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# Combining Possibilities and Negations

**Abstract.** Combining non-classical (or '*sub*-classical') logics is not easy, but it is very interesting. In this paper, we combine nonclassical logics of negation and possibility (in the presence of conjunction and disjunction), and then we combine the resulting systems with intuitionistic logic. We will find that Kracht's results on the undecidability of classical modal logics generalise to a non-classical setting. We will also see conditions under which intuitionistic logic can be combined with a non-intuitionistic negation without corrupting the intuitionistic fragment of the logic.

**Key words:** combining non-classical logics, intuitionistic logic, negation, possibility.

Many people are interested in logics of modal operators like 'necessarily' and 'possibly,' and their cousins taken from temporal, epistemic, doxastic and many other concerns. Quite a few people are also interested in *negative* modal operators, like classical boolean negation, but with some kind of 'modal' force. The idea with these sorts of operators is that to evaluate 'not $p$' at a point (world, information state, moment or whatever) you check the status of $p$ at some other class of points. Intuitionistic negation is one such negation: to check for 'not $p$' at a point, you examine whether $p$ fails at all of that point's successors. The de Morgan negation of relevant logics is also this sort of negation. For 'not $p$' is true at a point, you need $p$ to fail to be true at another particular point related to the first point. In this paper we will examine what happens when you put these things together. Specifically, we will see how combining logics of possibility and negation (or combining two possibilities, or two negations) can result in undecidability. We will also see what happens when these logics are combined with intuitionistic propositional logic.

The jumping-off points of this work are numerous. The results we will discuss are not only in the general scene of combining logics, but they are also an instance of combining logical *techniques*. From the side of classical modal logic, we will be using Marcus Kracht's elegant results giving examples of simple undecidable modal logics [7]. He shows that given a finitely presented semigroup, you can construct a finitely axiomatised modal logic (with a modality for every variable used in the presentation) which encodes the semigroup in a natural way. Deciding theoremhood in the logic is sufficient

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for deciding equations in the semigroup. As we know that there are finitely presented semigroups for which the word problem is undecidable, it follows that some of these simple modal logics are also undecidable. Since there are finitely presented semigroups with only *two* variables which are undecidable, there are particularly simple undecidable bimodal logics.

The question naturally arises: Do we need the full power of the classical logic underlying the modal structure to get undecidability? After all, there are more logics under heaven and earth than classical modal logics. There is much interest afoot in substructural logics, and in these, typically, boolean negation does not feature. There is also interest in *intuitionistic* modal logics. In deductive systems of these sorts (intuitionistic, or substructural systems) boolean negation is anathema because we wish to consider theories (or information states, or whatever) which are incomplete, and possibly, inconsistent. In those contexts we would not expect boolean negation to be present in the language under discussion. There is a subtle distinction here between languages you use to describe a model and the language you use 'inside' a model. Perhaps it is best illustrated in the context of a concrete model. Take the points in a model to be theories (closed under an appropriate consequence relation). Theories can expand, so there might well be theories $T$ and $T'$ where $T'$ asserts everything asserted by $T$, but it also asserts more. Say, the claim $A$. Now we know that $T$ doesn't assert $A$. It would be madness conclude from this that $T$ asserts $\sim A$ for some negation $\sim$, because, by construction $T'$ asserts everything asserted by $T$, and we would have $T'$ asserting both $A$ and $\sim A$. Now it is true that theories can be inconsistent — but they certainly need not in this case. Better to say that $T$ doesn't assert $A$ or $\sim A$. Of course, this doesn't mean that we cannot describe the theory $T$ by saying that $T \not\models A$. But there is no compulsion that this ought to be explicitly recorded in the theory as a fact asserted by the theory. It is a fact *about* the theory, not a fact *of* the theory.[1]

Once we renounce boolean negation we must answer the question: Does the decrease in expressive power mean that our modal logics become decidable? Or does Kracht's phenomenon occur? The problem is this: Kracht's proofs involve boolean negation in a number of important ways — most obviously in defining the material conditional, which is essential to the proof. So, we should ask — do the undecidability results remain when we renounce booelan negation?

[1]This perspective on logical theory is orthogonal to the 'Amsterdam Perspective' on modal logics. For people in that tradition logic is seen as a way of reasoning about structures. From that perspective, boolean negation is perfectly acceptable. I am not arguing against that tradition — it is quite fruitful and interesting in its own right — I simply point out that for some purposes, limiting our language is a necessity.

Further, we can ask, does Kracht's result work for different kinds of *negation*? There is increasing awareness that 'nots' are a kind of modal operator, just as much as boxes and diamonds. (For example, there is Goldblatt's pioneering work on orthonegation as a modal operator, and its generalisation due to Dunn [5].) But 'nots' are modal operators with a twist. They are *order inverting*. So, the question arises: Can we generalise Kracht's results to logics with different sorts of negations? In the presence of boolean negation the answer is clearly affirmative, because negative modal operators can be converted into positive operators with the addition of a boolean negation. This trick isn't available in the absence of boolean negation.

So, we look further afield to the literature on substructural logics. Our primary source of inspiration here is Mike Dunn's work on gaggle theory [3, 4, 5], in which he gives quite general conditions under which operators can have a frame semantics. In this work, we do not assume that boolean negation is present, so it is well suited for our own purposes.

In this note, we will examine these issues, and we will see that the landscape of logics in which we combine modalities and negations is indeed a rich one.

Section 1 of this paper is an introduction to the logics we will consider. Section 2 contains a definition of frames and models for these logics, and provides simple soundness, completeness and correspondence results. These proofs are reasonably standard, the innovation being the necessity to do without boolean nagation. Section 3 covers the undecidability results, and Section 4 considers the interaction of possibilities and negations with intuitionistic implication.

## 1. Logics of Negation and Possibility

Instead of assuming that the underlying logic is classical, we will work with the logic of distributive lattices with greatest and least elements. So, intuitionistic logic fits into our framework, as do relevant logics, and other substructural logics in their vicinity. Significant omissions include linear logic and quantum logic, because they lack the appropriate forms of distribution of conjunction over disjunction. For many applications, distribution is exactly what one would expect [1, 11]. However, for some applications distribution can get in the way, and for those applications the methods discussed in this paper will not work.

So, we assume that any logic **L** under consideration at least contains distributive lattice properties. In other words, its consequence relation $\vdash_{\mathbf{L}}$ (seen as a relation between formulae, and which we write $\vdash$ unless there is more than one logic in view at a time) is transitive, conjunction $\wedge$ and disjunction $\vee$ are associative, commutative, and idempotent, satisfying the absorbtion laws ($A \vdash A \wedge (A \vee B)$ and $A \vee (A \wedge B) \vdash A$) and distribution ($A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$). Finally, we have $\bot \vdash A$ and $A \vdash \top$ for all $A$.

The relation $\vdash$ can be extended to one between sets of formulae and sets of formulae by taking $\Sigma \vdash \Delta$ to be true if and only if some conjunction of formulae in $\Sigma$ entail some disjunction of formulae in $\Delta$.

Given that background, we can start asking questions of what it would be for a logic to possess a modal operator. A *positive modal operator* in our language must be *order preserving*.

$$\text{If } A \vdash B \text{ then } \Diamond A \vdash \Diamond B.$$

Under interpretation, that makes sense. If $A$ entails $B$, then the possibility of $A$ entails the possibility of $B$, similarly, the necessity of $A$ entails the necessity of $B$. $A$ obtaining in the future entails $B$ obtaining in the future, and so on.

We need not posit any other condition on $\Diamond$ for it to be a positive modality. Note that both $\Box$ and $\Diamond$ in any classical modal logic satisfy this condition. As well, modalities postulated in non-classical contexts invariably satisfy the ordering condition. It is simple to show that for any positive modality $\Diamond$ we must have $\Diamond(A \wedge B) \vdash \Diamond A \wedge \Diamond B$ and $\Diamond A \vee \Diamond B \vdash \Diamond(A \vee B)$. But we need not have equalities in the place of entailments here.

So, for the modal operator to be something like a 'possibility' we need two other conditions.

$$\Diamond(A \vee B) \vdash \Diamond A \vee \Diamond B \qquad \Diamond\bot \vdash \bot$$

Any positive modal operator satisfying these conditions is said to be a *p-type* modal operator. That is, it is something like a possibility.

We could analogously define an *n-type* positive modal operator to be one satisfying $\Box A \wedge \Box B \vdash \Box(A \wedge B)$ and $\top \vdash \Box\top$, but this doesn't exhaust the class of positive modalities: there are positive modal operators which are neither n-type nor p-type. For example, $\Box A \wedge \Diamond A$ will in general be neither n-type nor p-type, if $\Box$ is n-type and $\Diamond$ is p-type.

Negative modalities are similar. In the presence of boolean negation, we could define a p-type negative modality as the negation of an n-type positive modality, and *vice versa*. But we do not have that luxury. Instead, we can define them from scratch. A negative modality is a unary operator which is *order inverting*.

$$\text{If } A \vdash B \text{ then } {\sim}B \vdash {\sim}A.$$

Under interpretation this makes a lot of sense. If $A$ entails $B$, then ruling out $B$ (or evidence against $B$, or whatever) rules out $A$ (or is evidence against $A$, or whatever). Clearly boolean negation is of this form, as is intuitionistic negation and minimal negation, and the de Morgan negation of relevant logics, and so on. Note that if $\sim$ is a negative modality and and $\Diamond$ a positive modality, then $\sim\Diamond$ and $\Diamond\sim$ are negative modalities. Similarly, if $\blacklozenge$ is another positive modality and $\neg$ is another negative modality then $\Diamond\blacklozenge$, $\blacklozenge\Diamond$, $\sim\neg$ and $\neg\sim$ are all positive modalities.

An negative modality $\sim$ is an *n-type* negative modality if it satisfies

$$\sim A \wedge \sim B \vdash \sim(A \vee B) \qquad \top \vdash \sim\bot$$

Intuitionistic negation is an n-type negative modality, as is the negation present in relevant logics. For ease of reference, we will call n-type negative modalities *negations* and p-type positive modalities *possibilities*. And from now, we will restrict ourselves to considering logics with a family of negations and possibilities.

Given a language $\mathcal{L}_{m,n}$ with $m$ possibilities $\Diamond_i$ and $n$ negations $\sim_j$, we can form the *basic* logic $\mathbf{K}_{m,n}$. So, $\vdash_{\mathbf{K}_{m,n}}$ is the smallest relation closed under the conditions we have cited. This logic is the simple-minded way of combining $m$ copies of $\mathbf{K}_{1,0}$ together with $n$ copies of $\mathbf{K}_{0,1}$; identifying the underlying distributive lattice structure, while ensuring that the modal operators do not interact in any significant way. This is borne out by the frame semantics, which we introduce below.

## 2. Representations

Study of modal logics would be nothing like it is today were it not for the discovery of frames. It is reassuring to know that any logics extending $\mathbf{K}_{m,n}$ have a semantics in terms of frames.

### 2.1 Defining Frames and Models

A frame $\mathcal{F} = \langle U; R_1, \ldots, R_m; C_1, \ldots, C_n\rangle$ is a collection $U$ of points (worlds, information states, what-have-you) with binary relations $R_1, \ldots, R_m$ and $C_1, \ldots, C_n$ on $U$. We say $xR_iy$ just when relative to $x$, $y$ is $\Diamond_i$-possible, and $xC_jy$ when, relative $x$ is $\sim_j$-compatible with $y$.[2] Given a frame $\mathcal{F}$, a *model*

[2]In some circumstances it is natural to also consider partial ordering on the set of points, under which the truth of formulae is preserved, like the accessibility relation on a frame for intuitionistic logic. But we don't need it now. We will mention it again later when considering ways to extend this work to explicitly consider an intuitionistic conditional.

$\mathcal{M}$ determines a relationship between points and formulae in the following way. We start off with a map $V_{\mathcal{M}}$ which gives for every atomic proposition $p$ the set of points at which $p$ is true. Then we expand this to a relation between points and formulae inductively in the obvious way.

- $\mathcal{M}, x \models p$ if and only if $x \in V_{\mathcal{M}}(p)$.
- $\mathcal{M}, x \models \top$ always.
- $\mathcal{M}, x \models \bot$ never.
- $\mathcal{M}, x \models A \wedge B$ if and only if $\mathcal{M}, x \models A$ and $\mathcal{M}, x \models B$.
- $\mathcal{M}, x \models A \vee B$ if and only if $\mathcal{M}, x \models A$ or $\mathcal{M}, x \models B$.
- $\mathcal{M}, x \models \Diamond_i A$ if and only if for some $y$ where $xR_iy$, $\mathcal{M}, y \models A$.
- $\mathcal{M}, x \models \sim_j A$ if and only if for no $y$ where $xC_jy$, $\mathcal{M}, y \models A$.

Given a model $\mathcal{M}$, there is a notion of deduction associated with the model. Set $A \vdash_{\mathcal{M}} B$ to mean for every $x$, if $\mathcal{M}, x \models A$ then $\mathcal{M}, x \models B$. Note that this notion of deduction may well not satisfy substitution. If, according to $\mathcal{M}$, $p$ was true at every point but $q$ wasn't, then we would have $\top \vdash_{\mathcal{M}} p$ without $\top \vdash_{\mathcal{M}} q$, for example. A broader notion of deduction associated with any *frame* will satisfy substitution. We can define $A \vdash_{\mathcal{F}} B$ to mean that $A \vdash_{\mathcal{M}} B$ for every model $\mathcal{M}$ on $\mathcal{F}$.

## 2.2 Every Frame gives you a Logic (Soundness)

It is quite simple to show that for any frame $\mathcal{F}$, $\vdash_{\mathcal{F}}$ is really a logic.

THEOREM 1. *For any frame $\mathcal{F}$, $\vdash_{\mathcal{F}}$ is a logic, with each $\Diamond_i$ a possibility and each $\sim_j$ a negation.*

PROOF. Clearly conjunction, disjunction, top and bottom have distributive lattice properties. For possibilities and negations, we reason as follows. Suppose $A \vdash_{\mathcal{F}} B$, and take any model $\mathcal{M}$ on $\mathcal{F}$, and a point $x$ where $x \models \Diamond_i A$. Then we must have some $y$ where $y \models A$ and $xR_iy$. So, since $A \vdash_{\mathcal{F}} B$ we have $y \models B$ too, and hence, $x \models \Diamond_i B$ as desired. Similarly, if $x \models \sim_j B$ we must have no $y$ where $xC_jy$ satisfying $y \models B$. But then we couldn't have $y \models A$ (lest $y \models B$ as $A \vdash_{\mathcal{F}} B$) so $x \models \sim_j A$ as desired. We must also show that $\Diamond_i(A \vee B) \vdash_{\mathcal{F}} \Diamond_i A \vee \Diamond_i B$, and $\sim_j A \wedge \sim_j B \vdash_{\mathcal{F}} \sim_j(A \vee B)$. Suppose $x \models \Diamond_i(A \vee B)$. Then there's some $y$ where $xR_iy$ and $y \models A \vee B$. So, either $y \models A$ or $y \models B$, and hence $x \models \Diamond_i A$ or $x \models \Diamond_i B$. Either way, $x \models \Diamond_i A \vee \Diamond_i B$ as desired. Similarly, if $x \models \sim_j A \wedge \sim_j B$, we have for no $y$ where $xC_jy$, $y \models A$ and similarly, for no $y$ where $xC_jy$, $y \models B$. So, for no such $y$ does $y \models A \vee B$, giving $x \models \sim_j(A \vee B)$ as desired. ■

## 2.3 Every Logic gives you a Frame, and a Model (Completeness)

Given any logic $\mathbf{L}$, there is the corresponding frame $\mathcal{F}_{\mathbf{L}}$, in terms of particular sorts of *theories*.

A *theory* in a logic $\mathbf{L}$ is a set $a$ of formulas which is

- Closed under conjunction; if $A \in a$ and $B \in a$ then $A \wedge B \in a$.
- Closed under entailment; if $A \in a$ and $A \vdash B$ then $B \in a$.

A theory $a$ is said to be *non-trivial* if $\top \in a$ and $\bot \notin a$. A theory is said to be *prime* if whenever $A \vee B \in a$ then either $A \in a$ or $B \in a$.

Given a logic $\mathbf{L}$, the corresponding canonical frame $\mathcal{F}_{\mathbf{L}}$ is given as follows. Its set of points $U_{\vdash}$ is the set of all non-trivial prime theories in $\mathbf{L}$, and the relations $R_i$ and $C_j$ are determined as follows.

- $aR_ib$ if and only if whenever $A \in b$, $\Diamond_i A \in a$.
- $aC_jb$ if and only if whenever $A \in b$, $\sim_j A \notin a$.

Clearly the resulting structure $\langle U_{\vdash}; R_1, \ldots, R_m; C_1, \ldots, C_n\rangle$ is a frame. We call it the *canonical frame* of $\mathbf{L}$.

The interesting work is done in showing that there is a model on a canonical frame satisfying $\mathcal{M}, a \models A$ if and only if $A \in a$. The model $\mathcal{M}_{\mathbf{L}}$ on $\mathcal{F}_{\mathbf{L}}$ given by setting $V_{\mathcal{M}_{\mathbf{L}}}(p) = \{a : p \in a\}$.

THEOREM 2. *For any logic $\vdash$, and any prime theory $a$, $A \in a$ if and only if $\mathcal{M}_{\mathbf{L}}, a \models A$.*

To prove this, it is helpful to make use of the *Pair Extension Lemma* due to Meyer, Dunn and Leblanc [8] and independently, Gabbay [6]. For that we need the definition of a special kind of pair of sets of sentences. Relative to a background logic $\mathbf{L}$, $\langle b, c\rangle$ is said to be a *pair* if $b$ and $c$ are sets of formulae. The pair $\langle b, c\rangle$ is said to be $\mathbf{L}$-*exclusive* if for no $B_1, \ldots, B_k \in b$ and no $C_1, \ldots, C_l \in c$ do we have $B_1 \wedge \cdots \wedge B_k \vdash C_1 \vee \cdots \vee C_l$. The pair $\langle b, c\rangle$ is said to be *exhaustive* if $b \cup c = \mathcal{L}_{m,n}$. Finally, a pair $\langle b', c'\rangle$ extends $\langle b, c\rangle$ just when $b \subseteq b'$ and $c \subseteq c'$.

LEMMA 3. *For any logic $\vdash$ extending $\vdash_{\mathbf{K}_{m,n}}$ If $\langle b, c\rangle$ is a $\mathbf{L}$-exclusive pair, then there is an exhaustive $\mathbf{L}$-exclusive pair $\langle b', c'\rangle$ extending $\langle b, c\rangle$.*

PROOF. We must use the countability of the language $\mathcal{L}_{m,n}$ (or equivalently, the axiom of choice to well order the language if it is not countable).

Take an enumeration $D_1, D_2, \ldots$ of the language, and define $\langle b_i, c_i \rangle$ as follows: $\langle b_0, c_0 \rangle = \langle b, c \rangle$, and for any $i$, $\langle b_{i+1}, c_{i+1} \rangle = \langle b_i \cup \{D_{i+1}\}, c_i \rangle$ if this is $\vdash$-exclusive, or $\langle b_i, c_i \cup \{D_{i+1}\} \rangle$ otherwise.

We show that each $\langle b_i, c_i \rangle$ is **L**-exclusive. If one isn't, take the first such that isn't; say $\langle b_{j+1}, c_{j+1} \rangle$. (By hypothesis, $\langle b_0, c_0 \rangle$ is **L**-exclusive.) This can only fail to be **L**-exclusive if we have some $B_1, \ldots, B_k \in b_j$ and $C_1, \ldots, C_l \in c_j$ where both

$$B_1 \wedge \cdots \wedge B_k \wedge D_{j+1} \vdash C_1 \vee \cdots \vee C_l \quad \text{and} \quad B_1 \wedge \cdots \wedge B_k \vdash C_1 \vee \cdots \vee C_l \vee D_{j+1}$$

But by distributive lattice properties, this gives us $B_1 \wedge \cdots \wedge B_k \vdash C_1 \vee \cdots \vee C_l$, contradicting the **L**-exclusiveness of $\langle b_j, c_j \rangle$. So, each $\langle b_i, c_i \rangle$ must be **L**-exclusive, and hence, so must $\langle b', c' \rangle = \langle \bigcup_i b_i, \bigcup_i c_i \rangle$, which is exhaustive, and extends $\langle b, c \rangle$ as desired. ∎

The pair-extension lemma is important for us, because of the following result.

LEMMA 4. *If $\langle b, c \rangle$ is an **L**-exclusive, exhaustive pair, then $b$ is a prime theory.*

PROOF. Take $A, B \in b$. Clearly $A \wedge B \in b$, because $A \wedge B \vdash A \wedge B$, so we cannot have $A \wedge B \in c$. Take $A \in b$, and $A \vdash B$. Clearly we cannot have $B \in c$, so we must have $B \in a$. Finally, take $A \vee B \in b$. If neither $A$ nor $B$ were in $b$, they would both be in $c$, violating the **L**-exlcusiveness of $\langle b, c \rangle$, since $A \vee B \vdash A \vee B$. ∎

Now we can prove our completeness theorem.

PROOF. To show that $\mathcal{M}_{\mathbf{L}}, a \models A$ if and only if $A \in a$, we proceed by induction on the construction of $A$. The result holds by definition in the base case, and trivially for the lattice connectives $\bot$, $\top$, $\wedge$ and $\vee$ (since each $a$ is a non-trivial prime theory). Consider the cases for possibilities and negations. First for possibilities: $\mathcal{M}_{\mathbf{L}}, a \models \Diamond_i A$ if and only if there is some $b$ where $aR_iB$ and $A \in b$ (by induction hypothesis). Clearly if there is such a $b$, then we must have $\Diamond_i A \in a$, by the definition of $R_i$. To show that if $\Diamond_i A \in a$ then there is a complying $b$ where $A \in b$ and $aR_ib$, we proceed as follows. Set $b = \{B : A \vdash B\}$. It is clear that this is a theory. It is non-empty (since $A \in b$) and non-trivial ($\bot \not\in b$, because if $A \vdash \bot$, then $\Diamond_i A \vdash \Diamond_i \bot$, so $\Diamond_i \bot \in a$, but $\Diamond_i \bot \vdash \bot$ would give $\bot \in a$, contradicting the non-triviality of $a$). But it may not be prime. To 'beef' $b$ up to a prime theory, note that where $c = \{C : \Diamond_i C \not\in a\}$, $\langle b, c \rangle$ is a **L**-exclusive pair. To see this, note first that $c$ is closed under disjunction. If $\Diamond_i C_1, \Diamond_i C_2 \notin a$, then $\Diamond_i(C_1 \vee C_2) \notin a$ too. So, for all $B_1, \ldots, B_k \in b$ and $C_1, \ldots, C_l \in c$, $B_1 \wedge \cdots \wedge B_k \vdash C_1 \vee \cdots \vee C_l$ only when there is some $C \in c$ where $A \vdash C$. But this means $\Diamond_i A \vdash \Diamond_i C$, giving $\Diamond_i C \in a$, contradicting $C \in c$. So, $\langle b, c\rangle$ must be **L**-exclusive.

By the pair extension lemma, we must have a **L**-exclusive pair $\langle b', c'\rangle$ extending $\langle b, c\rangle$. This ensures that $b'$ is a non-trivial prime theory. And furthermore, $aR_ib'$, since whenever $\Diamond_i A \in b'$, we must have $A \in a$, because $\Diamond_i A \notin c$. This ensures that if $\Diamond_i A \in a$, then there is a $b'$ where $aR_ib'$ and $A \in b'$, establishing the induction case for possibilities.

The case for negation is similar. We have, by hypothesis, that $\mathcal{M}_{\mathbf{L}}, a \models {\sim_j} A$ if and only if for every $b$ where $aC_jb$, $A \notin b$. Clearly if ${\sim_j} A \in a$ then for any $b$ where $aC_jb$ we have $A \notin b$, by the definition of $C_j$. The interest is in the other direction. If ${\sim_j} A \notin a$, we want to find a $b$ where $aC_jb$, and $A \in b$. We start as before, with $b = \{B : A \vdash B\}$ a non-trivial theory satisfying our conditions. We note that with $c = \{C : {\sim_j} C \in a\}$, $\langle b, c\rangle$ is **L**-exclusive, and this gives us an exclusive $\langle b', c'\rangle$ extending $\langle b, c\rangle$. This makes $b'$ a prime theory (still non-trivial, as $\bot \in c'$, since $\top \vdash {\sim_j}\bot$, and $\top \in a$) and $aC_jb'$ by the construction of $c$. This completes the proof. ■

From this result it follows that $A \vdash_{\mathbf{K}_{m,n}} B$ if and only if $A \vdash_{\mathcal{M}} B$, where $\mathcal{M}$ is the canonical model for $\mathbf{K}_{m,n}$. But clearly, $A \vdash_{\mathcal{M}} B$ if $A \vdash_{\mathcal{F}} B$ (where $\mathcal{F}$ is the canonical frame), if $A \vdash_{\mathcal{F}} B$ for all frames $\mathcal{F}$. But if $A \vdash_{\mathcal{F}} B$ for all frames $\mathcal{F}$, we know that $A \vdash_{\mathbf{K}_{m,n}} B$, since the logic of any frame is at least the logic $\mathbf{K}_{m,n}$. So, each of the following are equivalent.

- $A \vdash_{\mathbf{K}_{m,n}} B$
- $A \vdash_{\mathcal{M}} B$, where $\mathcal{M}$ is the canonical model for $\mathbf{K}_{m,n}$
- $A \vdash_{\mathcal{F}} B$, where $\mathcal{F}$ is the canonical frame for $\mathbf{K}_{m,n}$
- $A \vdash_{\mathcal{F}} B$ for every frame $\mathcal{F}$

We will be interested in a number of logics extending the basic logic $\mathbf{K}_{m,n}$.

## 2.4 Correspondence

Consider any model $\mathcal{M}$ in which $\top \vdash_{\mathcal{M}} \Diamond_i \top$. This happens only when for every point $x$, there's some point $y$ where $xR_iy$. In this case, the accesibility relation $R_i$ is said to be *directed*. Conversely, if this is the case in the model $\mathcal{M}$, then we must have $\top \vdash_{\mathcal{M}} \Diamond_i \top$. So, the deduction $\top \vdash \Diamond_i \top$ corresponds to the condition that $R_i$ is directed.

Similarly, if $\sim_j \top \vdash_{\mathcal{M}} \top$, then $C_j$ must be directed, and conversely, if $C_j$ is directed in a model $\mathcal{M}$, then $\sim_j \top \vdash_{\mathcal{M}} \top$.

These two results are special, because they relate deductions to conditions which must hold in all models which validate those deductions. Very few conditions are like that. To see an example, consider

$$\Diamond_i p \wedge \Diamond_i q \vdash \Diamond_i (p \wedge q)$$

If this holds in a model $M$, we must have for every point $x$, $x \models \Diamond_i p \wedge \Diamond_i q$, only when $x \models \Diamond_i (p \wedge q)$. This might happen because $x \not\models \Diamond_i p \wedge \Diamond_i q$. It may tell us nothing about $R_i$ at all. This is not the case with *frames*. We will see that for any frame $\mathcal{F}$, $\Diamond_i p \wedge \Diamond_i q \vdash_{\mathcal{F}} \Diamond_i (p \wedge q)$ if and only if the frame is *single alternative* in $R_i$. That is, if and only if for every $x$, if $xR_iy$ and $xR_iz$, then $y = z$. To see this, note that if $x \models \Diamond_i p \wedge \Diamond_i q$ and $x \not\models \Diamond_i (p \wedge q)$ we must have $y, z$ where $xR_iy$ and $xR_iz$, $y \models p$, $z \models q$, and in addition, $y \neq z$. So clearly, if $\mathcal{F}$ is single alternative in $R_i$, then $\Diamond_i p \wedge \Diamond_i q \vdash_{\mathcal{F}} \Diamond_i (p \wedge q)$. Conversely, if $\mathcal{F}$ is not single aternative in $R_i$, then for some $x, y, z$ we have $xR_iy$, $xR_iz$ and $x \neq z$. Then construct a model in which $V(p) = \{y\}$, $V(q) = \{z\}$, and it follows that $x \models \Diamond_i p \wedge \Diamond_i q$ but $x \not\models \Diamond_i (p \wedge q)$, as desired.

Similarly, we can show that $\sim_j (p \wedge q) \vdash_{\mathcal{F}} \sim_j p \vee \sim_j q$ if and only if $\mathcal{F}$ is single alternative in $C_j$.

In the classical case, if **L** were a logic satisfying $\Diamond_i p \wedge \Diamond_i q \vdash \Diamond_i (p \wedge q)$, then the canonical frame would also be single alternative. We would argue that if $aR_ib$ and $aR_ic$, and $b \neq c$, then there is some $A$ where $A \in b$, and $\sim A \in c$ (using boolean negation). This means that $\Diamond_i A \wedge \Diamond_i \sim A \in a$, giving $\Diamond_i (A \wedge \sim A) \in a$, by the single alternative rule. This is impossible, as $\Diamond_i (A \wedge \sim A) \vdash \bot$, giving $\bot \in A$ which we know is not true.

Without boolean negation in our language we cannot reason in this way. In fact, there will be *many* failures of the single alternative condition in our canonical frames. This is because if $aR_ib$, then $aR_ib'$ for any $b' \subseteq b$. So all we must do to make the single alternative condition fail is to cut down our theories to strictly smaller theories.

However, all is not lost. Given our canonical frame $\mathcal{F}$, for a logic **L** with a possiblity satisfying single alternative, we can define a new frame $\mathcal{F}'$ which does satisfy single alternative, and in which there is a model $\mathcal{M}'$ satisfying $\mathcal{M}', a \models A$ if and only if $a \in A$. The construction is quite simple. Firstly, in the canonical frame $\mathcal{F}$, define $x^{\Diamond}$ as $\bigcup_{y:xRy} y$, if there is some $y$ where $xRy$. We will show that $x^{\Diamond}$ is a prime theory, satisfying $xRx^{\Diamond}$. That $xRx^{\Diamond}$ is simple. If $A \in x^{\Diamond}$ then $A \in y$ for some $y$ where $xRy$, and hence, $\Diamond A \in x$ as desired. That $x^{\Diamond}$ is closed under entailment, prime and non-trivial is immediate from its construction. That it is closed under conjunction is given by the single alternative axiom. Take $A, B \in x^{\Diamond}$, so $A \in y$ for some $y$ where $xRy$, and $B \in z$ for some $z$ where $xRz$. This means that $\Diamond A \in x$ and $\Diamond B \in x$, giving $\Diamond A \wedge \Diamond B \in x$ and hence $\Diamond(A \wedge B) \in x$, which ensures that for some $w$ where $xRw$, $A \wedge B \in w$; which in turn ensures that $A \wedge B \in x^{\Diamond}$ as desired. It follows that if we reduce $R_i$ to satisfy $xR_iy$ if and only if $y = x^{\Diamond}$, we have the reduced canonical frame. A similar construction helps us define $x^{\sim}$ if $\sim$ satisfies the single alternative axiom. We define $x^{\sim}$ as the union of all of the theory $y$ $C$-compatible with $x$. The set $x^{\sim}$ is clearly prime, non-trivial and closed under consequence. The only interesting detail is its closure under conjunction. But the single alternative rule $\sim(p \wedge q) \vdash \sim p \vee \sim q$ sees to that.

Let the logic $\mathbf{OA}_{m,n}$ ($\mathbf{OA}$ for 'one alternative') be the smallest logic extending $\mathbf{K}_{m,n}$ with the addition of $\top \vdash \Diamond_i \top$ and $\Diamond_i p \wedge \Diamond_i q \vdash \Diamond_i(p \wedge q)$ for each $i$ together with $\sim_j \top \vdash \bot$ and $\sim_j(p \wedge q) \vdash \sim_j p \vee \sim_j q$ for each $j$. We then have the following result, where we call a relation *functional* iff it is single alternative and directed.

THEOREM 5. *$A \vdash_{\mathbf{OA}_{m,n}} B$ if and only if $A \vdash_{\mathcal{F}} B$ in each frame $\mathcal{F}$ where each accessibility relation is functional.*

There is something interesting about frames in which each accessibility relation is functional. (From now we will call those *functional* frames.) We'll write the function associated each accessibility relation $R_i$ or $C_j$ as $r_i$ and $c_j$. In other words, $r_i(x)$ is the $y$ such that $xR_iy$ and $c_j(x)$ is the $y$ such that $C_jy$. So, for each frame $\mathcal{F}$ there is a corresponding semigroup $\mathcal{S}_{\mathcal{F}}$, generated by the functions $r_i$ and $c_j$, under composition. These functions together determine the behaviour of every modality in the language; not only the primitive ones. For example, $x \models \Diamond_1 \sim_2 p$ if and only if $c_2(r_1(x)) \not\models p$. This fact will become useful later. Note too that the semigroup $\mathcal{S}_{\mathcal{F}}$ corresponding to a frame is (isomorphic) to a quotient semigroup of the free semigroup $\mathcal{S}_{m+n}$ on $m+n$ generators. (It is generated by $m+n$ generators, and we know that every semigroup so generated is a quotient of $\mathcal{S}_{m+n}$.)

## 3. Undecidability

To extract an undecidability result from what we have so far, we can use the fact that the word problem for finitely presented semigroups is in general, unsolvable. In other words, for a given finitely presented semigroup $\mathcal{S} = \langle x_1, \ldots, x_n \mid eq_1, \ldots, eq_l \rangle$ where each $eq_j$ is an equation between words made up of the generators $x_i$, the problem of determining whether an arbitrary given equation $eq$ is true is not solvable. This is equivalent to the problem of determining whether $eq$ is true in every semigroup generated by $x_1, \ldots, x_n$ and satisfying the equations $eq_1, \ldots, eq_l$. We can transform this problem into a problem concerning logics extending $\mathbf{OA}_{m,n}$ in a simple way. First, we need to translate semigroup equations into sequents. For this, we associate with every word $w$ made from the alphabet $r_1, \ldots, r_m$, $c_1, \ldots, c_n$ a modality $\langle w \rangle$ as follows. Firstly, $\langle r_i \rangle = \Diamond_i$ and $\langle c_j \rangle = \sim_j$. Then $\langle w_1 w_2 \rangle = \langle w_1 \rangle \langle w_2 \rangle$. Given a word $w$ or a modality $\langle w \rangle$ we take its *character* to be *positive* if there is an even number of $c_j$s in $w$, and *negative* if there is an odd number of $c_j$s in $w$. It is simple to show that $\langle w \rangle$ is a possibility if $w$ is positive, and a negation if $w$ is negative. Note too that in a given one-alternative model $\mathcal{M}$, $\mathcal{M}, x \models \langle w \rangle A$ if and only if $\mathcal{M}, w(x) \models A$ (if $w$ is positive) or $\mathcal{M}, w(x) \not\models A$ (if $w$ is negative), where we interpret $w$ as a function of points in the frame in the obvious way.

We can tie together equations in the language of words on the elements $r_i$ and $c_j$ together with sequents in the logic in a simple way.

LEMMA 6. *Given a one-alternative frame $\mathcal{F}$, the semigroup over the frame $\mathcal{S}_\mathcal{F}$ satisfies the equation $w_1 = w_2$ if and only if*

- $\langle w_1 \rangle p \vdash_\mathcal{F} \langle w_2 \rangle p$*; if $w_1$ and $w_2$ have the same character.*
- $\langle w_1 \rangle p \wedge \langle w_2 \rangle p \vdash_\mathcal{F} \bot$*; if $w_1$ and $w_2$ have different character.*

We call the condition corresponding to the equation $\ulcorner w_1 = w_2 \urcorner$ the *condition corresponding to* $w_1 = w_2$, and we sometimes use '$[w_1 = w_2]$' as a shorthand for it. Conditions of this form are called *equational conditions*. We will call the equation $\ulcorner w_1 = w_2 \urcorner$ *balanced* if $w_1$ and $w_2$ have the same character, and *unbalanced* otherwise.

PROOF. The proof is quite simple. First, suppose $w_1$ and $w_2$ have the same character. Then if $\mathcal{S}_\mathcal{F}$ satisfies $w_1 = w_2$, then whenever $\mathcal{M}, x \models \langle w_1 \rangle p$ we must have $\mathcal{M}, w_1(x) \models ? p$ (where $\models ?$ is one of $\models$ and $\not\models$ depending on the character of $w_1$) and hence $\mathcal{M}, w_2(x) \models ? p$, giving $\mathcal{M}, x \models \langle w_2 \rangle p$ since $w_1$ and $w_2$ have the same character. Conversely, if $\mathcal{M}, x \models \langle w_1 \rangle p$ gives $\mathcal{M}, x \models \langle w_2 \rangle p$ we must have $w_1(x) = w_2(x)$ always, since we could otherwise construct a model in which $\mathcal{M}, x \models \langle w_1 \rangle p$ and $\mathcal{M}, x \not\models \langle w_2 \rangle p$.

The argument for the case where $w_1$ and $w_2$ have different character is quite similar. Without loss of generality, suppose $w_1$ is positive and $w_2$ negative. Firstly, suppose $w_1 = w_2$ in $\mathcal{S}_\mathcal{F}$. We must have $\mathcal{M}, x \models \langle w_1 \rangle p \wedge \langle w_2 \rangle p$ if and only if $\mathcal{M}, w_1(x) \models p$ and $\mathcal{M}, w_2(x) \not\models p$. This is impossible, if $w_1 = w_2$ in $\mathcal{S}_\mathcal{F}$. So, we must have $\langle w_1 \rangle p \wedge \langle w_2 \rangle p \vdash_\mathcal{F} \bot$ as desired. Conversely, suppose we never have $\mathcal{M}, x \models \langle w_1 \rangle p$ and $\mathcal{M}, x \models \langle w_2 \rangle p$. If $w_1 \neq w_2$ in $\mathcal{S}_\mathcal{F}$ we must have some $y$ where $w_1(y) \neq w_2(y)$. Let $\mathcal{M}$ be a model in which $p$ is true only at $w_1(y)$. This would give $\mathcal{M}, y \models \langle w_1 \rangle p$ (since $\mathcal{M}, w_1(y) \models p$) and $\mathcal{M}, y \models \langle w_2 \rangle p$ (as $\mathcal{M}, w_2(y) \not\models p$) contradicting our hypothesis. ∎

We will call any logic extending $\mathbf{OA}_{m,n}$ with a family of conditions $[w_{i1} = w_{i2}]$ a *semigroup logic*. Clearly for any finitely generated semigroup there is a corresponding semigroup logic. (There is actually more than one corresponding logic, because you have a choice of whether each generator is modelled by a possibility or a negation.) Given this we have the following result.

THEOREM 7. *For any semigroup logic $\mathbf{L}_{m,n}$ with a collection $[w_{i1} = w_{i2}]$ of equational conditions, $A \vdash_{\mathbf{L}_{m,n}} B$ if and only if $A \vdash_{\mathcal{F}} B$ for every one-alternative frame satisfying the equations $w_{i1} = w_{i2}$.*

This has as a simple corollary, our undecidability result.

COROLLARY 8. *There are undecidable finitely axiomatised semigroup logics entending $\mathbf{OA}_{m,n}$, for each $m, n$ where $m + n \geq 2$.*

PROOF. Let $\mathcal{S} = \langle r_1, \ldots, r_m, c_1, \ldots, c_n \,|\, eq_1, \ldots, eq_l \rangle$ be a finitely presented undecidable semigroup. (Such exist for every $m + n \geq 2$). Consider the logic $\mathbf{L}_{m,n}$ extending $\mathbf{OA}_{m,n}$ with the addition of the axioms $[eq_i]$ for each $i = 1, \ldots, l$.

We know that the semigroup $\mathcal{S}_{\mathcal{F}}$ corresponding to a frame $\mathcal{F}$ for the logic $\mathbf{L}_{m,n}$ must satisfy each equation $eq_i$. We also know that for any semigroup $\mathcal{S}'$ generated by $m + n$ generators satisfying the equations $eq_i$ gives us a frame $\mathcal{F}_{\mathcal{S}'}$ for the logic $\mathbf{L}_{m,n}$. So, the logic $\mathbf{L}_{m,n}$ neatly characterises the class of semigroups on $m + n$ generators satisfying our equations.

Take an equation $eq$ of words in the $r_i$ and $c_j$s. It holds in the semigroup $\mathcal{S}$ if and only if it holds in each semigroup corresponding to a frame in which the corresponding conditions $[eq_i]$ are valid. But this is equivalent to $[eq_i]$ holding in the logic $\mathbf{L}_{m,n}$. So, having a decision procedure for $\vdash_{\mathbf{L}_{m,n}}$ suffices for a decision procedure for the word problem for $\mathcal{S}$. As there is no such procedure for $\mathcal{S}$, we have no decision procedure for $\vdash_{\mathbf{L}_{m,n}}$ either. ∎

Let's take stock of what we have seen so far. This last result shows us how to construct an undecidable logic from a semigroup with undecidable word problem. The process goes as follows: given a finitely presented semigroup with generators $x_1$ to $x_k$, you decide which of the generators you wish to pair with possibilities and which with negations. They could all be possibilities, or all negations, if you like. However you choose, you will have $m$ possibilities and $n$ negations. So, your target logic will be an extension of $\mathbf{OA}_{m,n}$. Then, for each of the equations $eq_l$ in the semigroup presentation, you get the corresponding axiom $[eq_l]$. The logic extending $\mathbf{OA}_{m,n}$ with each axiom $[eq_l]$ will then be undecidable. Deciding $[w = w']$ in the logic is equivalent to deciding $\ulcorner w = w' \urcorner$ in the semigroup.

It is quite simple to prove that there is an exact equivalence between difficulty of deciding the semigroup and deciding the logic. Having at hand a decision procedure for the semigroup gives us a decision procedure for the logic in a rather simple fashion. Kracht [7] has one method, by normal forming sentences in the logic. Another, more directly suited to our purposes, is a simple tableaux method. To decide $A \vdash B$ in our logic, start a tableaux with $A : 0$ and $\overline{B} : 0$ at the root. The 0 indicates that we are at the root point in a model, and the overline indicates that $B$ is false at 0 (while $A$ is true) — this is a signed, labelled tableau. We employ the usual tableaux rules for conjunction, disjunction, $\bot$ and $\top$. For modalities, when we encounter $\langle w \rangle A : x$, we enter $A : wx$ where $w$ is positive, and $\overline{A} : wx$ where $w$ is negative. Similar rules apply for $\overline{\langle w \rangle A}$. We let $w0$ reduce to $w$, and we take a branch to close just when it features either $\bot : w$ or $\overline{\top} : w$ for some $w$, or $A : w$ and $\overline{A} : w'$ where $w = w'$ in our semigroup. It is clear that if the tableau of $A \vdash B$ has an open branch then we can construct a model invalidating $A \vdash B$. If the tableau closes, then $A \vdash B$ must be provable. So, deciding the semigroup (having an oracle deciding each $\ulcorner w = w' \urcorner$) gives us a decision procedure for the logic.

That completes our discussion of one level of combination, considering the weaving together of possibilities and negation, and the expressive power they give us. Now we will see what happens when we combinine these logics with Heyting's calculus **J**.

## 4. Intuitionistic implication and Combining frames

The modal logics we have been studying are all rather inexpressive. We have no way of forming conditionals. Metatheoretic statements (like '$A \vdash B$', telling us that $B$ follows from $A$) cannot be expressed in the language $\mathcal{L}_{m,n}$ itself. There are a number of ways of remedying this deficiency. Firstly, we can add boolean negation, and then the material 'conditional' will support a deduction theorem ($\Sigma, A \vdash B$ if and only if $\Sigma \vdash {\sim}A \vee B$). But as I have mentioned, there are uses for logics in which boolean negation is not present. More plausible for our own purposes is the addition of an intuitionistic conditional. For those interested in intuitionnistic logic this move needs no justification. For those interested in substructural logics a few words are needed. As I have argued elsewhere [10] the intuitionistic conditional is at home in substructural settings. Adding it is always a conservative extension of any traditional substructural logic (provided it has a distributive lattice $\wedge$, $\vee$, $\top$ and $\bot$ fragment like that we've been discussing). Furthermore, the moves we make in pasting together intuitionistic logic and modal logics will have analagous moves in the substructural setting. So, with those remarks out of the way, let's consider where the addition of the intuitionistic conditional leads us.

Firstly we must decide how we are to combine the intuitionistic and modal logics. Given a modal logic **M** of the sort we have been discussing, is there a logic **JM**, in the language $\mathcal{L}^{\supset}_{m,n}$ (extending $\mathcal{L}_{m,n}$ by adding the intuitionistic conditional) which conservatively extends **J** and which conservatively extends **M**. In other words, can we combine **J** and **M**, without distorting their underlying structure. This is not obviously affirmitive. A single-alternative negation like those present in **M** doesn't feature in **J**, and perhaps adding one might lead to some kind of collapse in the intuitionistic structure. Let's start the investigation with a definition. We'll take, for any logic **M**, the logic **JM** to be the smallest relation $\vdash$ on the language $\mathcal{L}^{\supset}_{m,n}$ which extends the consequence relations in **J** and in **M**, and is closed under uniform substitution.

To do any work with logics like **JM** it is helpful to consider their models and frames. And this is not completely trivial. Clearly any frame for a logic like **JM** will be of the form $\langle U; \leq; R_1, \ldots, R_m; C_1, \ldots, C_n \rangle$, where the $R_i$ and $C_j$ are as before, and $\leq$ is a partial order on $U$ for modelling the intuitionistic conditional. Models, however, add an extra subtlety. For any evaluation $V_{\mathcal{M}}$ we must be careful that it satisfy the heredity condition:

For all $p$, if $x \in V_{\mathcal{M}}(p)$ and $x \leq y$ then $y \in V_{\mathcal{M}}(p)$ too.

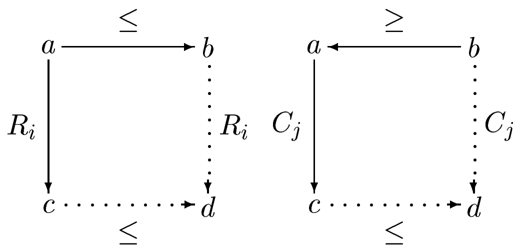
This is necessary if $\leq$ is to model informational inclusion on $U$. Then we can define $\models$ as a relation between points and formulae in the inductive manner as before, extending the definition with the usual clause for $\supset$

- $\mathcal{M}, x \models A \supset B$ if and only if for each $y$ where $x \leq y$ if $\mathcal{M}, y \models A$ then $\mathcal{M}, y \models B$.

However we will not be able to prove the general property of general monotonicity (if $x \models A$ and $y \geq x$ then $y \models A$ too) unless there are conditions relating the $R_i$s and $C_j$s to $\leq$. The usual inductive proof fails when you get to the modalities. In this way, the simple-minded combination of frame conditions does not capture the simple-minded combination of logics. To capture the combined logic we need do some more work. And the least amount of work possible are the following minimal conditions, the first of which was originally due to Božić and Došen [2].

- Whenever $a \leq b$ and $aR_ic$ there is some $d$ where $c \leq d$ and $bR_id$.
- Whenever $a \geq b$ and $aC_jc$ there is some $d$ where $c \leq d$ and $bC_id$.

Note that in the second of these conditions the ordering relation is twisted. This reflects the fact that negation is order inverting. The conditions are much more memorable when displayed diagrammatically.




Once we specify that any modal intuitionistic frame must satisfy these conditions, we can show that modal formulae are preserved upward in frames. The condition relating $\leq$ to $R_i$ is what we need to show that if $a \models \Diamond_i A$ and $a \leq b$ then $b \models \Diamond_i A$ too. The condition relating $\leq$ to $C_j$ ensures that if $a \not\models \sim_j A$ and $b \leq a$ then $b \not\models \sim_j A$ also.

It's not our place here to greatly further the study of intuitionistic modal logics. Rather, we'll simply consider the properties of **JM** where **M** is a semigroup logic. Specifically, we'll concern ourselves with two facts. First, **JM** is a conservative extension of **M** (that's trivial). Second, **JM** is a conservative extension of **J** (that's not so completely trivial).

First, we'll do away with the trivial facts. It is simple to see the following:

THEOREM 9. *For any semigroup logic* **M** *with a corresponding semigroup* $\mathcal{S} = \langle r_1, \ldots, r_m, c_1, \ldots, c_n | eq_1, \ldots, eq_l \rangle$, **JM** *is sound and complete with respect to the class of intuitionistic frames* $\mathcal{F}$, *such in which the modal accessibility relations are functional, and such that* $\mathcal{S}_{\mathcal{F}}$ *satisfies each equation* $eq_i$.

PROOF. Soundness is obvious. Any frame satisfying these conditions is a frame for **J**, and it is a frame for **M**. For completeness we need just show that the canonical frame is an intuitionistic modal frame, for we have already seen that it satisfies the modal conditions. In the frame we use $\subseteq$ to do work for the intuitionistic relation $\leq$. The standard proof shows that this gets the condition for the intuitionistic conditional right. The only interesting work involves showing that the relationship between $\leq$ and $R_i$ and $C_j$ is satisfied.

Recall that with single-alternative modalities we define $R'_i$ and $C'_j$ in the single-alternative canonical frame as follows: we set $x^{\Diamond_i} = \bigcup_{xR_iy} y$ (where $R_i$ is defined as usual as a relationship between theories. Clearly we have that if $x \leq x'$ then if $xR_iy$ we have $x'R_iy$ too (look back to the definition if you can't see this immediately). So, if $x \leq x'$ then $x^{\Diamond_i} \leq x'^{\Diamond_i}$. Similarly, we can show that $x^{\sim_j} \geq x'^{\sim_j}$. As we set $xR_iy$ if and only if $y = x^{\Diamond_i}$ and $xC_jy$ if and only if $y = x^{\sim_j}$ we have our result. The canonical frame is an intuitionistic modal frame. ∎

So, to prove our first fact (that **JM** conservatively extends **M**) we need only show that anything falsifiable in an **M**-frame is falsifiable in a **JM**-frame. But that's trivial. Take an **M** frame, and on it, set $\leq$ to be identity. That's your required **JM**-frame.

Now it is trickier to establish the result in the other direction. We need to show that for any thing falsifiable in a **J**-frame, there's a corresponding **JM**-frame in which it can be falsified. If **M** is made up only of possibilities, we can simply say that on the **J** frame we take each accessibility relation $R_i$ to be identity, ensuring both that the relations between each $R_i$ and $\leq$ hold, and that each semigroup equation holds (well, they all reduce to saying id = id). So, we'd be home. Proof theoretically this is just like saying: take any proof of anything you might suspect to be intuitionistically underivable. Replace all formulas of the form $\Diamond_i A$ by $A$. The result is still a proof. (Check all of the rules involving positive modalities. Squint so that you can't see the $\Diamond_i$s. The results are valid under this 'reading'.) So, what we suspected as being intuitionistically underivable isn't.

But life is not that simple in the presence of *negations*. And nor ought we to have expected it to be, because our negations satisfy things like $\vdash {\sim}(A \wedge B) \supset {\sim}A \vee {\sim}B$, ${\sim}\top \vdash \bot$ and no amount of squinting will make both of those valid at once. And there's good reason for that: some extensions of **J** with respect to single alternative negations aren't conservative at all. The obvious example is boolean negation. If we require that $\sim$ be single alternative, and that it satisfy ${\sim}A \wedge A \vdash \bot$, then we know that the relation $C$ collapses to identity. Then if $x \leq y$ we can argue that since $yCy$ there is some $z$ where $xCz$ and $z \geq y$. However, $xCz$ tells us that $z = x$, and hence that $\leq$ is an equivalence relation: we have argued that if $x \leq y$ then $y \leq x$. So, **J** collapses to classical logic. (Proof theoretically we can argue as follows: we have $A \wedge {\sim}A \vdash \bot$. Hence ${\sim}A \vdash A \supset \bot$, by residuation. Then $\top \vdash A \vee {\sim}A$ gives $\top \vdash A \vee (A \supset \bot)$, which is an intuitionistic undesirable.)

What assurance do we have that this doesn't occur in our semigroup logics? One simple fact. We have defined our logics in terms of semigroups, and we can define a model as follows. Take any **J** frame $\mathcal{F}$, on a set $U$, and add a point $u$ to $U$, making $U^+$. On $U^+$ we define $\leq$ as before, adding that $u \leq u$ (and $u$ is unrelated to any element of $U$). Clearly $\leq$ is still a partial order. Now we define each modality $R_i$ and $C_i$, by specifying that $xR_iy$ if and only if $xC_jy$ if and only if $y = u$. So what we have is that $u$ is the (only) modal alternative of any point in the original model. It is simple to show that this definition satisfies the intuitionistic modal conditions. It is almost as trivial to show that the resulting semigroup operations on the frame satisfy the semigroup equations. They must, because the semigroup operations are all identical: They are the constant function $f : U^+ \to \{u\}$. Clearly, then, under interpretation, each semigroup equation is satisfied. Under composition, modalities collapse. Not completely: $\Diamond{\sim}A$ is not the same as $\Diamond\Diamond A$, for the first is positive, and the other negative. However, $\Diamond{\sim}A$ and ${\sim}A$ are equivalent in this frame, as are $\Diamond A$ and $\Diamond\Diamond A$.

In this new frame $\mathcal{F}^+$ we can invalidate anything which doesn't hold in $\mathcal{F}$ by taking the same evaluation into $U$ and evaluating atomic formulae at $u$ in any way you like. This is still a countermodel for the intuitionistic formula, as $u$ is 'intuitionistically isolated' from $U$. So, we have proved the following.

THEOREM 10. *For any semigroup logic* **M**, **JM** *is a conservative extension of* **J**.

This result is not easily generalised. If we add the 'identity' modality $\langle\epsilon\rangle$, considering *monoid* actions on the frame,[3] then we have the difficulties we have sketched above. This difficulty is present if we allow *inverses* to our modalities, hence giving us *group* actions on frames. Assuming that the equations in our monoid or group are *balanced* however, we have a conservative extension. Recall that $\ulcorner w_1 = w_2 \urcorner$ is balanced if and only if $w_1$ and $w_2$ have the same character. We assume that $\epsilon$ is positive (if it were negative we'd have boolean negation, and that's giving up before we start) and that $w$ and $w^{-1}$ have the same character (if present). That means that when we use $r_i$ for a possibility, we use $r_i^{-1}$ for a possibility too, and similarly for $c_j$, $c_j^{-1}$ and negations. We assume this because otherwise, the presence of $c_j c_j^{-1} = \epsilon$ would give us unbalanced equations at the outset.

In any monoid (or group) with presentation $\langle r_1, \ldots, r_m, c_1, \ldots, c_n; eq_1, \ldots, eq_l\rangle$ in which all equations $eq_i$ are balanced, any other equations which hold in that structure must also be balanced. Note that these equations may be monoid equations (involving the identity $\epsilon$) or group equations (involving inverses). In this structure, because every equation is balanced we can be assured that the definition of elements being positive or negative is

[3]Monoids are semigroups with an identity. We have $\epsilon x = x\epsilon = x$ for every element $x$.

consistent. An element $w$ is positive when $\ulcorner w = \epsilon \urcorner$ is balanced, and negative, when that equation is unbalanced.

So, we can show that if the modal logic **M** has only balanced axioms, then **JM** is a conservative extension of **J**. We need just for any **J**-frame $\mathcal{F}$ to construct a **JM**-frame in which $\mathcal{F}$ is a subframe. But this is not difficult. Take the monoid $\mathcal{M} = \langle r_1, \ldots, r_m, c_1, \ldots, c_n; eq_1, \ldots, eq_l \rangle$, and define $\mathcal{MF}$ as follows. Its elements are ordered pairs $\langle x, w \rangle$ of $\mathcal{F}$ elements and $\mathcal{M}$ elements. We set $\langle x, w \rangle R_i \langle y, v \rangle$ just when $x = y$ and $wR_iv$. Similarly, $\langle x, w \rangle C_i \langle y, v \rangle$ just when $x = y$ and $wC_iv$. The trick is with $\leq$. We define $\langle x, w \rangle \leq \langle y, v \rangle$ just when $x \leq y$ and $w = v$ *when $w$ is positive*, and when $x \geq y$ and $w = v$ *when $w$ is negative*.

Before showing that this is truly a **JM**-frame, we will pause to give the reader more of and idea of what's actually going on. What we have done is taken a modal frame made from the monoid of the logic itself. Accessibility in this original frame is defined as follows: $xR_iy$ if and only if $r_ix = y$ and $xC_jy$ if and only if $c_jx = y$. The monoid (group) of actions of a monoid (or group) upon itself is just itself, so this is a model for our target modal logic. We replace each monoid element with the **J**-frame $\mathcal{F}$, keeping intuitionistic accessibility internal to each copy of $\mathcal{F}$, except that we *invert* the frame at negative points. Modal accessibility on this extended structure simply moves you between 'clusters' just as you would in the original modal frame, and it keeps you at the same point in the intuitionistic frame. Intuitionistic accessibility is as before, but inverted at negative points. The original frame is present at $\epsilon$ (at least) because that point is positive.

Now to verify that this is a **JM**-frame. Clearly $\leq$ is still a partial order, since the disjoint union of partial orders is a partial order, and the converse of a partial order is a partial order. The monoid (group) of actions on $\mathcal{MF}$ is simply $\mathcal{M}$, as it moves intuitionistic structures as *en masse*. We have only to prove that it satisfies the intuitionistic modal conditions. Suppose that $\langle x, w \rangle R_i \langle y, v \rangle$, and that $\langle x, w \rangle \leq \langle x', w' \rangle$. This means that $r_iw = v$ and $x = y$, that $w = w'$, and that if $w$ is positive, $x \leq x'$, and if $w$ is negative, $x' \leq x$. Consider the point $\langle x', v \rangle$. We have that $\langle x', w' \rangle R_i \langle x', v \rangle$, since $x' = x'$, and $r_iw' = r_iw = v$. Similarly, $\langle y, v \rangle \leq \langle x', v \rangle$, because if $w$ is positive, so is $v$ (since $r_i$ is positive and $r_iw = v$) so $y = x \leq x'$ gives us the result. Alternatively, if $w$ is negative so is $v$, so $y = x \geq x'$ gives us the result.

Similar reasoning works with $C_j$. If $\langle x, w \rangle C_j \langle y, v \rangle$ and $\langle x, w \rangle \geq \langle x', w' \rangle$, we must have $x = y$, $c_jw = v$, $w = w'$ and $x \geq x'$ if $w$ is positive or $x \leq x'$ if $w$ is negative. In either case $\langle x', v \rangle$ completes the square. We have $\langle y, v \rangle \leq \langle x', v \rangle$, as $y = x \geq x'$ when $w$ is positive, and $y = x \leq x'$ when $w$ is negative. If $w$ is positive then $v = c_j w$ is negative, so we have $x' \geq y$ as we wanted, and if $w$ is negative, then $v$ is positive, and $x' \leq y$ obtains, as we wanted. Finally, $\langle x', w' \rangle C_j \langle x', v \rangle$ as $x' = x'$ and $c_j w' = c_j w = v$.

So, we have the following extended result.

THEOREM 11. *For any modal logic* **M** *defined in terms of balanced equations, the logic* **JM** *conservatively extends* **J**.

That completes our small tour of combining logics in a non-classical setting. The results here generalise to other logics with a frame semantics, such as relevant logics. But I leave that generalisation for another time, and another place.[4]

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