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DISPLAY LOGIC AND GAGGLE THEORY

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Abstract This paper is a revised version of a talk given at the *Logic and Logical Philosophy* conference in Poland in September 1995. In it, I sketch the connections between Nuel Belnap's *Display Logic* and J. Michael Dunn's *Gaggle Theory*.

Display Logic and Gaggle Theory

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Nuel Belnap's *Display Logic* [1] is a neat, uniform method for providing a cut-free consecution calculus for a wide range of formal systems. Mike Dunn's *Gaggle Theory* [3] is a neat, uniform presentation of the semantics for a wide range of formal systems. In this paper I will show that the two live together happily — many gaggle-theoretically presented logics can be given a display proof theory, and that many logics with a display proof theory can be algebraically presented in gaggle theory.

1 Gaggle Theory

Dunn [3] introduced the notion of a *gaggle* as a way to unify many different logics — modal, intuitionistic, many-valued, and substructural logics are examples of those which fit the general scheme.

Consider propositions, partially ordered by $\leq$, the relation of *entailment*. In many different logics, there are properties relating the behaviour of connectives with the entailment relation. For example, in many logics containing a conditional $\rightarrow$, it is possible to show the following:

If $a \leq b$ then $c \rightarrow a \leq c \rightarrow b$.
If $a \leq b$ then $b \rightarrow c \leq a \rightarrow c$.

Furthermore, if the partial order has a greatest element $\mathbf{1}$ and a least element $\mathbf{0}$, then we also have

$$\mathbf{0} \rightarrow a = \mathbf{1} \text{ and } x \rightarrow \mathbf{1} = \mathbf{1}$$

We can symbolise this by saying that in these logics $\rightarrow$ has a "trace" of $(-,+) \mapsto +$, for the following reasons:

- The 'limiting value' of the conditional as a whole is $\mathbf{1}$, when the inputs reach limits. So, we say the conditional as a whole has a 'positive' value, and we give the output value a "+".
- Now if the consequent of the conditional is 'increased' then the value of the whole conditional is 'increased'. As a result, the value of the second place in the input should have the same value as the output. So we have the second "+".
- On the other hand, 'increasing' the antecedent 'decreases' the whole conditional. So the value of the first place in the input has the opposite sign to the output. This explains the "−" in the first place.

Note that the conditional of the 'conditional logics' of Lewis and Stalnaker do not get this trace (or any other) as in these cases we do not have that if $a \leq b$ then $b \rightarrow c \leq a \rightarrow c$.

In general, an n-ary connective f has a *trace* $(\tau_1, \ldots, \tau_n) \mapsto +$ if

- $f(c_1, \ldots, \mathbf{1}, \ldots, c_n) = \mathbf{1}$, if $\tau_i = +$ (where the $\mathbf{1}$ is in position i).
- $f(c_1, \ldots, \mathbf{0}, \ldots, c_n) = \mathbf{1}$, if $\tau_i = -$ (where the $\mathbf{0}$ is in position i).
- If $a \leq b$, and if $\tau_i = +$ then $f(c_1, \ldots, a, \ldots, c_n) \leq f(c_1, \ldots, b, \ldots, c_n)$.
- If $a \leq b$, and if $\tau_i = -$ then $f(c_1, \ldots, b, \ldots, c_n) \leq f(c_1, \ldots, a, \ldots, c_n)$.

We write this as $T(f) = (\tau_1, \ldots, \tau_n) \mapsto +$. On the other hand, the connective f has *trace* $(\tau_1, \ldots, \tau_n) \mapsto -$ if

- $f(c_1, \ldots, \mathbf{1}, \ldots, c_n) = \mathbf{0}$, if $\tau_i = +$ (where the $\mathbf{0}$ is in position i).
- $f(c_1, \ldots, \mathbf{0}, \ldots, c_n) = \mathbf{0}$, if $\tau_i = -$ (where the $\mathbf{1}$ is in position i).
- If $a \leq b$, and if $\tau_i = +$ then $f(c_1, \ldots, b, \ldots, c_n) \leq f(c_1, \ldots, a, \ldots, c_n)$.
- If $a \leq b$, and if $\tau_i = -$ then $f(c_1, \ldots, a, \ldots, c_n) \leq f(c_1, \ldots, b, \ldots, c_n)$.

We write this as $T(c) = (\tau_1, \ldots, \tau_n) \mapsto -$. A *tonoid*, then, is an algebra $\langle P, \leq, OP \rangle$, in which there is a partial order $\leq$ on a set P, and a family OP of operators, each of which has a trace.

Here are a few examples of traces of connectives. Conjunction-like connectives tend to be $(-, -) \mapsto -$, disjunction-like connectives tend to be $(+, +) \mapsto +$, necessity-like connectives tend to be $+ \mapsto +$, possibility-like connectives tend to be $- \mapsto -$, and negations can be either $+ \mapsto -$ or $- \mapsto +$ (and in many cases they are both).

Finally, we often have connectives not only related to the entailment relation, but they are also connected to each other by means of *residuation* properties. The most common example is the connection between a conditional $\rightarrow$, and a corresponding conjunction-like connective $\circ$. These are connected by residuation iff the following holds

$$a \circ b \leq c \text{ iff } a \leq b \rightarrow c$$

It is then simple to show that if $\rightarrow$ has trace $(-, +) \mapsto +$, then $\circ$ has trace $(-, -) \mapsto -$. But residuation-like connectives are not restricted to conditionals and conjunctions. In the context of tense-logics, a forward-looking necessitive operator $\Box$ (it always will be the case that ...) is tied together to a backward looking possibility operator $\blacklozenge$ (it sometimes was the case that ...) as follows:

$$a \leq \Box b \text{ iff } \blacklozenge a \leq b$$

This can be generalised in a simple way to arbitrary operators. First, we define $S(f, a_1, \ldots, a_n, b)$ as follows. If $T(f) = (\cdots) \mapsto +$, then $S(f, a_1, \ldots, a_n, b)$ is the condition $f(a_1, \ldots, a_n) \leq b$. If, on the other hand, $T(f) = (\cdots) \mapsto -$, then $S(f, a_1, \ldots, a_n, b)$ is $b \leq f(a_1, \ldots, a_n)$. Then, two connectives f and g are *contrapositives in place* j iff, if $T(f) = (\tau_1, \ldots, \tau_j, \ldots, \tau_n) \mapsto \tau$, then $T(g) = (\tau_1, \ldots, -\tau, \ldots, \tau_n) \mapsto -\tau_j$. (Where we define $-+$ as $-$ and $--$ as $+$.) Two operators f and g satisfy the *abstract law of residuation* iff f and g are contrapositives in place j, and $S(f, a_1, \ldots, a_j, \ldots, a_n, b)$ iff $S(g, a_1, \ldots, b, \ldots, a_n, a_j)$.

A collection of connectives in which there is some connective f such that every element of the collection satisfies the abstract law of residuation with f, is called a *founded family* of connectives.

Dunn defines a *partial-gaggle* to be an algebra $\langle P, \leq, OP \rangle$, in which OP is a founded family [4]. Many applications call for something more structured than a tonoid, but less so than a partial-gaggle. So, with this in mind, I will introduce another term. A *loose partial-gaggle* is an algebra $\langle P, \leq, F_1, \ldots, F_n \rangle$, such that each F_i is a founded family.

A partial gaggle in which $\leq$ is a distributive lattice ordering is called a *gaggle*. Dunn has shown that tonoids, partial-gaggles and gaggles are representable as concrete algebras of propositions on Kripke frames in which each n-ary family of connectives is modelled using a single $n+1$-ary relation. [3, 4].

2 Display Logic

The central ideas of Belnap's *Display Logic* [1, 2] are simple and elegant. Like other consecution proof theories, the calculus deals with structured collections of formulae, *consecutions*. In display logic, consecutions are of the form $X \vdash Y$, where X and Y are *structures*, made up from formulae. The formulae are just as you would expect — formulae come from some language or other, built up from primitive formulae by formula-connectives. The structures are new. Structures are made up of structure-connectives operating on structures, building up structures from smaller structures, in much the same way as formulae are built up by formula-connectives. But the base level of structures are the formulae.

However, there is one difference which will be helpful to mark. This difference does not feature in the main thrust of Belnap's work, though it is one which has become important in later applications of display logic [10]. Structures and their connectives have a polarity. They can be either *antecedent* or *consequent* structures. In Belnap's original formulation of display logic, each structure-connective could appear both in antecedent and consequent position — but connectives could be interpreted differently according to the positions in which they appeared. For example, Belnap's calculus features a binary structure-connective $\circ$, which is interpreted as conjunction-like in an antecedent position, and disjunction-like in a consequent position. The unary structure-connective $*$ is negation-like in both positions. However, if $*X$ is in antecedent position, then the X is interpreted as in conjunction position. Now there is no loss in generality in reading Belnap's formulation as actually using *two* connectives for each of $\circ$ and $*$, where Belnap's consecution

$$X \circ *(*Y) \vdash *(Z \circ *X) \circ Z$$

is read as

$$X \circ_a *_a(*_c Y) \vdash *_c(Z \circ_a *_a X) \circ_c Z$$

where $*_a$ and $*_c$ are $*$ in antecedent and consequent position respectively, and similarly for the circle. What we lose in an increased complexity, we gain in having each connective being interpreted univocally. We also gain in smoothly being able consider other structure-connectives which may appear only in antecedent position, or only in consequent position.[1]

From this perspective then, each n-ary structure-connective s comes with a *trace*, which is a sequence of the form $(p_1, \ldots, p_n) \mapsto p$, where each p_i is either `ant` or `cons`, and if s has that trace, then $s[X_1, \ldots, X_n]$ is a p-structure when each X_i is a p_i-structure. And to start the induction off, any formula is both an antecedent structure and a consequent structure.

The proper antecedent parts of $s[X_1, \ldots, X_n]$ are the X_is where p_i is `ant`, and the proper antecedent parts of any of the X_is. Its proper consequent parts are the the X_is where p_i is `cons`, and the proper consequent parts of any of the X_is.

If X is an antecedent structure, and Y is a consequent structure, then $X \vdash Y$ is a consecution. The antecedent parts of the consecution are X and the proper antecedent parts of X and Y. The consequent parts of the consecution are Y and the proper consequent parts of X and Y.

[1] This was motivated by the quest for a display formulation of relevant logics which does not appeal to Boolean negation [10].

The crucial feature of display logic is the *display property.* If $X \vdash Y$ is a consecution involving Z as a consequent substructure, then $X \vdash Y$ can be transformed into $W \vdash Z$ using *display rules* alone, and W depends only on the position of Z in the original consecution. Similarly if Z is an antecedent substructure, then the consecution can be transformed into $Z \vdash V$ for some appropriate V. Here, the *display rules* are a particular class of rules involving only the structure-connectives alone. For example, in Belnap's original formulation, the display rules were as follows:

$$X \circ Y \vdash Z \Longleftrightarrow X \vdash *Y \circ Z$$
$$X \vdash Y \circ Z \Longleftrightarrow X \circ *Y \vdash Z \Longleftrightarrow X \vdash Z \circ Y$$
$$X \vdash Y \Longleftrightarrow *Y \vdash *X \Longleftrightarrow **X \vdash Y$$

An important feature of all of the rules of Belnap's calculus is this: Each structure presented in the display equivalences keeps its 'position' across the display equivalences. For example, each instance of X appearing in the top row is in antecedent position, each instance of Y appearing in the bottom row is in consequent position, and so on.

In addition to rules which treat structure, there are rules which introduce connectives. Because of the display property, there is no loss of generality in assuming that the connective to be introduced can be either the entire antecedent or the entire consequent of the consecution. For example, these are Belnap's original rules for the conditional.

$$\frac{X \circ A \vdash B}{X \vdash A \to B} \qquad \frac{X \vdash A \quad B \vdash Y}{A \to B \vdash *X \circ Y}$$

We will consider another example of structure-connectives, and connective rules. Wansing [11] extended Belnap's original work by adding a unary structure $\bullet$ (in both antecedent and consequent position) with display rules

$$\bullet X \vdash Y \Longleftrightarrow X \vdash \bullet Y$$

with corresponding modal rules

$$\frac{X \vdash \bullet A}{X \vdash \Box A} \qquad \frac{A \vdash X}{\Box A \vdash \bullet X}$$

In his original paper, Belnap provides a list of eight easily checked conditions. If a display proof theory satisfies these conditions, then the rule *Cut* (from $X \vdash A$ and $A \vdash Y$ to infer $X \vdash Y$) is admissible in the system. That is, if there are proofs of $X \vdash A$ and $A \vdash Y$, then there is also a proof of $X \vdash Y$. We need not go in to the detail of these conditions here.

3 Displayed Logics tend to be Tonoids

Consider the connective rules for $\to$ and $\Box$. For $\Box$ it is clear that it 'mimics' the behaviour of $\bullet$ in consequent position. For the conditional, '$\to$', a similar thing becomes clear if we use a display-equivalence. The following is a proof:

$$\cfrac{\cfrac{X \vdash *A \circ B}{X \circ A \vdash B}}{X \vdash A \to B}$$

So, we see that the arrow $A \to B$ is a 'recoding' of the structure $*A \circ B$ in consequent position.

A general structure-connective is given by composing our original structure-connectives. We will write these in the form "$s[X_1, \ldots, X_n]$". We have already seen an example: $*$ is $\texttt{cons} \mapsto \texttt{ant}$, and $\circ$ is $(\texttt{ant}, \texttt{ant}) \mapsto \texttt{ant}$, then the composition $s[X_1, X_2]$, defined as $*X \circ Y$ (seen above) has trace $(\texttt{cons}, \texttt{ant}) \mapsto \texttt{ant}$. Any general structure connective will have a trace in just this way. (It must, in order to be well-formed.)

The behaviour of $\to$ and $\Box$ can be generalised. A connective f *mimics consequent structure* iff there is some general structure connective s (possibly complex) such that the following rules are admissible:

$$\frac{X \vdash s[A_1, \ldots, A_n]}{X \vdash f(A_1, \ldots, A_n)} \qquad \frac{C(X_1, A_1) \ \cdots \ C(X_n, A_n)}{f(A_1, \ldots, A_n) \vdash s[X_1, \ldots, X_n]}$$

Where the trace of s is some $(p_1, \ldots, p_n) \mapsto \texttt{cons}$, and $C(X_i, A_i)$ is $X_i \vdash A_i$ if p_i is $\texttt{ant}$, and is $A_i \vdash X_i$ if p_i is $\texttt{cons}$. (Note that the conditional fits this general scheme.) And f *mimics consequent structure* iff there is some general structure connective s (again, possibly complex) such that the following rules are admissible:

$$\frac{s[A_1, \ldots, A_n] \vdash X}{f(A_1, \ldots, A_n) \vdash X} \qquad \frac{C(X_1, A_1) \ \cdots \ C(X_n, A_n)}{s[X_1, \ldots, X_n] \vdash f(A_1, \ldots, A_n)}$$

where the trace of s is $(p_1, \ldots, p_n) \mapsto \texttt{ant}$, and as before, $C(X_i, A_i)$ is $X_i \vdash A_i$ if p_i is $\texttt{ant}$, and is $A_i \vdash X_i$ if p_i is $\texttt{cons}$.

It is a fact that nearly every connective in the display logic literature mimics structure. The exceptions are the structure-free rules for $\wedge$ and $\vee$, and those connectives which have side conditions on the applications of their rules — namely non-normal modal operators, and the exponentials ! and ? of Girard's linear logic [6].

Theorem 1 *If a connective f mimics structure, then it is a tonoid operator.*

Proof. If the connective c mimics consequent structure, and then rules of the form

$$\frac{X \vdash s[A_1, \ldots, A_n]}{X \vdash f(A_1, \ldots, A_n)} \qquad \frac{C(X_1, A_1) \ \cdots \ C(X_n, A_n)}{f(A_1, \ldots, A_n) \vdash s[X_1, \ldots, X_n]}$$

where the structure s has trace $(p_1, \ldots, p_n) \mapsto \texttt{cons}$. We will show that f has trace $(\downarrow p_1, \ldots, \downarrow p_n) \mapsto +$, where $\downarrow\texttt{ant}$ is $-$, and $\downarrow\texttt{cons}$ is $+$.

To do this, of course, we need to translate between the *syntactic* approach of display logic and the *algebraic* approach of gaggle theory. But this is the standard move to a Lindenbaum algebra. Given a particular displayed logic, we move to the algebra by identifying provably equivalent formulae, and by taking the partial order to be determined by the consecutions that are provable. In other words, $[A] \leq [B]$ iff $A \vdash B$ is provable.[2]

[2]For this to work, of course, we need to show that the connectives preserve provable equivalence, so that they are operations on equivalence classes as well as formulae. In other words, we need that if $A_i \dashv\vdash B_i$ for each i, then $f(A_1, \ldots, A_n) \dashv\vdash f(B_1, \ldots, B_n)$. It is not a difficult exercise to show that if f mimics structure then this must be the case.

Then to show that f has trace $(\downarrow p_1, \ldots, \downarrow p_n) \mapsto +$ first consider a place i where $\downarrow p_i$ is $+$ (so p_i is `cons`). We wish to show that if $A_i \vdash B_i$ is provable then so is $f(C_1, \ldots, A_i, \ldots, C_n) \vdash f(C_1, \ldots, B_i, \ldots, C_n)$. But this is simple. We have the following proof fragment:

$$\dfrac{\dfrac{C(C_1, C_1) \ \cdots \ C(B_i, A_i) \ \cdots \ C(C_n, C_n)}{f(C_1, \ldots, A_i, \ldots, C_n) \vdash s[C_1, \ldots, B_i, \ldots, C_n]}}{f(C_1, \ldots, A_i, \ldots, C_n) \vdash f(C_1, \ldots, B_i, \ldots, C_n)}$$

But each $C(C_j, C_j)$ is provable (it is simple to show that in any display logic, any consecution of the form $A \vdash A$ is provable). And since p_i is `cons`, $C(B_i, A_i)$ is $A_i \vdash B_i$, which we hypothesised to be provable.

Similarly, if $A_j \vdash B_j$ is provable, and p_i is `ant`, it is simple to prove $f(C_1, \ldots, B_i, \ldots, C_n) \vdash f(C_1, \ldots, A_i, \ldots, C_n)$

Next we need to check that the bound conditions are satisfied. To do this, we need the display proof theory equivalents of bounds. The details here are unimportant. All we need are that there are formulae $\top$ (corresponding to $\mathbf{1}$) and $\bot$ (corresponding to $\mathbf{0}$) such that $X \vdash \top$ and $\bot \vdash X$ are always provable.[3] Then, we need only show that $\top \vdash f(\ldots, \top, \ldots)$ is provable, where the $\top$ in f is in place i where $\downarrow p_i$ is $+$, and that $\top \vdash f(\ldots, \bot, \ldots)$ is provable, where the f is in place j where $\downarrow p_j$ is $-$. Let's take the second of these cases.

We have a proof of $\top \vdash f(\ldots, \bot, \ldots)$ if we have a proof of $\top \vdash s[\ldots, \bot, \ldots]$. But here, the display property comes in to play. In particular, we know that there is a display equivalent consecution in which the $\bot$ presented here is displayed. This consecution will be either $\bot \vdash Z$ for some Z, or it will be $W \vdash \bot$ for some W, depending on the position of $\bot$ in $s[\ldots, \bot, \ldots]$. But we know the trace of s, and that p_j is `ant` (as $\downarrow p_j$ is $-$). As a result, when displaying $\bot$, it remains in antecedent position (as positions of structures are preserved when displaying formulae) and so, we have a proof of $\top \vdash s[\ldots, \bot, \ldots]$, provided we have a proof of the display equivalent $\bot \vdash Z$. But this is immediate, by the behaviour of $\bot$. So, the bounds condition works in this case. The bounds condition works in the other case as well, as a completely parallel argument shows.

Similarly, we can show that if the trace of s is $(p_1, \ldots, p_n) \mapsto$ `ant`, then if f mimics s, it has trace $(\downarrow p_1, \ldots, \downarrow p_n) \mapsto -$. With that, we can pronounce the theorem proved. $\dashv$

So, we can immediately infer that most of the logics which we give a display proof theory (those in which the connectives mimic structure) are tonoids (when viewed algebraically). Of course we can do more by considering how much like a partial-gaggle we can make the logic. And the results here are also encouraging. If I have some n-ary connective f which mimics s (let's choose s to an antecedent structure this time, to be fair), then I can introduce another connective g which is the generalised contrapositive of f in the ith place. Note that f is introduced on the left as follows

$$\dfrac{s[A_1, \ldots, A_n] \vdash X}{f(A_1, \ldots, A_n) \vdash X}$$

[3] These may be introduced as axioms, or they may "fall out" of other properties of the system. Exactly how is unimportant for our purposes.

Now $s[A_1, \ldots, A_n] \vdash X$ has a display equivalent consecution in which A_i is displayed. Let's take A_i to be in the consequent position of $s[A_1, \ldots, A_n]$. So the consecution is display equivalent to $Z \vdash A_i$ for some Z. But the consecution Z contains each A_j ($j \neq i$) as substructures. It also contains X as a substructure. So, let's take Z to be $t[A_1, \ldots, X, \ldots, A_n]$, where we write the X in place i. It is simple to show that X is in consequent position and each A_j is in the same position as before. So if s had trace $(p_1, \ldots, +, \ldots, p_n) \mapsto -$, then t has trace $(p_1, \ldots, -, \ldots, p_n) \mapsto +$. It is then straightforward to show that a connective g which mimics the structure t satisfies the abstract law of residuation with our original connective f.

Of course, the same sort of thing happens if we choose an f which mimics consequent structure. Instead of pursuing this line, we will move on to consider the other direction. If I have a tonoid (or a partial-gaggle, or a gaggle) can I construct a display proof theory for it?

4 Tonoids tend to be Displayable

This title of this section is an exaggeration. Only a particular class of tonoids is displayable (by straightforward means at least). Marcus Kracht has shown in the context of displayed modal logics [9] that only a particular class of systems extending the basic modal system can be given a display proof theory in which the connective $\Box$ mimics structure. For the conditions defining the behaviour of $\Box$ correspond to extra rules in the system, and these may not satisfy Belnap's conditions in all cases.

However, it is the case that many tonoids which you come across are displayable. I will indicate this by showing how a particular tonoid can be given a display proof theory (and one which has not been seen before). This is Dunn's logic of a *split negation* [5]. This logic has two negation connectives $\sim$ and $\neg$, tied together by the condition

$$a \leq {\sim} b \text{ iff } b \leq \neg a$$

It is simple to see that both negations have the trace $- \mapsto +$, and they satisfy the abstract law of residuation with respect to each other. So, to give this logic a display proof theory, we introduce unary consequent structure connectives $\sharp$ and $\flat$ which have the following display equivalence

$$X \vdash \flat Y \text{ iff } Y \vdash \sharp X$$

These structure connectives have trace `ant` $\mapsto$ `cons`. Then we can fix $\neg$ to mimic $\sharp$ and $\sim$ to mimic $\flat$. We have the following connective rules:

$$\frac{X \vdash \sharp A}{X \vdash \neg A} \qquad \frac{X \vdash A}{\neg A \vdash \sharp X} \qquad \frac{X \vdash \flat A}{X \vdash {\sim} A} \qquad \frac{X \vdash A}{{\sim} A \vdash \flat X}$$

These provide a cut-free proof theory of the free algebra of a pair of split negations. The fact that the proof theory is cut-free is a straightforward corollary of Belnap's original work. The fact that it is sound and complete for the free logic of a split pair of negations is a simple algebraic verification. The rules are valid under interpretation (so it is sound); and the algebra of equivalence classes of provably equivalent propositions, partially ordered by entailment is an algebra of a pair of split negations, as a result anything not provable in the proof theory does not hold in the free algebra (so it is complete). Here is one example proof, of $A \vdash {\sim}\neg A$.

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A \vdash \flat A}}{A \vdash \sharp\neg A}}{A \vdash \neg{\sim}A}$$

We can add structure-free rules for $\wedge$, $\vee$, $\top$ and $\bot$ without any pain whatever.[4] These then ensure that the algebraic relation $\leq$ corresponding to $\vdash$ is a lattice ordering.

Then are some conditions on negation which correspond to structural rules. For example, the axiom $A \wedge \neg A \vdash \bot$, or equivalently $A \wedge \neg A \vdash X$ for arbitrary X, corresponds to the structural rule

$$\dfrac{X \vdash \sharp X}{X \vdash Y}$$

As we then have the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A \vdash \flat A}}{A \wedge \neg A \vdash \flat A}}{A \vdash \sharp A \wedge \neg A}}{A \wedge \neg A \vdash \sharp A \wedge \neg A}}{A \wedge \neg A \vdash \bot}$$

5 Discussion

What does all of this mean? Firstly, the proof that displayed logics (in which all connectives mimic structure) are tonoids gives us a semantic story for displayed logics. We can immediately use Dunn's representation results to give a Kripke-style proof theory for displayed logics. Conversely, we can also infer that if a particular logic is *not* a tonoid (for example, non-normal modal logics, conditional logics and the exponential fragment of linear logic) then the connectives cannot simply mimic structure. Something else is needed — for instance, rules with side conditions.

Or putting it another way, we have exploited the similar polarities which feature in each framework. In display logic it is the polarity of antecedent and consequent position. In gaggle theory it is the polarity of positive and negative positions. These notions are essential to the behaviour of generalised residuation on the one hand, and a display-style proof theory on the other. If a connective does not 'respect' polarities, then including it in gaggle theory, on the one hand, and display logic, on the other, is going to be more involved.[5]

[4] The conjunction and disjunction rules are as follows.

$$\dfrac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B} \quad \dfrac{A \vdash X}{A \wedge B \vdash X} \quad \dfrac{B \vdash X}{A \wedge B \vdash X} \quad \dfrac{A \vdash X \quad B \vdash X}{A \vee B \vdash X} \quad \dfrac{X \vdash A}{X \vdash A \vee B} \quad \dfrac{X \vdash B}{X \vdash A \vee B}$$

[5] Thanks to Rajeev Goré (whose earlier papers [8, 7] also broached these issues) for helpful discussions on these topics, and to Nuel Belnap, Mike Dunn, Heinrich Wansing, and Marcus Kracht for comments and advice.

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