EXISTENCE, DEFINEDNESS, AND THE SEMANTICS OF POSSIBILITY AND NECESSITY

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Abstract: In this paper, I will address just some of Professor Williamson's treatment of necessitism in his *Modal Logic as Metaphysics* [28]. I will give an account of what space might remain for a principled and logically disciplined contingentism. I agree with Williamson that those interested in the metaphysics of modality would do well to take quantified modal logic—and its semantics—seriously in order to be clear, systematic and precise concerning the commitments we undertake in adopting an account of modality and ontology. Where we differ is in how we present the semantics of that modal logic. I will illustrate how *proof theory* may play a distinctive role in elaborating a quantified modal logic, and in the development of theories of meaning, and in the metaphysics of modality.

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How do the quantifiers ($\exists$ and $\forall$, the existential and universal quantifiers, in particular) interact with modal operators ($\Diamond$ and $\Box$)? If it is possible ($\Diamond$) that something ($\exists$) is F, does it follow that there is something that is possibly F? If it could be (*as far as we know*) that something is F, then is there something such that it could be (*as far as we know*) that *it* is F? In this paper, I aim to give a suitable *proof-theoretic* setting to understand what we are asking when we ask such questions, and to help us understand what is involved in the difference between contingentism, the view that it is contingent which objects there are, and necessitism, the opposing view, according to which, of necessity, all objects there are, exist necessarily.

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The most elaborate, significant and nuanced recent discussion of necessitism and contingentism is found in Professor Williamson's *Modal Logic as Metaphysics* (hereafter MLM) [28]. In this book many subtle questions are raised and many options for elaborating both contingentism and necessitism are raised. I have no space to address even 5% of those questions in this paper—I am in sympathy with nearly all of them, and in agreement with many. I fundamentally agree with Professor Williamson that the best way to address these deep and fundamental—and *metaphysical*—issues of modality and quantification is to do serious work in quantified modal logic, and to be completely explicit and disciplined about the commitments we undertake when we do so. In what follows, I will outline a parallel track to explore when traversing this landscape, using the resources in the proof theory of modal logic for addressing these questions.

To be precise concerning the scope of my discussion here. Williamson is careful to explain that there are different formulations of necessitism and contingentism, depending on the vocabulary involved in our underlying logic. In this short paper I will not have the space to consider questions concerning the identity predicate, or matters of higher order logic. The discussion here will be limited to first order quantified modal logic without identity, though this is not an essential limitation of the approach. Higher order logic and identity can be treated in a similar way to the modal predicate logic discussed here, though to do so would require more pages than are available.

My focus will be on the proper understanding of the semantics of the quantifiers and modal operators, the status of model theory, and options for the contingentist. In particular, I aim to introduce into the discussion the use of *proof theory* in giving the meaning of modal operators, and the options that are opened up to contingentists if they do so. Along the way, we will see that *epistemic* modalities—of the kind explored in two dimensional modal logics—prove a challenge to everyone, and a significant challenge to the necessitist. So, in the next section (Section 2), I introduce the Barcan formula, and its treatment in models for quantified modal logic, for contingentists and for necessitists, in Section 3, I introduce the hypersequent proof theory for the modal logic S5, and explain how it can be used as a formal account of the meaning for modal vocabulary. In Section 4, I introduce the treatment of quantifiers in this hypersequent calculus, and the options for contingentists and necessitists, showing that there is a very natural way for the contingentists to interpret the quantifiers in am anner consonant with their commitments. In Section 5, we reexamine variable domain models for contingentist quantified modal logic, and we see that the *proof theorist*'s view of models provides an alternate explanation for why those models accurately give an account of the behaviour of modal concepts without committing the contingentist to a seemingly problematic ontology of merely possible objects. However, the proof theory of quantified modal logic raises its own questions, and in Section 5, I will explore the resources in the proof theory itself to define wider quantifiers—quantifiers that put pressure on contingentist commitments. However, we will see that at least some of the quantifiers definable seem *equally* problematic to the necessitist. In Section 6, I explain how the proof theory has the resources to model a two-dimensional modal logic, with counterpossible *epistemic* or *indicative* modalities, and these modalities raise their own questions for necessitism. In the final section, I conclude.

With that map before us, let's begin.

1 MODAL MODEL THEORY AND THE BARCAN FORMULA

There are a number of different ways to characterise the difference between necessitism and contingentism. One way to do so is the *Barcan Formula*, introduced into the discussion of quantified modal logic by Ruth Barcan Marcus [1–3]. What is now known as the Barcan Formula is this thesis, connecting necessity and the universal quantifier

$$(\forall x)\Box F(x) \supset \Box(\forall x)F(x)$$

for a given open sentence $F(x)$, to the effect that if everything is necessarily F, then it is necessary that everything is F. By necessitist lights, this seems true. If everything that is exists necessarily, then if everything is necessarily F, then no matter how things go, everything is F, because if (even if things had gone differently) there *were* something that weren't F, then (as a matter of fact) that thing (which in that other circumstance fails to be F) is something that isn't necessarily F.

The contingentist need not be convinced by that reasoning, for she may respond that just because it is *possible* that something fails to be F, any such thing that fails to be F in that circumstance may not exist in *this* circumstance. Perhaps everything (that exists) is necessarily F, but this doesn't preclude the possibility *other* things existing which fail to be F. The Barcan Formula is one site of disagreement between contingentists and necessitists.

Williamson prefers a different formula, equivalent to the Barcan Formula, to draw the distinction between necessitism and contingentism, a formula using $\Diamond$ and $\exists$:

$$\Diamond(\exists x)A(x) \supset (\exists x)\Diamond A(x)$$

This formula is eqiuvalent to the Barcan Formula (for $\neg F(x)$) when necessity and possibility are tied together as de Morgan duals (so $\neg\Box A$ is equivalent to $\Diamond\neg A$) and the universal and existential quantifier are also duals (so $\neg(\forall x)A$ is equivalent to $(\exists x)\neg A$), and the propositional logic is classical. In what follows, I will slide between both formulations, and call them both 'the' Barcan Formula, or BF for short.

In Sections 3.5–3.7 of MLM, Williamson gives an extended discussion of models for modal logic which allow for BF to fail. He shows that such models raise significant questions and pressing questions for the contingentist. Williamson's treatment there is extended and subtle, and I will not reiterate all of it here. The gist is straightforward. In any possible worlds model for a contingentist modal logic, we require a world at which $\Diamond(\exists x)A(x)$ holds and at which $(\exists x)\Diamond A(x)$ fails. If $\Diamond(\exists x)A(x)$ holds at a world w_1, then there is some (accessible[1]) world w_2 where $(\exists x)A(x)$ holds. For that, we require some object d where $A(x)$ holds *of* d, at w_2.[2] For $(\exists x)\Diamond A(x)$ to fail at w_1, we need there to be no objects d' where $\Diamond A(x)$ holds of d' at w_1. Now, since $A(x)$ holds of d at w_2, it looks very much like d could do the job of witnessing the truth of $(\exists x)\Diamond A(x)$ at w_1. What can the contingentist say? The obvious response is that the object d doesn't *exist* at w_1, so it is not present there to witness the quantifier. For a quantified statement $(\exists x)B(x)$ to hold at w_1 we need some object e *at* w_1 where $B(x)$ holds of e at w_1. The objects available at w_1 to interpret the quantifiers (call them the domain at w_1, or $D(w_1)$) need not be the same as the objects available at other worlds—like w_2. If the object d, which witnesses $(\exists x)A(x)$ at w_2 is not present in $D(w_1)$, we cannot so swiftly move from $\Diamond(\exists x)A(x)$, which commits us to the presence of an A-object at *some* world, to $(\exists x)\Diamond A(x)$, which commits us to the presence *here* of an object that is an A-object at some world.

So much is standard, when it comes to variable domain models for modal logics. They are the well-understood way to give an account of contingentist quantified modal logic—quantified modal logic in which BF fails.

Now, as Williamson points out in MLM, these models put pressure on the contingentist when one is careful to pay attention to the role models play in interpreting the formal language. One can be a necessitist and take there to be models that give counterexamples to BF. The mere presence of such models is not enough to decide between contingentism and necessitism. For that, you need to more. In Chapter 3 of MLM, Williamson turns to *metaphysical universality* as a constraint for deciding between selecting models. A formula in the language of modal predicate logic is *metaphysically universal*, for Williamson, when its universal generalisation (in a higher order logic, for *all* non-logical constants are to be generalised) is true on its intended interpretation. So, for example, $Fa \supset Fa$ is metaphysically universal, because under the intended interpretation, $(\forall X)(\forall x)(Xx \supset Xx)$ is true. Williamson then notes that $(\exists y)a = y$ is metaphysically universal, because $(\forall x)(\exists y)x = y$ is true. Williamson then argues that metaphysical universality is a good constraint for logicality. An *intended* model structure for modal predicate logic should be sound and complete for metaphysically universal sentences. If a sentence is valid in that structure, it should be metaphysically universal, and *vice versa*.

With these pieces in place, Williamson then argues that BF holds in all in-

[1] If the modal logic is weaker than S5, or if it is a multi-modal logic, we may need to keep track of an accessibility relation, but in what follows, this is not important, so I will drop consideration of modal accessibility until it is important.

[2] One way to render "$A(x)$ holds of d at w" is in terms of a Tarski-style satisfaction relation between an open formula and an assignment of values to variables. Another is to extend the language with a suitable stock of canonical terms denoting objects, and replacing the free variable x in $A(x)$ with a term for d. The difference between these treatments of quantifiers is not material for our discussion here, so I will not choose between them.

tended model structures. Since $(\exists y)x = y$ is metaphysically universal, it holds in all intended model structures. So, given some $d \in D(w_2)$, and let x be assigned to d. Since $(\exists y)x = y$ holds at w_1 (by metaphyiscal universality), there must be some $e \in D(w_1)$ where when y is assigned to e, $x = y$ holds at w_1. This can happen (given a standard semantics for identity) only when $e = d$. The d chosen from $D(w_2)$ was arbitrary, so we have shown that $D(w_2) \subseteq D(w_1)$. This suffices for BF to hold at w_1. We have shown that BF holds in our intended model structure.

This is not presented as a decisive argument against the rejection of BF or against contingentism—there are a number of different moves the contingentist can make, both concerning metaphysical universality and the details of the semantics for the quantifiers and identity—but it does pose a significant challenge to the contingentist. To respond to this challenge, and other challenges concerning the status of modal model theory, the contingentist needs the resources to understand modal semantics better. Forms of semantics beyond model theory will help here, so it is time to turn to *proof theory*.

2 SEQUENTS AND DEFINING RULES

In a series of papers [15–22], I have explored an approach to the proof theory of classical propositional logic and its extensions to modal logic, and first and second order predicate logic, in which the proof theory is not just a technical device for generating the logically valid formulas, but is intimately connected to the rules for use for the logical connectives and quantifiers. The proof theory's rules for the connectives and quantifiers are understood as governing assertions and denials using those concepts. They can be seen as rules for the use of $\wedge, \vee, \supset, \neg, \forall, \exists, \Box$ and $\Diamond$—that is, a *semantics*. In this paper, I will use these resources elaborate the scope for a rigorous, coherent and defensible semantics for contingentism.

The central idea in the proof theory is that of a *sequent* $\Gamma \succ \Delta$ consisting of two finite collections of sentences Γ and Δ from our formal language. A sequent can be seen as marking out a *position* one could take in a discourse or in a reasoning situation—a position in which each sentence in Γ is asserted an each sentence in Δ is denied. A sequent is said to be *valid* if that position is *out of bounds*.

What does it mean for a position to be out of bounds? One straightforward answer is that it is impossible for each sentence in Γ to be true and each sentence in Δ to be false. That would be true enough, but this explanation uses concepts we are in a position to *explain*—in particular, the notion of *possibility*, or logical satisfiability, at its heart. Rather, we build up the notion of why a position is out of bounds from the simplest cases.[3] The most simple position that is out of bounds is the self-undermining position

$$A \succ A$$

[3] Rather in the same way that we recursively explain truth in a model from the simplest cases in model theory.

in which the one and the same sentence is asserted and denied. This is the *identity* sequent for A. It expresses the heart of the notion of a position being out of bounds. At its heart, a position is out of bounds when it is self-undermining. When we attempt to give with one hand (by asserting A) and take with the other (by denying it). This condition depends only on the identity of the assertion and the denial, and not on any of the structure or content of what is asserted or denied. The same holds for some other constraints on the bounds. These are the other so-called *structural rules*. First, *weakening*:

$$\frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta}\,[KL] \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}\,[KR]$$

according to which if a position is out of bounds, it remains out of bounds when either more assertions or denials are added. The only way return to the 'field of play' is to withdraw some assertions or denials, not to dig yourself in further. The rule of *contraction*:

$$\frac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta}\,[WL] \qquad \frac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}\,[WR]$$

simply makes explicit that repetitions of assertions or denials have no distinct force.[4] The most significant structural rule is the *Cut* rule:

$$\frac{\Gamma \succ A, \Delta \qquad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}\,[Cut]$$

according to which if a position $\Gamma \succ \Delta$ is *in* bounds (if it isn't out of bounds: that is, there is no clash in asserting each member of Γ and denying each member of Δ) then if there were a clash involved in denying A (if $\Gamma \succ A, \Delta$ is out of bounds), that is, if A is *undeniable* (relative to $\Gamma \succ \Delta$) then there is no clash invovled in asserting A (relative to $\Gamma \succ \Delta$). In other words, if A is undeniable (relative to the position in which Γ is asserted and Δ is denied), then asserting A is simply making explicit what is already implicit in $\Gamma \succ \Delta$. Adding the assertion of A to $\Gamma \succ \Delta$ is no more out of bounds than $\Gamma \succ \Delta$ itself.

The form of the [*Cut*] rule I have given is the so-called *additive* cut rule, in which the side formulas Γ and Δ are shared between both premises of the rule. A *multiplicative* [*Cut*] rule

$$\frac{\Gamma \succ A, \Delta \qquad \Gamma', A \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,[\mathrm{m}Cut]$$

is possible, where we allow for a *Cut* on sequents with distinct side-formulas.[5] The

[4] If the 'collections' of assertions and denials were *sets* and not *multisets* or *lists*, this rule would be redundant. It is good to make it explicit, because in most proof theory it is simpler to assume that the premises (left-hand side) and conclusions (right-hand side) of sequents form *multisets* and not sets.

[5] The 'side-formulas' in a sequent used in a rule are the formulas other than the formula directly operated upon by that rule. The side formulas in a [*Cut*] inference from $\Gamma \succ A, \Delta$ and $\Gamma, A \succ \Delta$ to $\Gamma \succ \Delta$ are those formulas other than the displayed cut-formula A—the formulas in Γ and Δ.

multiplicative *Cut* rule is justified by way of the weakening rule and additive *Cut*.

$$\dfrac{\dfrac{\Gamma \succ A, \Delta}{\Gamma, \Gamma' \succ A, \Delta, \Delta'}[K] \qquad \dfrac{\Gamma', A \succ \Delta'}{\Gamma, \Gamma', A \succ \Delta, \Delta'}[K]}{\Gamma, \Gamma' \succ \Delta, \Delta'}[Cut]$$

In the rest of this paper, I will switch between additive and multiplicative *Cut* rules as needed withou further mention.

These rules, [*Id*] (the identity rule), [*K*] (weakening), [*W*] (contraction) and [*Cut*] together form the *structural rules*, which govern the behaviour of sentences as such—we do not appeal to the structure or content of the sentences involved. To go beyond these structural features to the distinct behaviour of the connectives and quantifiers, we need to appeal to rules for those connectives and quantifiers. In this section, I will examine the behaviour of classical propositional connectives, in the next, the modal operators, and then in the section after that, the quantifiers.

The classical connectives can be introduced uniformly with a series of *invertible* rules, which can be applied from top to bottom or from bottom to top.

$$\frac{\Gamma \succ A, \Delta}{\Gamma, \neg A \succ \Delta}[\neg Df] \quad \frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}[\wedge Df] \quad \frac{\Gamma \succ A, B, \Delta}{\Gamma \succ A \vee B, \Delta}[\vee Df] \quad \frac{\Gamma, A \succ B, \Delta}{\Gamma \succ A \supset B, \Delta}[\vee Df]$$

These rules show how to interpret sequents involving the new connectives in terms of their *constituents*. So, we can use these as rules to interpret assertions or denials of our involving the *defined* connective in terms of assertions or denials of their constituents. The classical negation rules, governed by [$\neg Df$] show that the asserting $\neg A$ has the same force (as far as positions are concerned) as a denial of A. For conjunction, an assertion of $A \wedge B$ has the saem force as the assertion of A and the assertion of B. The denial of $A \vee B$ has the same force as the denial of A and denial of B. To deny $A \supset B$ has the same force as asserting A and denying B. This suffices to fix the behaviour of the connectives involved, insofar as two concepts introduced with rules of the same shape (say, for example, $\vee_1$ and $\vee_2$, both disjunctions), then there is no open position where one could assert a 1-disjunction and deny a 2-disjunction, or *vice versa*.

$$\dfrac{\dfrac{A \vee_1 B \succ A \vee_1 B}{A \vee_1 B \succ A, B}[\vee_1 Df]}{A \vee_1 B \succ A \vee_2 B}[\vee_2 Df] \qquad \dfrac{\dfrac{A \vee_2 B \succ A \vee_2 B}{A \vee_2 B \succ A, B}[\vee_2 Df]}{A \vee_2 B \succ A \vee_2 B}[\vee_1 Df]$$

The same goes for the other connectives, too. These defining rules govern the behaviour of the propositional connectives by uniquely characterising them—the rules characterise the concepts, rather than merely describing some constraints they satisfy.[6] A longer argument shows that a language governed by a conse-

[6] I have in mind here the distinction between these defining rules and the axioms for a modal operator, such as an S5 necessity. These axioms describe constraints satisfied by the $\Box$ in question without uniquely characterising it. We can have two non-equivalent necessity operators both satisfying the S5 axioms.

quence relation satisfying our rules of identity, weakening, contraction and *Cut* can be *conservatively extended* by the propositional connectives given by these defining rules. That argument uses Gentzen's *cut elimination* argument, together with the fact that Gentzen's left and right rules for each connective can be recovered from each defining rule, like this. The derivation below

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \vee B \succ A \vee B}\,[Id]}{A \vee B \succ A, B}\,[\vee Df] \quad \Gamma, A \succ \Delta}{\Gamma, A \vee B \succ B, \Delta}\,[Cut] \quad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[Cut]$$

Shows how the traditional disjunction left rule:

$$\frac{\Gamma, A \succ \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[\vee L]$$

can be justified in terms of [*Id*], [$\vee$*Df*], [*K*] and [*Cut*], and as with unique definability, this conservative extension argument works for each connecitve in the vocabulary. We can justify the usual Gentzen rules for negation conjunction and the conditional

$$\frac{\Gamma, A \succ \Delta}{\Gamma \succ \neg A, \Delta}\,[\neg R] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta}\,[\wedge R] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[\supset L]$$

in terms of the defining rules, given [*Id*] and [*Cut*]. The rules [$\vee$*L*], [$\wedge$*R*] and [$\supset$*L*] differ from the defining rules for the connectives. They are not invertible—and neither should we expect them to be. The rule [$\vee$*L*] tells us some of what is involved in asserting the disjunction $A \vee B$. We should not expect this to be equivalent to any combination of assertions and denials involving A and B. After all, there are some things we can *do* with a disjunction that we could not do without it. Having a single item we can deny whose denial has the same significance as the denial of A and the denial of B gives us something *new*. Its *assertion* allows us to say something we may not have been able to say without that concept. The defining rule [$\vee$*Df*] defines disjunction in the sense that it uniquely characterises the bounds for assertions and denials of disjunctions.

In this way, we have a kind of *semantics* for a vocabulary involving the classical propositional connectives, in that we have defined rules for the coherence of positions involving assertions and denials in that vocabulary. The resulting relation of coherence is exactly the same as that delivered by truth tables for propositional logic,[7] but we have not started with the notion of truth. This is not to say that the notion of truth is altogether absent from this style of proof theory. If we move

[7]Gentzen's sequent calculus, which we have defined in a roundabout way, is sound and complete for classical propositional logic. A sequent $\Gamma \succ \Delta$ is derivable if and only if there is no evaluation which assigns each member of Γ true and each member of Δ false.

from the referee's position, where we stand apart and judge positions like $\Gamma \succ \Delta$ for coherence, to the *player's* standpoint, where we *make* those assertions and denials, we see that someone who asserts Γ and denies Δ is (in some sense[8]) taking each member in Γ to be *true* and each member of Δ to be false. Further, if we have taken up the position $\Gamma \succ \Delta$ and if $\Gamma \succ A, \Delta$ is out of bounds, then there is a sense in which A, too, is taken to be *true* in that position, since it is undeniable—the only coherent option to take a stand on it is to assert it, and that option *is* coherent if the position $\Gamma \succ \Delta$ is coherent. This does not go far enough to ground a full-blooded and robust notion of truth, but we can go so far as to draw the connection between truth in a position and truth in a model, and that connection becomes very tight when we move from finite positions to *idealised* positions for which we fill Γ and Δ out to take a stand on *each* sentence of the language [19]. None of this moves us beyond truth in a model to truth *per se*, because nothing tells us which of these idealised positions counts as the truth. However, we can say a little more. To take the position $\Gamma \succ \Delta$—to assert each member of Γ and to deny each member of Δ is to take each member of Γ to be true and each member of Δ to be false, and to take the whole truth (in that language) to be given by one of the ideal positions extending $\Gamma \succ \Delta$. What we have taken to be true in asserting Γ and denying Δ is whatever is true in each of those ideal positions extending $\Gamma \succ \Delta$. *Which* of those positions is the truth? To single one out is to move beyond $\Gamma \succ \Delta$ to a stronger sequent, adding more assertions or denials, and choosing between some of the ideal sequents extending $\Gamma \succ \Delta$.

This understanding of proof theory and its use in semantics is fit for the *normative pragmatist* [6, 7], who takes a semantic theory to be formulated in rules for use. The account is *pragmatist* in the sense that the theory governs acts—in this case, acts of assertion and denial—and it is *normative* in that the theory gives *rules* or *norms* governing those acts—in this case, the norms governing positions, combinations of assertions and denials, and the outer boundary of the space of such positions. There is more to say concerning the norms governing assertion (and denial), and the role of such norms in semantics, but this is enough to go on with for the moment.[9] It is time to consider the modal operators.

3 HYPERSEQUENTS, POSSIBILITY AND NECESSITY

Let's consider modal reasoning, and the norms governing how we interpret modal operators such as $\Box$ and $\Diamond$. Suppose that it's possible that the disjunction—either A or B—holds (that is, suppose $\Diamond(A \vee B)$). Then we can reason like this: Since it's

[8] Since we can 'try on' assertions and denials under suppositions, or when taking someone else's position as a starting point in our reasoning, this 'taking to be true' need not involve *belief* or a *commitment* any wider than the scope of the dialogue or that supposition.

[9] For example, I have said *nothing* concerning norms governing correct assertion, and the large literature discussing these norms [5, 8, 10, 12, 26, 27]. The fact that I have not discussed these does not mean that I take them to be unimportant or unrelated to the bounds of assertion and denial discussed here.

possible that either A or B, in that possibility (which is not necessarily how things *really* go, but it's *possible*) the disjunction A or B holds (so, in that possibility: $A \vee B$). So, there are two cases there: A, and B. In the first case, considering things from our original point of view, it turns out that it was A that's possible (so, we have $\Diamond A$—not holding in *that* possibility, but back 'here' where we started), or in the other case, it turns out here that it's B that is possible (so, we have $\Diamond B$, back here). In either case, we have either A is possible or B is possible (that is, $\Diamond A \vee \Diamond B$).

That was a small piece of modal reasoning, from the premise $\Diamond(A \vee B)$ to the conclusion $\Diamond A \vee \Diamond B$. We moved from the complex premise $\Diamond(A \vee B)$ to reason with the constitudent claim $A \vee B$—which we asserted (under our initial premise, which we granted) but not in the same way that we supposed $\Diamond(A \vee B)$. We asserted $A \vee B$ 'in that possibility'. With that $A \vee B$ asserted, we split into two different cases. In the A case, in that possibility, we had A, and back in the home context, we conclued $\Diamond A$. In the other B case, in the possibility, we had B, and and back in the home context, we conclued $\Diamond B$. So in either case, we have $\Diamond A \vee \Diamond B$. This is a straighforward example of modal reasoning.

In modal reasoning, we typically transform a modalised statement $\Box A$ into A—understood some range of alternate circumstances. The proof theory of modal logic can take this very seriously, using *hypersequents* allowing for formulas to be asserted and denied not just in one circumstance, but multiply. A simple hypersequent (or simply, a *hypersequent*) has the form:

$$\Gamma_1 \succ \Delta_1 \mid \cdots \mid \Gamma_n \succ \Delta_n$$

It is a multiset of sequents. It represents a position (in a discourse or a piece of reasoning) where we have asserted the members of Γ_1 and denied the members of Δ_1 relative to one 'possibility', asserted Γ_2 and denied Δ_2 in another, etc. Keeping track of such 'possibilities' is a matter of syntax in the discourse, in rather the same way that keeping track of how the use of pronouns in a discourse trace reference to the same discourse item. The kinds of markers we use for swtiching from one 'zone' to another are phrases like 'if *that* were the case then …' or 'in *this* circumstance …', and the like. In such discourse we don't typically quantify over such 'cases' or 'circumstances,' but we use such words to mark off contexts or 'zones' in which assertions and denials are made. In the kind of modal reasoning for bare possibility and necessity, we keep track only of the different zones, and not any notion of 'relative possibility' or 'nearness'. For a hypersequent proof theory for S5 we need only keep track of a number of different zones, not anything more than that.

In referring to hypersequents we have some new syntax. We will use $\mathcal{H}$ as a variable ranging over for hypersequents, and $\Gamma \succ \Delta \mid \mathcal{H}$ is the hypersequent $\mathcal{H}$ with the sequent $\Gamma \succ \Delta$ added, analagously to Γ, A being the multiset Γ with the formula A added. Before looking at how the structural rules are to be understood in this setting, we start with the defining rules for necessity and possibility:

$$\frac{\Gamma \succ \Delta \mid \; \succ A \mid \mathcal{H}}{\Gamma \succ \Box A, \Delta \mid \mathcal{H}}\,[\Box Df] \qquad \frac{\Gamma \succ \Delta \mid A \succ \; \mid \mathcal{H}}{\Gamma, \Diamond A \succ \Delta \mid \mathcal{H}}\,[\Diamond Df]$$

The idea is straightforward: to deny $\Box A$ in a zone of a discourse is coherent (with other commitments, given in $\Gamma \succ \Delta$ and $\mathcal{H}$) iff denying A in *some* zone is coherent (relative to those commitments). That is, $\Box A$ is undeniable in a given zone iff A is undeniable in *any* zone. To assert $\Diamond A$ in a zone of a discourse is coherent iff asserting A in *some* zone is coherent. Possibility and necessity trade on this shift of zones.

The hypersequent structure in the proof theory is important. Possibility and necessity are *not* logical constants in the same sense as the boolean propositional connectives. They are not constant—not only in the sense that there are many different modal logics, but in the stronger sense that even if we fix on one logic as the correct account of necessity, one can have a multi-modal logic in which there is more than one 'necessity' operator satisfying that logic. The axioms and theorems governing necessity are not enough to fix its meaning. In a multi-modal logic (say, given by a model with two different accessibility relations governing each necessity operator). Nonetheless, these defining rules *define* the connectives, relative to the hypersequent structure. If we agree to interpret $\Box$ using $[\Box Df]$, then you and I agree on the meaning of $\Box$, even though we could have very different views on what formulas of the form $\Box A$ are true. Agreement on modal operators trades on coordination on the context shifts used in our reasoning with them.

The structural rules for hypersequents can be motivated in the usual way. *Identity* is as before

$$A \succ A$$

since an assertion of A clashes with a denial of A—in the same zone. Of course, there need be no clash between an assertion of A in one zone and a denial of A in another. Weakening comes in more forms, allowing for internal and *external* weakening.

$$\frac{\Gamma \succ \Delta \mid \mathcal{H}}{\Gamma, A \succ \Delta \mid \mathcal{H}}\,[KL] \qquad \frac{\Gamma \succ \Delta \mid \mathcal{H}}{\Gamma \succ A, \Delta \mid \mathcal{H}}\,[KR] \qquad \frac{\mathcal{H}}{\Gamma \succ \Delta \mid \mathcal{H}}\,[KE]$$

For $[KE]$, if a position is incoherent, then adding extra zones (in which other things are asserted and denied) is not going to help restore coherence. The *contraction* rules could be understood both internally and externally too:

$$\frac{\Gamma, A, A \succ \Delta \mid \mathcal{H}}{\Gamma, A \succ \Delta \mid \mathcal{H}}\,[WL] \qquad \frac{\Gamma \succ A, A, \Delta \mid \mathcal{H}}{\Gamma \succ A, \Delta \mid \mathcal{H}}\,[WR] \qquad \frac{\Gamma \succ \Delta \mid \Gamma \succ \Delta \mid \mathcal{H}}{\Gamma \succ \Delta \mid \mathcal{H}}\,[WE]$$

If it is incoherent to assert Γ and deny Δ in two different zones of the discourse, it's incoherent to assert Γ and deny Δ in one. For asserting Γ and denying Δ in two different zones of the discourse commits you to nothing more than you are committed to in asserting Γ and denying Δ in one, if there is nothing different in the two zones—and there isn't, since we have individuated those zones purely in terms of what is asserted and denied in them.[10]

The additive *Cut* rule is a straightforward generalisation of the rule in the classical sequent context:

$$\frac{\Gamma \succ A, \Delta \mid \mathcal{H} \qquad \Gamma, A \succ \Delta \mid \mathcal{H}}{\Gamma \succ \Delta \mid \mathcal{H}}\,[\mathrm{a}\mathit{Cut}]$$

according to which if A is *undeniable* in first zone in the coherent context $\Gamma \succ \Delta \mid \mathcal{H}$, then adding it as an assertion in that zone is coherent. The multiplicative variant of *Cut*, in which contexts are merged

$$\frac{\Gamma \succ A, \Delta \mid \mathcal{H} \qquad \Gamma', A \succ \Delta' \mid \mathcal{H}'}{\Gamma, \Gamma' \succ \Delta, \Delta' \mid \mathcal{H} \mid \mathcal{H}'}\,[\mathrm{m}\mathit{Cut}]$$

is equivalent to the additive variant in the presence of contraction and weakening:

$$\frac{\dfrac{\dfrac{\Gamma \succ A, \Delta \mid \mathcal{H} \qquad \Gamma, A \succ \Delta \mid \mathcal{H}}{\Gamma, \Gamma \succ \Delta, \Delta \mid \mathcal{H} \mid \mathcal{H}}\,[\mathrm{m}\mathit{Cut}]}{\Gamma, \Gamma \succ \Delta, \Delta \mid \mathcal{H}}\,[\mathit{WE}]}{\Gamma \succ \Delta \mid \mathcal{H}}\,[\mathit{WL/WR}]$$

$$\frac{\dfrac{\dfrac{\Gamma, \succ A, \Delta \mid \mathcal{H}}{\Gamma, \Gamma' \succ A, \Delta, \Delta' \mid \mathcal{H}}\,[\mathit{KL/KR}]}{\Gamma, \Gamma' \succ A, \Delta, \Delta' \mid \mathcal{H} \mid \mathcal{H}'}\,[\mathit{KE}] \qquad \dfrac{\dfrac{\Gamma', A \succ \Delta' \mid \mathcal{H}'}{\Gamma, \Gamma', A \succ \Delta, \Delta' \mid \mathcal{H}'}\,[\mathit{KL/KR}]}{\Gamma, \Gamma', A \succ \Delta, \Delta' \mid \mathcal{H} \mid \mathcal{H}'}\,[\mathit{KE}]}{\Gamma, \Gamma' \succ \Delta, \Delta' \mid \mathcal{H} \mid \mathcal{H}'}\,[\mathrm{a}\mathit{Cut}]$$

So, as before, we have a suite of structural rules. We extend them with the modal rules, and the other connective rules by allowing for hypersequents rather than sequents, but keeping the rules as before. So here, for example, are the rules for conjunction, disjunction and negation:

$$\frac{\Gamma, A, B \succ \Delta \mid \mathcal{H}}{\Gamma, A \land B \succ \Delta \mid \mathcal{H}}\,[\land \mathit{Df}] \qquad \frac{\Gamma \succ A, B, \Delta \mid \mathcal{H}}{\Gamma \succ A \lor B, \Delta \mid \mathcal{H}}\,[\lor \mathit{Df}] \qquad \frac{\Gamma, A \succ \Delta \mid \mathcal{H}}{\Gamma \succ \neg A, \Delta \mid \mathcal{H}}\,[\neg \mathit{Df}]$$

[10] Consider the analogy: If $Fa, Fb, \Gamma \succ \Delta$ is out of bounds, where Γ and Δ say nothing more about a or b, then so is $Fa, \Gamma \succ \Delta$ since nothing in the position $Fa, Fb, \Gamma \succ \Delta$ says that a and b must be different things. So if $Fa, Fb, \Gamma \succ \Delta$ is out of bounds, so is $Fa, \Gamma \succ \Delta$. In the same way, in taking up the position $\Gamma \succ \Delta \mid \Gamma \succ \Delta \mid \mathcal{H}$ we are simply commiting ourselves to the possibility of everything in Γ holdsding and everything in Δ failing, and the possibility of everything in Γ holding and everything in Δ failing, and everything in $\mathcal{H}$. That is no more and no less than the possibility of everything in Γ holding and everything in Δ failing, and everything in $\mathcal{H}$, which is what is said by $\Gamma \succ \Delta \mid \mathcal{H}$.

As before, these rules are uniquely defining and conservatively extending, once we have moved to the broader setting of hypersequent positions. As before, two connectives introduced with rules of the same shape are interderivable, and hence, indistinguishable as far as the bounds of positions are concerned. Similarly, rules of the form of Gentzen's *left* and *right* rules for each connective may be defined in terms of our defining rules, *identity* and *Cut*, and a Cut elimination argument proved for he resulting system. The result is a conservative extension fact, showing that any position ruled out of bounds may be done so on the basis of the concepts occuring in that position. Adding new concepts governed by defining rules does not interfere with the bounds for positions in the prior vocabulary. Concepts given by defining rules are free additions to our vocabulary in the sense that they are uniquely defined (relative to the hypersequent structure) and they do not interfere with any prior positions.

Here is an example derivation, using defining rules and the structural rules.

$$
\begin{prooftree}
\AxiomC{$A \vee B \succ A \vee B$}
\RightLabel{$[\vee Df]$}
\UnaryInfC{$A \vee B \succ A, B$}
\AxiomC{$\Diamond A \succ \Diamond A$}
\RightLabel{$[\Diamond Df]$}
\UnaryInfC{$A \succ \ \mid\ \succ \Diamond A$}
\RightLabel{$[\mathrm{m}Cut]$}
\BinaryInfC{$A \vee B \succ B \mid\ \succ \Diamond A$}
\AxiomC{$\Diamond B \succ \Diamond B$}
\RightLabel{$[\Diamond Df]$}
\UnaryInfC{$B \succ \ \mid\ \succ \Diamond B$}
\RightLabel{$[\mathrm{m}Cut]$}
\BinaryInfC{$A \vee B \succ \ \mid\ \succ \Diamond A \mid\ \succ \Diamond B$}
\RightLabel{$[K,W]$}
\UnaryInfC{$A \vee B \succ \ \mid\ \succ \Diamond A, \Diamond B$}
\RightLabel{$[\Diamond Df]$}
\UnaryInfC{$\Diamond(A \vee B) \succ \Diamond A, \Diamond B$}
\RightLabel{$[\vee Df]$}
\UnaryInfC{$\Diamond(A \vee B) \succ \Diamond A \vee \Diamond B$}
\end{prooftree}
$$

This derivation gives us another account of how to get from $\Diamond A \vee \Diamond B$ from $\Diamond(A \vee B)$—it has a similar structure to the everyday reasoning given at the introduction to this section, though the particulars are different. Take the intermediate hypersequent in the derivation

$$A \vee B \succ \ \mid\ \succ \Diamond A \mid\ \succ \Diamond B$$

The fact that this hypersequent is derivable means that asserting $A \vee B$ (in one zone), while denying $\Diamond A$ (in another) and denying $\Diamond B$ (in another) is out of bounds. We have given an account of a proof theory for the modal logic S5 in which the connectives are defined by way of rules for use, governing assertion and denial of modal formulas—in different zones, as one would expect in modal reasoning. There is more to say about hypersequent proof theory for propositional modal logic [4, 14, 17], but instead of stopping there, we will move on to the logic of quantifiers.

4 QUANTIFIERS, DEFINEDNESS AND THE BARCAN FORMULAS

Combining the rules for the modality with the classical rules for quantification are a recipe for immediately delivering the *Barcan formula*. Defining rules for the classical quantifiers are simple.

$$\frac{\Gamma \succ A(n), \Delta \mid \mathcal{H}}{\Gamma \succ (\forall x)A(x), \Delta \mid \mathcal{H}}\,[\forall Df] \qquad \frac{\Gamma, A(n) \succ \Delta \mid \mathcal{H}}{\Gamma, (\exists x)A(x) \succ \Delta \mid \mathcal{H}}\,[\exists Df]$$

In these rules the name n must be absent from the premise hypersequent, except for its use in the formula $A(n)$. These rules suffice for classical predicate logic. With this quantifiers defined in this way, we can derive all of the first order classical validities, and more—we can prove BF.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{(\forall x)\Box A(x) \succ (\forall x)\Box A(x)}{(\forall x)\Box A(x) \succ \Box A(n)}\,[\forall Df]}{(\forall x)\Box A(x) \succ \quad \mid \quad \succ A(n)}\,[\Box Df]}{(\forall x)\Box A(x) \succ \quad \mid \quad \succ (\forall x)A(x)}\,[\forall Df]}{(\forall x)\Box A(x) \succ \Box(\forall x)A(x)}\,[\Box Df]}{\succ (\forall x)\Box A(x) \supset \Box(\forall x)A(x)}\,[\supset Df]$$

If we are contingentists, then this derivation should not strike us as compelling. While it would be a mistake to assert $(\forall x)\Box A(x)$ and deny it at the same time (the first sequent indeed is out of bounds), is it a mistake to assert $(\forall x)\Box A(x)$ and to deny $\Box A(n)$? This depends on the status of the term n. If n possibly fails to denote (as it might do, even if we grant that n as a matter of *fact* denotes), then would be coherent to deny $\Box A(n)$ while asserting $(\forall x)\Box A(x)$ in some zone. This argument can be broken if we move to a free logic in which we not only allow terms to fail to denote (so much is standard) but we allow denotation failure to vary from zone to zone in a hypersequent derivation. This is exactly the shift the contingentist desires, for what counts as a suitable substitution for a quantifier differs from zone to zone. Exploring this possibility, we generalise the defining rule for the quantifiers for free logic to the hypersequent case as follows:

$$\frac{\Gamma, n \succ A(n), \Delta \mid \mathcal{H}}{\Gamma \succ (\forall x)A(x), \Delta \mid \mathcal{H}}\,[\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta \mid \mathcal{H}}{\Gamma, (\exists x)A(x) \succ \Delta \mid \mathcal{H}}\,[\exists Df]$$

Notice that now we extend our zones in our hypersequents to not only keep track of those formulas asserted (on the left) and denied (on the right) but we have, in these rules added the term n to the left hand side of a zone. We allow terms in sequents to keep track of which terms are suitable substitutions for the quantifiers. To rule a term *in* as suitable for substitution (relative to some zone), it is added left hand side of that zone. To rule it *out* we add it to the right. Then the defining rules for the quantifier are motivated. To *deny* $(\forall x)A(x)$ is to take there to be something (which we call n) that doesn't satisfy $A(x)$. That is, in this zone we rule n *in*, and deny A of *it*. To *assert* $(\exists x)A(x)$ is to take there to be something (again, call it n) that satisfies $A(x)$. We rule n in and assert A of it. With these quantifier rules, the derivation of BF breaks down, and it does so in an informative way. We can proceed this far with the derivation

$$\dfrac{\dfrac{(\forall x)\Box A(x) \succ (\forall x)\Box A(x)}{n, (\forall x)\Box A(x) \succ \Box A(n)}\,[\forall Df]}{n, (\forall x)\Box A(x) \succ \quad | \quad \succ A(n)}\,[\Box Df]$$

but to go further, to generalise on the n in the zone containing $A(n)$ to conclude $(\forall x)A(x)$, we need our hypersequent to contain n in LHS in *that* zone, not in the other zone: we wouldd need to have derived $(\forall x)\Box A(x) \succ \quad | \; n \succ A(n)$ because then we could continue:

$$\dfrac{\dfrac{(\forall x)\Box A(x) \succ \quad | \; n \succ A(n)}{(\forall x)\Box A(x) \succ \quad | \quad \succ (\forall x)A(x)}\,[\forall Df]}{(\forall x)\Box A(x) \succ \Box(\forall x)A(x)}\,[\Box Df]$$

In the absence of a rule that would allow terms to migrate from zone to zone, such as this:

$$\dfrac{t, \Gamma \succ \Delta \;|\; \Gamma' \succ \Delta' \;|\; \mathcal{H}}{\Gamma \succ \Delta \;|\; t, \Gamma' \succ \Delta' \;|\; \mathcal{H}}\,[t\ \textit{Migration}]$$

the standard derivation of BF is blocked, and it is blocked in a principled way. In fact, in the system with the standard structural rules and the defining rules for $\Box$ and the free quantifiers $\forall$ and $\exists$ there is no derivation of the Barcan formula. This hypersequent, for example, cannot be derived:

$$a, Fa, \Box Fa, (\forall x)\Box Fx \succ b, Fb, \Box(\forall x)Fx \;|\; a, b, Fa \succ Fb, (\forall x)Fx$$

and in fact, this is this position is, in an important sense, *closed*. Any complex formula in the position has its consequences spelled out in that position. For example, $(\forall x)\Box Fx$ is asserted in the first zone. Its only instance is $\Box Fa$, since the only term ruled *in* in that zone is a, and that instance $\Box Fa$ is asserted in that zone. This assertion takes Fa to be necessary, and Fa is indeed asserted in all zones, both the first, and the second. The same holds for the denial of $\Box(\forall x)Fx$ in that zone. This denial is spelled out by way of the denial of $(\forall x)Fx$ in the second zone, and that denial is spelled out by way of the denial of its instance Fb, which is now a fitting instance of the quantified formula $(\forall x)Fx$ in this zone, since there, b is ruled in.

This simple position corresponds to a Kripke model with two worlds, one of which has as domain $\{a\}$ (which bears property F in that world) and the other, domain $\{a, b\}$ (where the extension of F is $\{a\}$). In the first world, $(\forall x)\Box Fx$ is true since $\Box Fa$ is true (and this world's domain is $\{a\}$), while $\Box(\forall x)Fx$ is false since in the second world $(\forall x)Fx$ fails, since Fb fails (and b is in the domain at this world).

This construction is perfectly general. The procedure of filling out an underivable hypersequent (a position that is in bounds) into a closed position by decomposing complex formulas and finding instances of quantifiers, results in a systematic and canonical model construction for a variable domain quantified S5.

A hypersequent system of this form is consistent with contingentist's motivation in rejecting BF—even though the free logic modifications to quantification used here[11] are motivated by other concerns [9]. However, once we admit that a singular term might succeed in one world but fail in another, then the hypersequent calculus provides us an apt interpretation, fit for giving a normative pragmatic semantics for a contingentist—it gives a semantics in which the variable domain models for a contingentist modal logic have their place, but they are explained by and grounded in something prior, the rules for use for the modal operators and quantifiers.

5 THE STATUS OF CONTINGENTIST MODELS

So, now we have the tools to respond to Williamson's arguments concerning semantics for contingentists:

> Although contingentists may tell some sort of representational story to connect the model theory with the intended modal interpretation, their story is too indirect to make the model theory much more than a complicated digression. By contrast, necessitists can connect the model theory to the intended interpretation more directly… That asymmetry is not decisive in favour of necessitism, because the model theory has no explanatory priority in semantics. Nevertheless, necessitism has a theoretical advantage over contingentism in giving a clearer, simpler, and more satisfying account for quantified modal languages of the relation between truth in a model and truth. [28, page 129]

It is immediately clear that the proof theorist's options are rather different from those that are canvassed here. No single modal model is to be *identified* an intended modal interpretation. However, the connection between modal model theory—and truth in a model—and *truth* is quite direct. If we fix on a particular language, and we take a set of *truths* Γ in that language, we can fill out the sequent $\Gamma \succ$ into a closed position in a canonical way. This process generates witnesses for quantifiers and zones for modal operators, in a familiar manner. The result is a well-behaved model. Let's consider what kind of models are delivered by this construction.

Consider a *model* for the failure of the Barcan Formula. The contingentist, who asserts $\Diamond(\exists x)A(x)$ and denies $(\exists x)\Diamond A(x)$ can agree that there is a *model* with worlds w_1 and w_2, where, according to w_1 there is no object o for which $\Diamond A(o)$, according to w_2 there is an object n where $A(n)$. They can go further, and say that this model, is equipped with a domain for each world, and—talking about the model, from the *outside*, as it were—the object n is in the domain $D(w_2)$ of w_2-objects, and, acording to w_2, $A(n)$ holds, while n is not to be found in the domain $D(w_1)$.

[11] … and explored in my paper "Generality and Existence 1" [22].

This much is relatively standard. However, the *normative pragmatist* contingentist can say more about the status of models like these. There is no need to identify any such model with modal reality as such, because for the normative pragmatist for whom the meaning of the modal operators and the quantifiers is given by rules for use such as [$\Box Df$] and [$\forall Df$] there is no such thing as a single intended modal model. If a sequent, such as $\Diamond(\exists x)A(x) \succ (\exists x)\Diamond A(x)$ is in bounds, then that position can be filled out into an ideal (closed) position—in some extension of the language in question, with a sufficiently large stock of extra constants to be used as witnesses for the quantifiers—using the method sketched in the previous section. The *model* is an idealisation of a position in discourse, where that position is filled out completely.

What are the *worlds* of such models, and what are their *domains*? As is usual in such completeness proofs, they are syntactic constructions, relative to the language being modelled. In our case, the worlds are the different *zones* in the fully refined position, where a zone is the pair $[\Gamma : \Delta]$ of items ruled in and ruled out. The sentences in Γ are the sentences *asserted* in that zone, and the terms in Γ are those terms that are *ruled in* in that zone. The terms in Γ form the *domain* for that zone.[12] The conditions for refinement of positions ensure that if $(\exists x)A(x)$ is in Γ then so is $A(n)$ for some term n where n is also in Γ. The domain of quantification for each zone is a set of names, which represent what that zone takes there to *be*.

Does this give pressure to a kind of ontological inflation for the normative pragmatist contingentist? I think that the answer here is mixed. In one sense, there is *some* pressure, for the usual reason that the models give us some way to interpret wide, necessitist quantifiers. However, as we will see in the next section, this pressure is not new pressure from the existence of models which allow for the interpretation of necessitist quantifiers. The normative pragmatist contingentist has enough resouces to define necessitist quantifiers on their own terms, and that question will be faced in the next section.

The pressing question concerning models is whether they provide any reason themselves for the normative pragmatist contingentist to acknowledge objects that they take to be non-existent. After all, the contingentist, when doing model theory, talks about domains like $D(w_2)$ which are domains of "non-existent objects". But a fair reading of the model theory shows that this pressure is very light, if there is any pressure at all.

Consider an analogy in the model theory of set theory. Suppose I am a proponent of standard Zermelo–Fraenkel set theory with the axiom of choice—ZFC. I agree that ZF¬C—Zermelo–Fraenkel set theory with the *negation* of the axiom of choice—is consistent, and it also has models. I can (using the resources of ZFC) construct models of ZF¬C, whose domain contains 'sets' without choice functions. Are these objects *sets*? In some sense, they are. Do they have choice functions? According to the model, they don't. If the objects *are* sets, then (outside that model)

[12] When the language is extended with an identity predicate, we use equivalence classes of terms under identity rather than the terms themselves.

they have cohice functions. But they are represented inside the model as having no choice functions. That model is a representation of how the world would be were sets as ZF¬C takes them to be.

The worlds in a modal model work in similar way: these zones represent how the world could be. It may represent $A(n)$ to be true, where it represents there to *be* such an n where $A(n)$, even if, as a matter of fact, there is no such object n. So much is standard—the worlds in a model are representations of how things could be. There is no pressure for us to take the term n and to *use* it as if it were a denoting term when (in our vocabulary as we use it) there is no such n where $\Diamond A(n)$. The fact that we can use a term which *could* have denoted some object had things been different does not mean that we need to treat that term as if it *does* denote an object which could been A. Perhaps there is some reason to make that transition, but it is clearly a transition. The items in modal models delivered to us by the completeness proof are representations, and need have no ontological import above and beyond that.

This makes the normative pragmatist about modality sound rather like the 'ersatzist' (such as Alvin Plantinga [13] or Robert Stalnaker [24, 25]) who identifies possible worlds with abstract representations. However, this similarity is superficial. The normative pragmatist takes the rules governing the modal operators to fix their meaning, and the models (and any 'worlds' arising out of such models) are language-relative constructions out of such positions. The modal operators can be used in larger and larger languages without any apparent limit, so there is no requirement that there be any single given 'intended' model which determines the outer limits of modal reality or modal truth. There is no identification of any particular model with the class of all worlds. Perhaps there *is* such a model, but nothing in the normative pragmatist picture given so far seems to commit us to such a thing. (If we were to move from the model theory of modal logic to giving the semantics language which encriches the vocabulary to include singular terms for possible worlds, and quantifiers ranging over worlds, then we would have more scope for judging what commitment to worlds might involve.)

Modal models are useful representations which can give us insight into the behaviour of modal and quantificational concepts. Given that the semantic labour of interpreting the connectives, quantifiers and modal operators is discharged by the proof theory and not the model theory, and the model theory is also grounded in the proof theory, we have an independent explanation of how it is—and why it is—that Kripke models can give us insight into the interaction between modal concepts and quantifiers. There is nothing in such an explanation that leads the normative pragmatist away from any contingentist sympathies they may have had. The ontology of domains of non-actual worlds does not, in itself, lead the normative pragmatist to necessitism.

However, this does not mean that there is no push to necessitism to be found in the normative pragmatist's commitments. Wider quantifiers, satisfying the Barcan Formula, seem to be definable on the normative pragmatist's *own* terms, without going through the machinery of model theory, and it is this that puts *greater* pressure on the contingentist to move beyond the contingentist world-bound quanitifers. However, we will see that it does not necessarily lead away from contingentism to Williamson's necessitism.

6 QUANTIFIERS—WIDER AND STILL WIDER

The quantifier rules in our proof system are zone-relative. This choice suited the contingentist, but other choices are available for defining quanifiers. We could relax the requirement that the name be defined in the zone of application to a weaker requirement that the name be defined *somewhere*, to define some wider quantifiers, $\forall^\Diamond$ and $\exists^\Diamond$ as follows:[13]

$$\frac{n \succ \ \mid\ \Gamma \succ A(n), \Delta \ \mid\ \mathcal{H}}{\Gamma \succ (\forall^\Diamond x)A(x), \Delta \ \mid\ \mathcal{H}}\,[\forall^\Diamond Df] \qquad \frac{n \succ \ \mid\ \Gamma, A(n) \succ \Delta \ \mid\ \mathcal{H}}{\Gamma, (\exists^\Diamond x)A(x) \succ \Delta \ \mid\ \mathcal{H}}\,[\exists^\Diamond Df]$$

These are defining rules for the quantifiers, with exactly the same properties of unique definability and conservative extension as the contingentist quantifiers. If defining rules suffice to give meaning to an expression for a normative pragmatist, then these quantifiers make sense. But this gives rise to a puzzle for the contingentist. With the quantifiers defined by these more liberal rules, the *Barcan formula* is derivable again.

$$\cfrac{\cfrac{\cfrac{\cfrac{(\forall^\Diamond x)\Box A(x) \succ (\forall^\Diamond x)\Box A(x)}{n \succ \ \mid\ (\forall^\Diamond x)\Box A(x) \succ \Box A(n)}\,[\forall^\Diamond Df]}{n \succ \ \mid\ (\forall^\Diamond x)\Box A(x) \succ \ \mid\ \succ A(n)}\,[\Box Df]}{(\forall^\Diamond x)\Box A(x) \succ \ \mid\ \succ (\forall^\Diamond x)A(x)}\,[\forall^\Diamond Df]}{(\forall^\Diamond x)\Box A(x) \succ \Box(\forall^\Diamond x)A(x)}\,[\Box Df]$$

If we have both $\forall^\Diamond$ and $\forall$ around, which is more suited to be considered the genuinely *universal* quantifier? In some sense, $\forall^\Diamond$ is the more universal of the quantifiers, since it has wider scope. If we add to our vocabulary a one place 'predicate' $\downarrow$, the existence predicate, defined in the natural way, making explicit as an assertion what is implicit in ruling in a term as defined

$$\frac{\Gamma, t \succ \Delta \ \mid\ \mathcal{H}}{\Gamma, t{\downarrow} \succ \Delta \ \mid\ \mathcal{H}}\,[{\downarrow}Df]$$

then we can use the predicate to can define the narrower contingentist quantifier $(\forall x)$ in terms of $(\forall^\Diamond x)$ using the existence predicate $\downarrow$ as a scope restirctor.

[13] I use "$\forall^\Diamond$" and "$\exists^\Diamond$" for these quantifiers, since from the contingentist's perspective, the requirement for n to be an appropriate substitution into $(\forall^\Diamond x)A(x)$ is that is defined *somewhere*, that is, it *possibly* denotes. From the contingentist's perspective, they are possibilist quantifiers. This is just a notation—nothing stands against reading the quantifiers as a necessitist would.

$(\forall x)A(x)$ is equivalent to $(\forall^{\Diamond}x)(x{\downarrow} \supset A(x))$:

$$\begin{array}{cl}
\Gamma \succ (\forall x)A(x), \Delta \mid \mathcal{H} & \\ \hline\hline
\Gamma, n \succ A(n), \Delta \mid \mathcal{H} & [\forall Df] \\ \hline\hline
n \succ \mid \Gamma, n \succ A(n), \Delta \mid \mathcal{H} & [K,W] \\ \hline\hline
n \succ \mid \Gamma, n{\downarrow} \succ A(n), \Delta \mid \mathcal{H} & [{\downarrow}Df] \\ \hline\hline
n \succ \mid \Gamma \succ n{\downarrow} \supset A(n), \Delta \mid \mathcal{H} & [\supset Df] \\ \hline\hline
\Gamma \succ (\forall^{\Diamond}x)(x{\downarrow} \supset A(x)), \Delta \mid \mathcal{H} & [\forall^{\Diamond} Df]
\end{array}$$

So far, so good. This puts pressure on the contingentist to turn to necessitism. The wider quantifiers $\forall^{\Diamond}$ and $\exists^{\Diamond}$ are suitable for the necessitist's purposes, for all objects that fall under such quanifiers (in any zone) do so necessarily (they do so in any other zone). However, the quantifiers are not the end of the story for the normative pragmatist. It is possible to totally *unrestricted* quantifiers, in which the restriction to terms being defined somewhere is no longer in force:

$$\begin{array}{cl}
\Gamma \succ A(n), \Delta \mid \mathcal{H} & \\ \hline\hline
\Gamma \succ (\Pi x)A(x), \Delta \mid \mathcal{H} & [\Pi Df]
\end{array}
\qquad
\begin{array}{cl}
\Gamma, A(n) \succ \Delta \mid \mathcal{H} & \\ \hline\hline
\Gamma, (\Sigma x)A(x) \succ \Delta \mid \mathcal{H} & [\Sigma Df]
\end{array}$$

These quantifiers, also, are uniquely defined and conservatively extending. They have been given defining rules, and if defining rules are enough to grant meaning, these, too, are meanginful. This is where the puzzle concerning the quantifiers becomes most pointed, for now the quantifiers $\forall^{\Diamond}$ and $\exists^{\Diamond}$ are not the widest possible quantifiers. $\forall^{\Diamond}$ is not quite universal enough. Given the *absolutely* unrestricted quantifier (Πx), we could define $(\forall^{\Diamond}x)A(x)$ as $(\Pi x)(\Diamond x{\downarrow} \supset A(x))$.

$$\begin{array}{cl}
\Gamma \succ (\forall^{\Diamond}x)A(x), \Delta \mid \mathcal{H} & \\ \hline
n \succ \mid \Gamma \succ A(n), \Delta \mid \mathcal{H} & [\forall^{\Diamond} Df] \\ \hline\hline
n{\downarrow} \succ \mid \Gamma \succ A(n), \Delta \mid \mathcal{H} & [{\downarrow}Df] \\ \hline\hline
\Gamma, \Diamond n{\downarrow} \succ A(n), \Delta \mid \mathcal{H} & [\Diamond Df] \\ \hline\hline
\Gamma \succ \Diamond n{\downarrow} \supset A(n), \Delta \mid \mathcal{H} & [\supset Df] \\ \hline\hline
\Gamma \succ (\Pi x)(\Diamond x{\downarrow} \supset A(x)), \Delta \mid \mathcal{H} & [\Pi Df]
\end{array}$$

So now, we have a *totally* unrestricted quantifier, which allows substitution with non-denoting terms, such as ‘$\frac{1}{0}$’ If $\frac{1}{0}$ is a singular term—in this case, it is a singular term that *necessarily* fails to denote—then the inference from $A(\frac{1}{0})$ to $(\Sigma x)A(x)$ is valid.

It is important to underline that these rules for the wider and unrestricted quantifiers are not given by a straightforward *substitutional* semantics for quantification. We at no point identify the truth of $(\Sigma x)A(x)$ with the truth of some instance $A(t)$. We can have $(\Sigma x)A(x)$ true without there being any instance $A(t)$ true for any term t in our vocabulary.

What are we to make of these quantifiers? It seems to me that there are two viable options for the normative pragmatist using the hypersequent calculus. One option is to be *totally* liberal, and to allow all three kinds of quantification. In that case, the widest quantification does not carry existential import (no-one should think that $\frac{1}{0}$ *exists*), and the question then remains whether existential import is to go with the necessitist quantifier $\exists^{\Diamond}$ or with the original quantifier $\exists$. A further argument is required in order to connect existential import with the quantifiers. The definability of wider quantifiers like Σ or $\exists^{\Diamond}$ does not mean these rules must themselves be treated as existentially committing. More is required to get to that conclusion.

Another option is to demur at the step where we took [$\exists^{\Diamond}$] or [ΣDf] to actually *define* a meaningful concept. Perhaps there are further constraints on definitions of concepts like these quantifiers. For example, there may be constraints on what is involved in ruling a term *in* (or *out*) which conflict with any attempt to use that term as a witness for a quantifier without also ruling it in, and hence, would prevent treating the rules [$\exists^{\Diamond}$] or [ΣDf] as fully legitimate definitions of *quantifiers*.[14] Such considerations are mere sketches and hints of a possible line for the contingentist to pursue.

Now we must address a siginficant remaining issue with this approach. Admitting non-denoting terms like $\frac{1}{0}$ into our vocabulary seems to violate Williamson's appeal to metaphysical universality, and his defence of that appeal. Since $(\exists y)\frac{1}{0} = y$ is metaphysically universal (since $(\forall x)(\exists y)x = y$ is true), if metaphysical universality implies truth, then there is some object that is identical to $\frac{1}{0}$. But this is what the contingentist, and the free logician, denies. What does Williamson say in favour of this constraint?

> For a restriction to completely free logic undermines the application of scientific method by permitting one to hold on to a universal generalization after one of its instances has been refuted: one denies Ga but still asserts $\forall x Gx$ by also denying $\exists y\ a = y$, still retaining the constant a in the language. We assume that the formal languages under consideration in this chapter are well designed in the relevant sense, so that metaphysical universality implies truth. For our present aim is neither to model natural languages, for example in their use of fictional and mythological names…nor to stick to what is knowable *a priori* in some sense, which might exclude whether some names refer. Rather, our business is to clarify the structure of metaphysical universality in a broadly scientific spirit. Non-referring uses of 'Pegasus' have no more place in such an enquiry than they have in physics or zoology. Of course, the term 'phlogiston' did occur in scientific language, but if it failed to refer (rather than referring to an empty kind) then its presence in any scientific theory was a defect in that theory. Consequently, we should not distort our formal language by allowing for such a term. [28, pages 131–132]

There is a lot here, but a response seems at hand for the contingentist and the de-

[14]That would not necessarily prevent using them as 'quasi-quantifiers'—quantifier-like concepts that violate those constraints on quantifier witnesses.

fender of free logic. The use of a free logic does not undermine the application of scientific method, because there is an ambiguity in the expression "hold on to a universal generalization after one of its instances has been refuted." We assert $(\forall x)\ x{\downarrow}$ (everything exists) while denying $\frac{1}{0}{\downarrow}$. Have we held on to the the generalisation and denied one of its instances? In the bare syntactic sense, yes, where $G(a)$ is an instance of $(\forall x)G(x)$ for any singular term a. However, this is another sense in which $(\forall x)\ x{\downarrow}$ is *not* refuted by $\neg(\frac{1}{0}{\downarrow})$, since the term $\frac{1}{0}$ is ruled out as an appropriate substitution for the quantifier in the zone. We take $\frac{1}{0}$ to not refer, and so, it is not a counterexample to the universal quantifier—it is not an instance at all.

More serious is Williamson's appeal to scientific discourse in the defence of metaphysical universality. This, it seems to me, is the core of his defence of necessitism, and the appeal to scientific discourse seems to me misplaced, at least when it comes to mathematics. Mathematical discourse is shot through with what the mathematicians take to be non-referring terms, like these:

$$\frac{1}{0} \qquad \{x : x \notin x\} \qquad \lim_{x\to 0} \frac{\sin x}{x}$$

And expressions like these, which are found in the most scientific of texts (whether pure mathematics, applied mathematics or one of the other sciences), sometimes refer, and sometimes do not, depending the behaviour of the components.

$$\lim_{n\to\infty} a_n \qquad \sum_{n=0}^{\infty} a_n \qquad f'(x) \qquad \int_a^b f(x)\,dx \qquad \{x : \phi(x)\}$$

The same goes for recursion theory and computer science. Reasoning about termination of algorithms and the definedness of functions is widespread. Now, of course it is in some sense *possible* for us to strip our mathematical and scientific discourse of such terms—or rather, it's possible for us to stop doing mathematics in the usual manner, and restrict ourselves to a much more limited language, free of undefined and non-denoting terms. However, it is by no means clear that the language which admits of partial functions and non-denoting terms is in any sense more defective than a language which does away with them. The free logic of these non-denoting terms is thoroughly classical, and straightforward to work with [9]. And if non-denoting terms have their use in *those* contexts, it seems to less problematic to allow for the possibility that terms which *do* denote (like names for seemingly contingently existing objects) would have failed to denote had things gone differently. The hypersequent calculus treats non-denoting mathematical terms and names for contingently existing objects in the same manner. The discipline of ruling terms *in* our *out* in zones in a hypersequent keeps track of those terms that are appropriate instances for the quantifiers in those zones, and it is done in a way which respects the strictures and conventions of scientific discourse.

7 EPISTEMIC MODALS

Modal alternatives come in a number of different varieties. There is broad metaphysical necessity like we have considered so far, but there are other alternatives to consider, such as alternative *times*, alternative states of non-deterministic system, alternative outcomes of a choice setup. Do any of those modal alternatives go beyond the scope of metaphysical necessity? Can we go beyond into the *counterpossible*? Or is the modality encompassed by the $\Box$ and $\Diamond$ discussed until now—let's call it *metaphyiscal necessity*—the broadest possible modality [29]?

I think that there is reason to think that some kinds of supposition allow us to coherently and consistently suppose the *impossible*, and that at least some of the modal contexts we use in discourse give rise to modal operators with logical structure as coherent and precise as the $\Box$ and $\Diamond$ of metaphysical necessity, but which are essentially counterpossible. In this section I will examine the argument for necessitism in the light of these modalities.

Here is an example of counterpossible reasoning. Suppose you and I disagree about some necessary truth. You take it that A is necessary and I deny that it is. When reasoning together, I could (in the context of the discussion) suppose that you are right, and grant your commitment that $\Box A$, and reason with you for a while. Turning the tables, you could grant my commitment that $\neg\Box A$, and you could reason with me. We could discover that many of our commitments survive on either supposition, and some hold only if $\Box A$ and others only if $\neg\Box A$. Upon reflection, and understading each other's positions more, we may decide to *withdraw* our commitments to $\Box A$ and to $\neg\Box A$ respectively, and become agnostic between these two alternatives.

Now, strictly speaking, if the logic of necessity is S5, then one of $\Box A$ and $\neg\Box A$ is *impossible*. But now, we don't know which it is. We have two alternatives we are considering: $\Box A$ and $\neg\Box A$, and when we swap between *your* initial position and *mine*, this kind of alternativeness is very much like the *modal* alternativeness of metaphysical necessity, but it has significantly different features. The alternatives are not considering how things could have gone (otherwise)—a kind of *subjunctive* alternative—but rather, considering how things might *be* (as far as we know)—they a *indicative* alternatives.

One way this kind of alternativeness is when you and I might disagree over whether it is actually the case that A. On a standard logic of actuality $S5_@$, what is *actually true* is *necessarily* so. The sequent $@p \succ \Box @p$ is derivable. If p holds in the actual world, then from the point of view of any world at all, p holds back at the actual world. Or to consider hypersequents, the rules for actuality are straightforward, when we allow for a special zone in a hypersequent to be marked off as the 'actual' zone.

$$\Gamma \succ_@ \Delta \;\mid\; \Gamma' \succ \Delta' \;\mid\; \cdots$$

Then, we can introduce a defining rule for @ as follows:

$$\frac{\Gamma, A \succ_{@} \Delta \mid \Gamma' \succ \Delta' \mid \mathcal{H}}{\Gamma \succ_{@} \Delta \mid \Gamma', @A \succ \Delta' \mid \mathcal{H}}\,[@Df]$$

according to which an assertion of @A in a zone has the same significance as the assertion of A in the zone tagged with @. Then, we can derive hypersequents inovlving the special properties of the actuality operator.

$$\cfrac{\cfrac{\cfrac{@A \succ @A}{A \succ_{@} \quad \mid \quad \succ @A}\,[@Df]}{A \succ_{@} \quad \mid \quad \succ \Box @A}\,[\Box Df]}{@A \succ \Box @A}\,[@Df]$$

So, to assert @A (in any zone) has the same force as asserting A in the actual zone, and to deny @A has the same force as denying A in the actual zone. To assert @A and to deny $\Box$@A, then, involves a clash, since to deny $\Box$@A is coherent only if denying @A in some zone or other is coherent, and asserting @p in one zone and denying @A in another zone (or the same zone) is to assert p and deny p in the actual zone.

Now, it is coherent for you and me to disagree over @A. But then, given the logic $S5_{@}$ this is another example of disagreeing over necessary truths. It follows that indicative alternativeness incorporate different zones both considered to be candidates for the *actual* zone. Let's mark this kind of alternativeness with a double vertical line.

$$\Gamma_1 \succ_{@} \Delta_1 \;\|\; \Gamma_2 \succ_{@} \Delta_2$$

In our discussion where we keep track of *your* commitments and *mine*, and where we reason together about the two views before us, we have two indicative alternative contexts, each with a zone marked out as the actual zone. If you and I disagree about modal claims, and unpack those claims, then each indicative alternative zone might come with other subjunctive alternatives, and the result is a thouroughly two-dimensional hypersequent [21].

$$\Gamma_1 \succ_{@} \Delta_1 \mid \Gamma_1' \succ \Delta_1' \mid \cdots \;\|\; \Gamma_2 \succ_{@} \Delta_2 \mid \Gamma_2' \succ \Delta_2' \mid \cdots$$

Given indicative alternatives like this, we could idealise away from the constraints of taking an indicative alternative zone to be a position that you or I take as a live *option*, to generalise away to arbitrary alternative views of the world. The only constraint over such alternatives is that each alternative is marked as *actual* zone (when we make an indicative alternative supposition, we are supposing that things are *actually* as described), and we apply the same semantic constraints on what is said as are applied in other alternative zones.[15] Given this broad idealisation of the

[15]To suppose that something is actually green is to depend on the meaning of 'green', not to vary the interpretation of that word. This doesn't matter much when it comes to modal *logic*, but when we are applying these rules to an interpreted language with meaning conditions this will play more of a role.

rules of supposition, it makes sense to define an *indicative* necessity, which stands to indicative alternatives as $\Box$ stands to subjunctive alternatives[16]

$$\frac{\mathcal{H}[\Gamma \succ \Delta \mid\ \succ A]}{\mathcal{H}[\Gamma \succ \Box A, \Delta]}\,[\Box Df] \qquad \frac{\mathcal{H}[\Gamma \succ \Delta \parallel\ \succ_{@} [i]A]}{\mathcal{H}[\Gamma \succ [i]A, \Delta]}\,[[i]Df]$$

Indicative necessity can be understood as a kind of disciplined and well-behaved kind of *a priori knowability*,[17] since something to be indicatively necessary, its denial is out of bounds in any indicative alternative. If I take something to be indicatively necessary, I take it to hold in all indicative alternative circumstances, including any circumstaces which I would count as live epistemic alternatives.

Here are two example derivations using the rules for indicative necessity.

$$\dfrac{\dfrac{\dfrac{\succ\ \parallel A \succ_{@} A}{\succ\ \parallel A \succ_{@} @A}\,[@Df]}{\succ\ \parallel\ \succ_{@} A \supset @A}\,[\supset Df]}{\succ [i](A \supset @A)}\,[[i]Df] \qquad \dfrac{\dfrac{\dfrac{[i]A \succ [i]A \mid\ \succ}{[i]A \succ\ \mid\ \succ\ \parallel\ \succ_{@} A}\,[[i]Df]}{[i]A \succ\ \mid\ \succ [i]A}\,[[i]Df]}{[i]A \succ \Box[i]A}\,[\Box Df]$$

The first shows that (as we would hope), $A \supset @A$ is indicatively necessary. In any indicative alternative at all, $A \supset @A$ holds. However, in a (counterfactual) subjunctive alternative, $A \supset @A$ can fail—$A \supset @A$ is not, in general *necessary*. The second derivation shows that, with the rules we have given, indicative necessities are (metaphyiscally) *necessarily* indicatively necessary, for the reason that our sequent structure takes the indicative alternatives for any subjunctive alternative to be the same.[18]

Indicative necessity seems well behaved and defensible on normative pragmatist lines.[19] Now consider BF for *indicative* modalities. Just as the contingentist may wish to block any derivation from $\Diamond(\exists x)Fx$ to $(\exists x)\Diamond Fx$, there seems to be equally good reason to block a derivation from $\langle i \rangle(\exists x)Fx$ to $(\exists x)\langle i \rangle Fx$. For example, we may think that it's an *option to consider* that the Morning Star differ from the Evening Star: $\langle i \rangle((\exists x)(\exists y)(Mx \mathbin{\&} Ey \mathbin{\&} x \neq y)$. Does it follow from this

[16] Now, rules are stated with the notation $\mathcal{H}[\Gamma \succ \Delta]$, which is a hyperseqent in which $\Gamma \succ \Delta$ is included as some zone; $\mathcal{H}[\Gamma \succ \Delta \mid \Gamma' \succ \Delta']$ where $\Gamma \succ \Delta$ and $\Gamma' \succ \Delta'$ are subjunctive alternatives within $\mathcal{H}$, and $\mathcal{H}[\Gamma \succ_{@} \Delta \parallel \Gamma' \succ \Delta']$, where $\Gamma \succ_{@} \Delta$ is an indicatie alternative of $\Gamma' \succ \Delta'$

[17] In "A Cut-Free Sequent System for Two-Dimensional Modal Logic" [21], I used 'APK' instead of '$[i]$' for this modality. That is cumbersome, especially when we consider the dual operator '$\neg$APK$\neg$' for a priori epistemic 'permissibility.' In what follows, we will use '$\langle i \rangle$'

[18] If this seems problematic, it is possible to generalise sequent structure to loosen this connection, but I cannot see a compelling reason for doing this.

[19] Mark Lance and Heath White have argued for the importance of recognising indicative and subjunctive supposition as distinct acts with distinct normative structures [11]. They argue that for social creatures like us, who *act* on the basis of our *views*, it is deeply important to be able to subjunctively suppose (to consider what would happen *if we do this…*) and to indicatively suppose (to consider what is involved *if you are right…*). I make a little more of the distinction and spell out the connection with two dimensional hypersequents elsewhere [21].

that there is something such that it's open that *it* differs from the evening star? Must we have $(\exists x)\langle i\rangle((\exists y)(Mx \ \& \ Ey \ \& \ x \neq y)$? Furthermore, do we have $(\exists x)(\exists y)\langle i\rangle(Mx \ \& \ Ey \ \& \ x \neq y)$? What are these objects x and y such that it's not given to us that they are identical, one the evening star and the other, the morning star? What could such objects be? It seem like a significant ontological inflation on *this* world, to introduce distinct this-worldly counterparts to the Morning Star and the Evening Star, when it is no cost to say that the theorist who tries to distinguish the Morning Star and the Evening Star, takes there to be different entities, so *their* domain includes distinct objects, while ours does not.[20] Simiarly, if we take it there is nothing *inconsistent* about finitism (the view that there are only finitely many things)—and by the lights of first order (or higher order) classical logic, there is no inconsistency in there being finitely many items in the first-order domain—then it would seem coherent to consider the views of some finitist, who took there to be finitely many things. Similarly, we could consider the infinitist. However, to take these alternatives to be genuinely indicative alternatives to one another would seem to require rejecting the Barcan formula for the indicative modalities, lest the domain of the infinitist alternative overrun the domain of the finitist.

I see no good reason to rule indicative modalities out as having a semantics like this, so I take this to be a reason against wholesale adoption of necessitism and the Barcan Formula. And if the argument can be resisted for indicative modalities, the appeal is lessened for the metaphysical modalities too. I offer this as a challenge for necessitists and contingentists alike.

8 CONCLUDING THOUGHTS

I have offered these thoughts as a *compliment* and a *complement* to Williamson's *Modal Logic as Metaphysics*. I concur with Williamson that the tools of modal logic are a fruitful setting for understanding modal metaphysics. We do need the discipline of a formal theory to keep ourselves honest, and to help us understand exactly what theory it is we're adopting, and what its consequences *are*. *Modal Logic as Metaphysics* is a fruitful example of the best work in this tradition.[21] The *complement* in this work is the attempt to bring to bear the neglected half of logic—proof theory—to the task of modal semantics. I hope that it can bring just as small part of the rigour and insight found in Williamson's *Modal Logic as Metaphysics*.

[20] The same considerations apply for epistemic scenarios where we mistakenly identify objects, as well as when we mistakenly distinguish them. It seems clear that it may well be that $\langle i\rangle(a = b)$ even though a and b are as a matter of fact, distinct. In the scenario where $a = b$ holds, what is in the domain of quantification? Are the denotations of the terms a and b different there, or the same?

[21] We are living at a good time for this kind of work. I single out Rini and Cresswell's *The World-Time Parallel* [23] is another recent example of a logically disciplined and rigorous treatment of the metaphysics of modality.

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