ If knowledge of A, in the salient sense, requires some appropriate access to or grasp of *how A is made true*, then we can see how possession of a proof from something known already known to A provides the resources required for knowledge, but if we do not possess that proof, we may be ignorant.

OBSERVATION 3: *Proofs transfer warrant.* As is the case with truth, the definition of proof *also* does not involve warrant or justification. ¶ However, given plausible (less minimal) assumptions about the nature of warrant, it follows that if there is a proof for $X \succ A, Y$ then given (*conclusive*) warrant for each member of X and (*conclusive*) warrant against each member of Y, then that proof provides the means to grasp (*conclusive*) warrant for A. The particular shape of an account like this are to be worked out, but the general scheme (in the case of intuitionistic logic) is given by recent work of Dag Prawitz [6].

Caveat: matters are delicate when it comes to *defeasible* warrant. Consider the lottery paradox, where for any ticket we have defeasible reason to believe, that this ticket will not win, but we also have reason to believe that some ticket will win. The position

$$[(\exists x)(Tx \land Wx), (\forall x)(Tx \leftrightarrow x = t_1 \lor \cdots \lor x = t_{1\,000\,000}) : Wt_1, \ldots, Wt_{1\,000\,000}]$$

is out of bounds, but each particular component of the position is highly likely.

OBSERVATION 4: Consider (in)famous dialogue between Achilles and the Tortoise [4]. The general problem is this: consider the argument from $A, A \to Z$ to Z. What if someone accepts A and $A \to Z$ and does not accept Z? What should we do concerning such a person? ¶ Carroll shows that adding the premise, *if* A *and* $A \to Z$ *then* Z will not do, for in the same way one could accept these premises and still resist Z. ¶ On this approach to logical consequence the following responses seem appropriate. (1) There is nothing inconsistent or problematic in accepting A and $A \to Z$ and simply *failing* to accept Z. (2) If, however, one accepts A and $A \to Z$ and *rejects* Z, then this position is out of bounds—or '$\to$' is being used in violation of its defining rule. (3) What if Z has neither been accepted nor rejected, but the question concerning Z has been *asked*? We have not connected the norms governing positions, assertions, denials and defining rules with those governing questions and answers. It is plausible, however, in just the same way that asserting $A \to Z$ and A is inconsistent with denying Z, then asserting A and $A \to Z$ counts as a (conclusive) answer to *why* Z. ¶ In general, the right sort of answer to the puzzle of Achilles and the Tortoise is to show how norms of questions and answers (or giving reasons) are connected to defining rules and proofs.

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