Assertion, Denial, Commitment, Entitlement, and Incompatibility (and some inference)

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This talk is a loosely connected string of vignettes on related themes.

Semantic anti-realists have a number of different options for their explanatory primitives, when it comes to articulating the behaviour of logical consequence, incompatibility, and related notions.

I will explore some of these options (in particular, choices for how to connect consequence and incompatibility), and I will defend a set of tools for looking at these connections.

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$$\forall C(\text{if } \perp B, C \text{ then } \perp A, C)$$

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However, it is not available for the friend of intuitionistic logic, if $\vdash_I$ is intuitionist logical consequence, and $\perp$ is intuitionist incompatibility.

Recall that for intuitionists $\sim\sim A \nvdash A$, and $\perp B, C$ iff $C \vdash \sim B$.

We'll show that $\sim\sim A \vdash_I A$.

Suppose $\perp A, C$. Then $C \vdash \sim A$. Since $\sim A \vdash \sim\sim\sim A$, $C \vdash \sim\sim\sim A$. It follows that $\perp \sim\sim A, C$.

Therefore, $\sim\sim A \vdash_I A$.

For friends of intuitionistic logic, content and consequence can't be characterised by incompatibility, because A and $\sim\sim A$ are incompatible with exactly the same statements.

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We have one consideration, then, on purely anti-realist grounds, to reject intuitionist logic. (These aren't *strong* grounds, of course.)

If you take incompatibility to be one of the basic materials for the construction of your theory of concepts (as the "Hegelian" Brandom of *Holism & Idealism* does), and if you take consequence to be related to incompatibility in the way we've seen, then intuitionistic logic is not for you.

So, let's see what happens when we connect incompatibility and consequence in a different way. We'll try to define incompatibility in terms of consequence.

We will assume that we have a consequence relation $\vdash$, at our disposal, relating a premises to a conclusion.

So, it makes sense to say that $X \vdash A$, where X is a set of premises and A is a single conclusion.

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Here is a straightforward attempt at defining incompatibility from consequence:

ATTEMPT 1

$\bot X$ iff $X \vdash C$ for *every* C.

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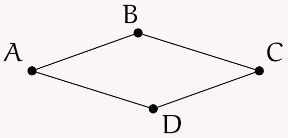
As a *static* analysis, this has something going for it. However, if we are interested in the case where a language is *augmented* by new material, it fails.

Consider a language with just four primitive statements, A, B, C and D, ordered by ⊢ as follows. We have ⊥A, C. Then *extend* the language with four more statements. Now A and C are no longer incompatible.

⊢ is preserved in the move to the large structure, but ⊥, as defined in ATTEMPT 1 is not.

I take that to be a consideration against ATTEMPT 1.



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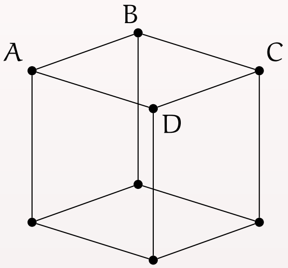

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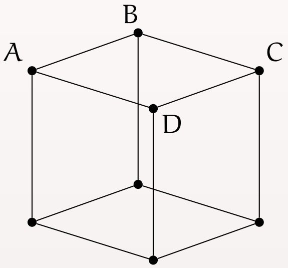

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Entailing f

Now for another attempt. Instead of defining $\perp$ in terms of $\vdash$ alone, we bring in a special statement f.

ATTEMPT 2

$\perp X$ iff $X \vdash f$.

Now *this* is preserved when we go from one structure to a larger one, as whatever entails f in the small structure entails f in the large one.

But I don't think this is the *heart* of the matter either, because it only works in languages where we have this special statement f. It seems that we can define incompatibility in *more* cases than those.

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For example: Consider a 'language' with two statements A and B where we have $A \nvdash B$ and $B \nvdash A$, and $\perp A, B$.

This *seems* coherent. But, according to ATTEMPT 2, A and B aren't incompatible since there is no statement that both entail.

(Of course, this 'language' is, deficient in some sense, because it cannot express the conjunction of A and B. You might want to say that the statement f is *implicit*, rather than *explicit*.)

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Entailing f

How can we respect this possibility? One way to do this is to allow f to be outside the language. (f is, if you like, an 'ideal element.')

Now, consider what 'A, B ⊢ f' might mean. If 'A, B ⊢ C' is the trace of a deduction starting at A and B and ending at C, then 'A, B ⊢ f' is the trace of a deduction from A and B and ending at … at *what?*

A deduction for A, B ⊢ f (which we might call a *refutation* of A, B) is a proof starting at A and B and *without a concluding formula*. (Consider proofs that end "Contradiction!" You can think of them as stepping from a particular contradiction to *nowhere*.)

This should not be foreign to you. You have already seen proofs with *no premises*. It is not much of a jump to think of A, B ⊢ f as A, B ⊢ .

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A deduction might end in a conclusion, or not

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If you want to think of incompatibility as defined by consequence, then think of consequence as not only relating premises not just to a single conclusion ($X \vdash A$) but to also allow *refutations* ($X \vdash$), which allow premises without a conclusion.

But if you allow (1) deductions with many premises and a single conclusion and (2) deductions with many premises and *no* conclusion, then what is stopping us from considering (3) deductions with many premises and many conclusions?

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Opponents of multiple-premise–multiple-conclusion consequence reject the idea for a number of different reasons.

The most substantial is that it is hard to read $X \vdash Y$ in terms of preservation of warrant to assert.

If $X \vdash A$, then if you have warrant to assert each member of X, you have warrant to assert A.

If $X \vdash$, then you don't have warrant to assert each member of X.

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Warrants

Well, the one thing you *don't* have warrant for is to assert $A \vee B$ and to *deny* both of A and B.

In general, $X \vdash Y$ iff there could be no warrant to assert each member of X and deny each member of Y. For this to work, you need a few principles connecting assertion and denial. (In particular, denying A will not necessarily be asserting $\sim A$, at least for the friend of truth-value gaps or gluts.)

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There are other ways to introduce "bilateral" elements into one's account of concepts. The Brandom of *Making it Explicit* and *Articulating Reasons* does so in terms of *commitment* and *entitlement*.

For Brandom, an agent's commitments and entitlements help constitute the dialectical *score* in the game of giving and asking for reasons.

Incompatibility is then defined in terms of commitment and entitlement: $\perp A, B$ iff *commitment to* A *precludes entitlement to* B.

I find this account suggestive, but obscure. (1) It is hard to get to grips with the formal properties of commitment and entitlement, and (2) *preclusion* is also underspecified.

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If $\vdash A, B$ (that is, $A, B \vdash$) then if $[X : Y]$ is committed to A (that is, $X \vdash A, Y$) then by *cut*, $X, B \vdash Y$ (that is $[X : Y]$ is not entitled to B).

Conversely, if commitment to A doesn't preclude entitlement to B, there is some coherent position $[X : Y]$ at which A is a commitment and B is not an entitlement. If follows that $X \vdash A, Y$ and $X, B \vdash Y$.

So, if A and B *are* incompatible, then $A, B \vdash$, and the cut rule would give $X \vdash Y$, contrary to the coherence of $[X : Y]$.

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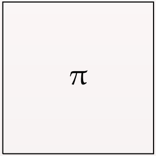
It's one thing to have a bilateral account of consequence. If a statement of consequence is a record of a proof, then what kind of proof might be recorded in a multiple-premise/multiple-conclusion sequent?

Can the picture be as *attractive* as intuitionistic natural deduction?

Can it satisfy the kinds of theoretical constraints (normalisation, separability, etc.) appropriate for natural deduction system?

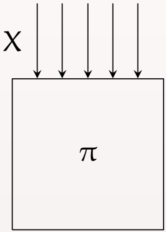


A *proof* for $X \vdash Y$ has each element of X as an *input* and each element of Y as an *output.*

It will be simplest to represent proofs as directed graphs, where formulas label *edges* and rules label *nodes.*

The simplest proof is the *identity* proof with the single premise and conclusion A. It is a naked edge labelled with A.

From multiple-premise/multiple-conclusion consequence to proofs



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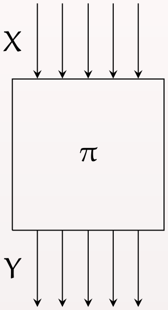

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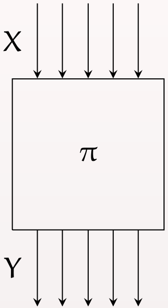

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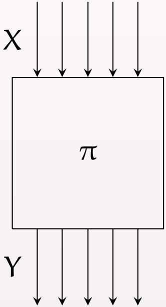

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Cut

Given two proofs: π_1 and π_2, where A is an output of π_1 and an input of π_2, we get a new proof chaining π_1 to π_2. The *inputs* of the new proof are all of the inputs of π_1, together with the inputs of π_2 other than the A input we singled out, and the *outputs* are the outputs of π_2 together with all of the outputs of π_1 other than the A output.

This is the *cut* rule: If $X \vdash Y, A$ and $A, X' \vdash Y'$ then $X, X' \vdash Y, Y'$.

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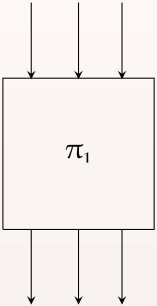

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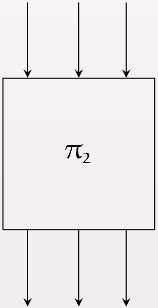

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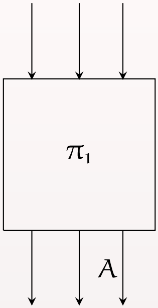

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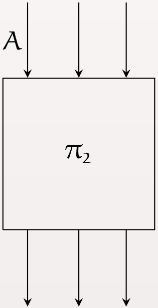

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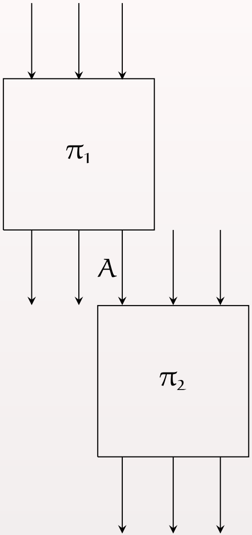

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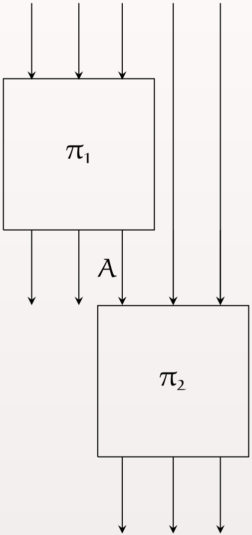

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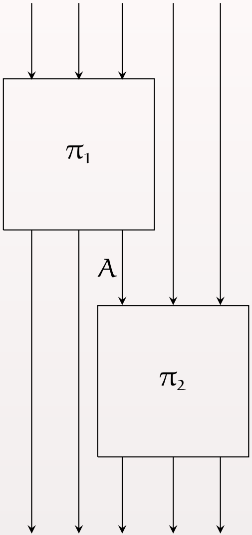

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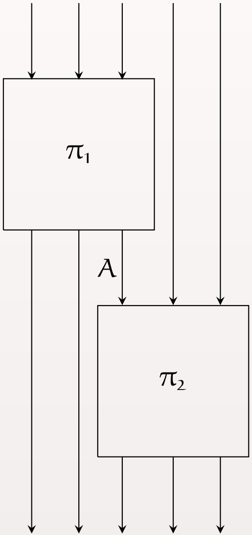

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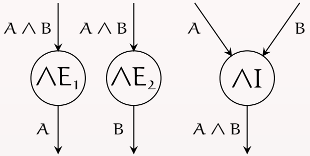

The conjunction rules are straightforward. Pairs of ∧I/∧E links can be removed in a *normalisation* process.

(It's simple and *local* if you add 'weakening' links, which take two inputs, and return one of the inputs as an output.)

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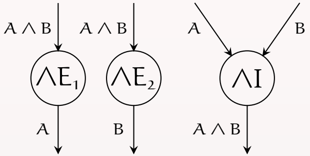

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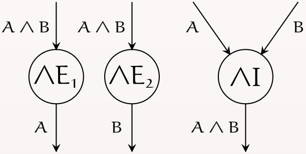

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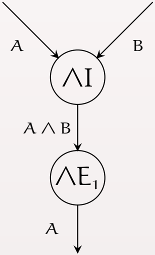

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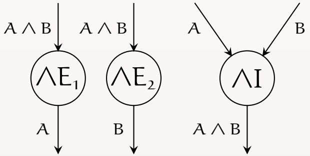

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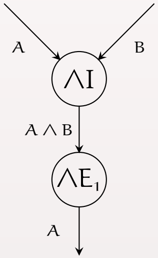

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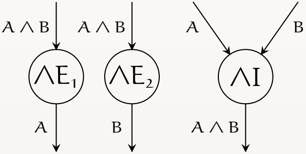

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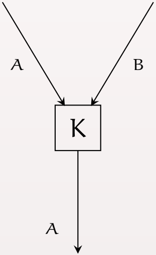

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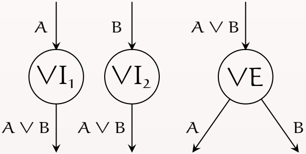

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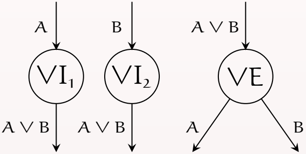

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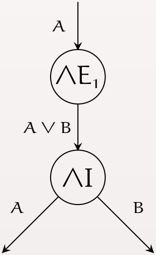

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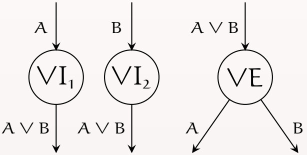

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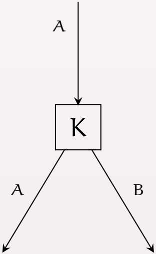

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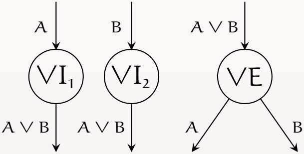

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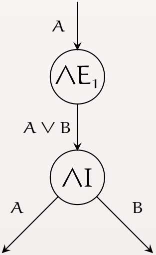

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Negation

The negation rules take their cue from the classical Gentzen rules: if $X \vdash A, Y$ then $X, {\sim}A \vdash Y$ (an output A can be traded in for an input ${\sim}A$) and if $X, A \vdash Y$ then $X \vdash {\sim}A, Y$ (an input A can traded in for an output ${\sim}A$).

(Think of these nodes in use. For (${\sim}$E), you can plug an ouptut A of a proof π into the (${\sim}$E) node, which uses up that output, and gives the proof a new A input. Similarly for (${\sim}$I).)

Notice that normalisation is trivial (an introduction and elimination step for ${\sim}A$ is rewritten by an A arrow).

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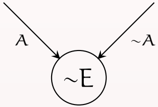

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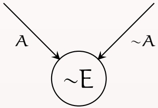
The negation rules take their cue from the classical Gentzen rules: if $X \vdash A, Y$ then $X, {\sim}A \vdash Y$ (an output A can be traded in for an input ${\sim}A$) and if $X, A \vdash Y$ then $X \vdash {\sim}A, Y$ (an input A can traded in for an output ${\sim}A$).



(Think of these nodes in use. For (${\sim}$E), you can plug an ouptut A of a proof π into the (${\sim}$E) node, which uses up that output, and gives the proof a new A input. Similarly for (${\sim}$I).)

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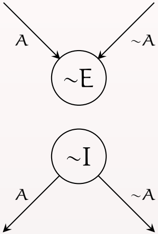

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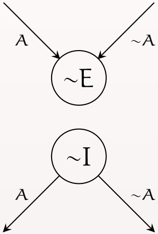

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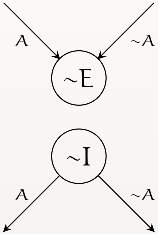

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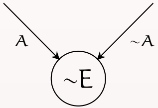

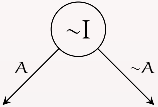

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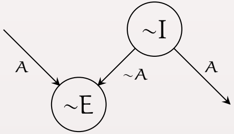

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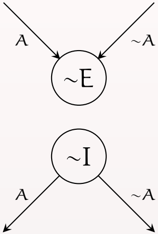

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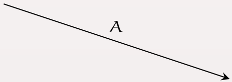
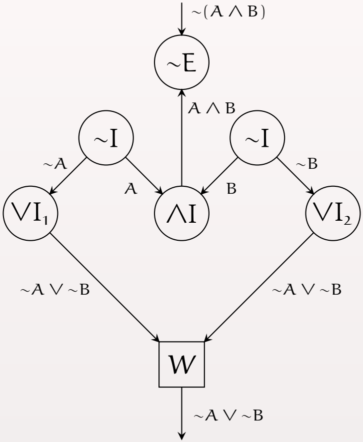


Here is a proof and a corresponding sequent derivation, for the intuitionistically unprovable $\sim(A \wedge B) \vdash \sim A \vee \sim B$.



$$\begin{array}{c} A \vdash A \quad B \vdash B \\ \hline A, B \vdash A \wedge B \\ \hline \sim(A \wedge B), A, B \vdash \\ \hline \sim(A \wedge B), A \vdash \sim B \\ \hline \sim(A \wedge B) \vdash \sim A, \sim B \\ \hline \sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B \\ \hline \sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B \\ \hline \sim(A \wedge B) \vdash \sim A \vee \sim B \end{array}$$

Example Proof

Here is a proof and a corresponding sequent derivation, for the intuitionistically unprovable $\sim(A \wedge B) \vdash \sim A \vee \sim B$.

$$
\begin{array}{c}
A \vdash A \qquad B \vdash B \\
\hline
A, B \vdash A \wedge B \\
\hline
\sim(A \wedge B), A, B \vdash \\
\hline
\sim(A \wedge B), A \vdash \sim B \\
\hline
\sim(A \wedge B) \vdash \sim A, \sim B \\
\hline
\sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B \\
\hline
\sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B \\
\hline
\sim(A \wedge B) \vdash \sim A \vee \sim B
\end{array}
$$

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A \vdash A \quad B \vdash B \\ \hline
A, B \vdash A \wedge B \\ \hline
\sim(A \wedge B), A, B \vdash \\ \hline
\sim(A \wedge B), A \vdash \sim B \\ \hline
\sim(A \wedge B) \vdash \sim A, \sim B \\ \hline
\sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B \\ \hline
\sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B \\ \hline
\sim(A \wedge B) \vdash \sim A \vee \sim B
\end{array}$$

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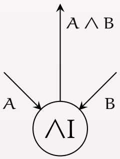

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{\color{red}{A, B \vdash A \wedge B}}}{\sim(A \wedge B), A, B \vdash}}{\sim(A \wedge B), A \vdash \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B}$$

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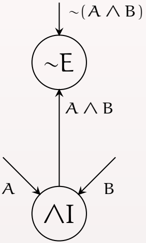

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}}{{\color{red}\sim(A \wedge B), A, B \vdash}}}{\sim(A \wedge B), A \vdash \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B}$$

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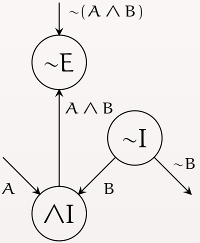

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}}{\sim(A \wedge B), A, B \vdash}}{\color{red}{\sim(A \wedge B), A \vdash \sim B}}}{\sim(A \wedge B) \vdash \sim A, \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B}
$$

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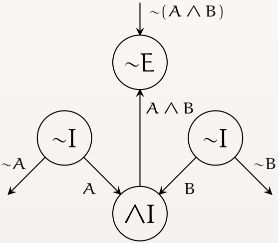

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}}{\sim(A \wedge B), A, B \vdash}}{\sim(A \wedge B), A \vdash \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B}$$

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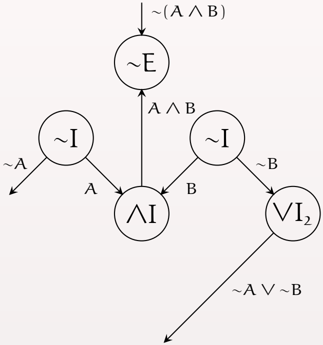

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}}{\sim(A \wedge B), A, B \vdash}}{\sim(A \wedge B), A \vdash \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B}$$

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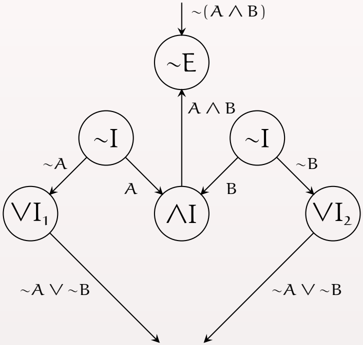

$$
\begin{array}{c}
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\hline
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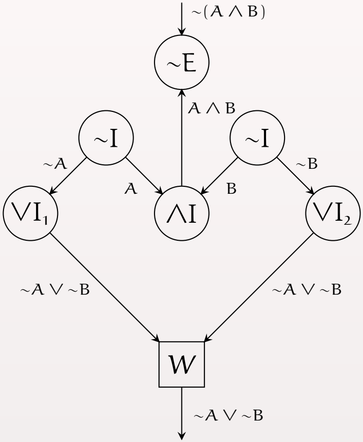

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}}{\sim(A \wedge B), A, B \vdash}}{\sim(A \wedge B), A \vdash \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim B}}{\sim(A \wedge B) \vdash \sim A, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B, \sim A \vee \sim B}}{\sim(A \wedge B) \vdash \sim A \vee \sim B}}{}
$$

Tentative Conclusion 2

Multiple-premise, multiple-conclusion consequence relations are not only defensible on inferentialist grounds, they also have a proof theory with nice properties.

(You don't have to put up with *ad hoc* proof theory for classical logic.)

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Any justification of a logic on proof-theoretic grounds depends *crucially* on assumptions made about the *structure* of proof.

(All the better, then, to be *explicit* about those assumptions.)

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