Interpreting and Applying Proof Theories for Modal Logic

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Abstract

Proof theory for modal logic has blossomed over recent years. Many extensions of the classical sequent calculus have been proposed in order to give natural and appealing accounts of proof in modal logics like K, T, $\mathsf{S4}$, $\mathsf{S5}$, provability logics, and other modal systems. The common feature of each of these different proof systems consists in the general structure of the rules for modal operators. They provide *introduction* and *elimination* rules for statements of the form $\Box \mathsf{A}$ (and $\Diamond \mathsf{A}$), which show how those statements can feature as *conclusions* or as *premises* in deduction. These rules give an account of the deductive power of a modal formula $\Box \mathsf{A}$ (and $\Diamond \mathsf{A}$) in terms of the constituent formula A.

The distinctive feature for the modal operators in contemporary proof systems for modal logics is that the step introducing or eliminating $\Box \mathsf{A}$ is at the cost of introducing or eliminating some kind of extra *structure* in the proof. In this way, the proof rules for modal concepts such as $\Box$ run in parallel with the truth conditions for these concepts in a Kripke model, in which the truth of $\Box \mathsf{A}$ stands or falls with the truth of A, but at the cost of checking that truth *elsewhere*, at points in the model accessible from the point at which $\Box \mathsf{A}$ is evaluated.

In this paper we will introduce these recent advances in proof theory for a general audience, and then we will show how they are connected with different—metaphysical and epistemic—conceptions of modality. We will show that these different pictures are nothing but different ways of connecting the statement $\Box \mathsf{A}$ with the statement A, of showing the significance of *modalising*. In the light of this result we will draw conclusions about the link between analyticity and modality, and about the nature of a proof system for modal logics.

1 Introduction

Modal logic is traditionally the logic obtained by adding to basic propositional logic, like classical logic, the concepts of necessity ($\Box$) and possibility ($\Diamond$). There is a wide consensus on which are the main systems of modal logic—systems such as K, KT, KB, $\mathsf{S4}$, $\mathsf{S5}$ and the provability logic GL—and their canonical interpretation, Kripke models. Beyond that, there is little consensus. In particular, there is little consensus on the way to understand what it is to *prove* a statement like $\Box \mathsf{A}$. While we have a systematic and rigorous formal account of truth conditions of modal statements (in Kripke models with points and accessibility relations with different properties, underwriting different principles governing $\Box$ and $\Diamond$ and their interaction), we have no such consensus on what the basic items of *deduction* in modal vocabulary are.

If our goal is to find a well-behaved proof system in which each item of vocabulary is governed by its own introduction and elimination rules, in which each of those rules is *separable*, so we can specify the rules for one concept independently of the rules for an of the others, and in which proofs can be composed, manipulated and transformed in natural ways, then the situation has not been promising for very many years. As Serebriannikov said in 1982:

> Gentzen's proof-theoretical methods have not yet been properly applied to modal logic. [23, p. 79]

This is a sad state of affairs, as Gentzen's sequent methods have a number of desirable properties, which bear a very close relationship to the pleasant properties of truth conditional semantics such as are given in Kripke models. In sequent calculi for propositional logics, each connective is governed by an introduction and elimination rule independent of each other connective, in just the same manner in which each connective is given truth conditions independently of each other connective. The aim is to construct modal proof systems with the same kind of systematic generality as Kripke models, and which also provide a natural account of what counts as a *proof* of a modal statement. This debt, which was still was owing in 1982, has only now begun to be discharged. In this paper we will explain recent advances in modal proof theory, and what they mean.

In recent years, something has changed. Indeed there is an increasing number of journal and conference papers devoted to the subject of modal proof theory. More precisely, we can count several new generalisations of the classical sequent calculus that handle modal logical systems. The common feature of these generalisations consists in the introduction of new *objects* or *structures* to the meta-language of the classical sequent calculus.[1] By exploiting these new elements, we can formulate the rules that introduce the constants $\Box$ and $\Diamond$ on the left and on the right of a sequent. In addition, by means of these elements, several structural rules naturally arise, which correspond in a tight way to different modal principles in just the same manner as conditions satisfied by an accessibility relation on a frame correspond to the same sort of modal principles.

The first aim of this paper is to present in a clear and general way certain of these recent advances in proof theory for modal logic. More precisely we are going to introduce the reader to the following three methods for extending the classical sequent calculus: *display logic*, *labelled sequent calculi* and the *tree-hypersequent method*.

Why did we chose these generalisations and not others? First, labelled calculi [11] and the tree-hypersequent method [12, 13, 14] are the most recent two generalisations of the classical sequent calculus; and they both apply to a wide number of modal logics and each of them is representative of a different way of extending the classical sequent calculus: the semantical and the purely

[1] Statements in the classical sequent calculus, such as $\mathsf{A} \vee \mathsf{B} \Rightarrow \mathsf{A}, \mathsf{B}$ or $(\forall x)(\mathsf{F}x \vee \mathsf{G}x) \Rightarrow (\forall x)\mathsf{F}x, (\exists x)\mathsf{G}x$ are in the meta-language of classical logic, for these are statements about validity or consequence, between object language statements.

syntactical way, respectively. As for the display method, we think that it now represents a sort of milestone of proof theory not only of modal logics but a wide range of classical and non-classical systems [1, 9, 24], so it is also worth considering, especially as it has distinctive properties differing from both labelled and tree-hypersequent calculi.

After having introduced these three extensions Gentzen's original calculus, we will show how they are connected with different conceptions of modality. We will attempt to show that these different pictures are nothing but different ways of connecting the judgement that $\Box A$ with the judgement that A. In the light of this result we will draw conclusions about the link between analyticity and modality.

Along the way, we will address important issues in the philosophy of modal logics and proof theory. Many, like Dummett [7] and Prawitz [15, 16] have had the thought that there is an important application of proof theory to issues of meaning. Can any such connection be made in the case of modal logic? Or is the distinctive nature of modal proof theory a sign that modal vocabulary should not count as properly *logical*?

2 Display Sequents

In Display Logic [1, 2, 3] sequents have the usual form $M \Rightarrow N$, but M and N are not the usual multisets, sets or sequences of formulas but much more general *structures*.[2] Structures are made up of formulas and structure connectives: a structure is constructed from smaller structures, in quite the same way as formulas are constructed from formula connectives. The structure connectives for propositional logic are three: the empty structure I, and the unary and binary operations $*$ and $\circ$ respectively. Structures have a kind of *polarity*: they can be either positive or negative structures, depending on the positions in which they occur in a sequent. Such polarity is reflected and made explicit in the object language by connectives. The structure-connective $\circ$ is interpreted as a conjunction in negative position and as a disjunction in positive position; this fact becomes clear by means of the following rules and for the two connectives $\wedge$ and $\vee$:

$$\frac{A \circ B \Rightarrow N}{A \wedge B \Rightarrow N}\wedge L \qquad \frac{M \Rightarrow A \circ B}{M \Rightarrow A \vee B}\vee R$$

These rules trade in structure connectives—which combine formulas into structures and are not themselves found inside formulas—with formula connectives. The connection between $\circ$ in negative position and conjunction, and $\circ$ in positive position and disjunction remains with the rules for the connectives on the other side of the sequent arrow.

$$\frac{M \Rightarrow A \quad M' \Rightarrow B}{M \circ M' \Rightarrow A \wedge B}\wedge R \qquad \frac{A \Rightarrow N \quad B \Rightarrow N'}{A \vee B \Rightarrow N \circ N'}\vee L$$

These splitting rules pair a formula connective on one side of the arrow with the structure connective on the other. For negation, we have another structure

[2] Structures in display sequents are not to be confused with the structures of model theory. Structures here are syntactic items in the meta-language, combining formulas in new ways. These syntactic items allow us to construct more general kinds of sequents.

connective, which is paired with negation in *both* positions.

$$\frac{*A \Rightarrow N}{\neg A \Rightarrow N}\,\neg L \qquad \frac{M \Rightarrow *A}{M \Rightarrow \neg A}\,\neg R$$

However $*$ inverts the position of those structures occurring in its scope (as you would expect if it acts like negation), so if $*M$ occurs in a positive position, then the embedded M occurs in negative position, and if $*M$ occurs in negative position, then the M under its scope is in positive position.

The the 'empty' structure I should be understood as $\top$ (logical truth) if it occurs in negative position, and as $\perp$ (logical falsity) if it occurs in positive position. This fact appears clear with the following two rules:

$$\frac{I \Rightarrow N}{M \Rightarrow N} \qquad \frac{M \Rightarrow I}{M \Rightarrow N}$$

If M is a negative structure, and N is a positive structure, then $M \Rightarrow N$ is a sequent.[3] The structure M (N) is the antecedent (succedent) of $M \Rightarrow N$. A substructure P is an antecedent (succedent) part of a sequent $M \Rightarrow N$, if it occurs positively (negatively) in M or negatively (positively) in N.

Given this family of structures and sequents, there are rules that determine the simple and clear inferential behaviour of the new structural connectives. These are called *basic structural rules* and are the following:

$$\begin{array}{c} M \circ S \Rightarrow N \\ \hline\hline M \Rightarrow N \circ S^* \\ \hline\hline S \Rightarrow M^* \circ N \end{array} \qquad \begin{array}{c} M \Rightarrow N \\ \hline\hline N^* \Rightarrow M^* \\ \hline\hline M \Rightarrow N^{**} \end{array} \qquad \begin{array}{c} M \Rightarrow N \circ T \\ \hline\hline M \circ T^* \Rightarrow N \\ \hline\hline N^* \circ M \Rightarrow T \end{array}$$

If two sequents are interderivable by means of these rules, then these sequents are said to be *structurally equivalent*. One of the crucial features of display logic, by which it takes its name, is the *display property*. This is the property to the effect that any substructure of a given sequent s may be DISPLAYED, either as the entire antecedent (if the original structure was an antecedent part) or as the entire succedent (if the original structure was a succedent part) of a structurally equivalent sequent s', and that this equivalence can be achieved by means of the basic structural rules. For example, the following string of structural equivalences convert the sequent $X \circ (Y^* \circ (Z \circ U)^*) \Rightarrow V$ (in which V is the entire consequent) into the equivalent sequent $Z^* \circ (Y^{**} \circ (X^* \circ V^*)) \Rightarrow U$ in which the embedded U is displayed as the entire consequent.

$$\begin{array}{c} X \circ (Y^* \circ (Z \circ U)^*) \Rightarrow V \\ \hline\hline Y^* \circ (Z \circ U)^* \Rightarrow X^* \circ V \\ \hline\hline (Z \circ U)^* \Rightarrow Y^{**} \circ (X^* \circ V) \\ \hline\hline (Z \circ U)^* \Rightarrow (Y^{**} \circ (X^* \circ V))^{**} \\ \hline\hline (Y^{**} \circ (X^* \circ V))^* \Rightarrow Z \circ U \\ \hline\hline Z^* \circ (Y^{**} \circ (X^* \circ V))^* \Rightarrow U \end{array}$$

The display property ensures that the rules for the connectives already seen are as general as they need to be. A rule, like $\vee L$ which tells us how to introduce or

[3]This is a statement of syntactic well-formedness, not a statement about validity. $A \Rightarrow A$ and $B \Rightarrow A$ are both sequents, the first valid and the second, invalid.

eliminate $\mathsf{A} \vee \mathsf{B}$ as the entire antecedent of a sequent suffices to tell us how to introduce or eliminate $\mathsf{A}\vee\mathsf{B}$ in an arbitrary negative position, since a disjunction in negative position may be *displayed* as the entire antecedent.

In order to introduce the modal operator $\Box$[4] we need a further structural connective. This is going to be the unary operator $\bullet$, was introduced by Wansing [24].[5] The rules for $\Box$ correspond to $\bullet$ in positive position:

$$\frac{\mathsf{A} \Rightarrow \mathsf{M}}{\Box\mathsf{A} \Rightarrow \bullet\mathsf{M}}\,\Box\mathsf{L} \qquad \frac{\bullet\mathsf{M} \Rightarrow \mathsf{A}}{\mathsf{M} \Rightarrow \Box\mathsf{A}}\,\Box\mathsf{R}$$

The basic structural rule that determines the behaviour of the structure-connective $\bullet$ is this:

$$\frac{\bullet\mathsf{M} \Rightarrow \mathsf{N}}{\mathsf{M} \Rightarrow \bullet\mathsf{N}}$$

which suffices for the proof of the display property for structures involving $\bullet$. The structure operator $\bullet$ in positive position acts like necessity, and in antecedent position, it acts like a kind of possibility, but this possibility is the *converse* of the usual possibility. If necessity is understood in a temporal setting as truth at *all future* times, then the possibility is truth at *some past* time. The display equivalence is then straightforward. If at any moment t, at which M is true at some earlier time, it follows that N is true then it follows that for any moment t' at which M is true, then at any later time, N must hold — and *vice versa*. The two statements $\bullet\mathsf{M} \Rightarrow \mathsf{N}$ and $\mathsf{M} \Rightarrow \bullet\mathsf{N}$ describe the same connection between earlier and later moments, but one ($\bullet\mathsf{M} \Rightarrow \mathsf{N}$) is from the perspective of the later moment, looking back to where M held true, while the other ($\mathsf{M} \Rightarrow \bullet\mathsf{N}$) is from the perspective of the earlier moment, looking forward to times at which N holds. Here is an example modal derivation in the display system:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{A} \Rightarrow \mathsf{A}}{\Box\mathsf{A} \Rightarrow \bullet\mathsf{A}}\,\Box\mathsf{L}}{\Box\mathsf{A} \circ \Box\mathsf{B} \Rightarrow \bullet\mathsf{A}}\,\textit{weaken}}{\bullet(\Box\mathsf{A} \circ \Box\mathsf{B}) \Rightarrow \mathsf{A}}\,\textit{display} \qquad \dfrac{\dfrac{\dfrac{\mathsf{B} \Rightarrow \mathsf{B}}{\Box\mathsf{B} \Rightarrow \bullet\mathsf{B}}\,\Box\mathsf{L}}{\Box\mathsf{A} \circ \Box\mathsf{B} \Rightarrow \bullet\mathsf{B}}\,\textit{weaken}}{\bullet(\Box\mathsf{A} \circ \Box\mathsf{B}) \Rightarrow \mathsf{B}}\,\textit{display}}{\dfrac{\bullet(\Box\mathsf{A} \circ \Box\mathsf{B}) \Rightarrow \mathsf{A} \wedge \mathsf{B}}{\dfrac{\Box\mathsf{A} \circ \Box\mathsf{B} \Rightarrow \Box(\mathsf{A} \wedge \mathsf{B})}{\Box\mathsf{A} \wedge \Box\mathsf{B} \Rightarrow \Box(\mathsf{A} \wedge \mathsf{B})}\,\wedge\mathsf{L}}\,\Box\mathsf{R}}\,\wedge\mathsf{R}$$

In this derivation, we show how $\Box(\mathsf{A} \wedge \mathsf{B})$ follows from $\Box\mathsf{A} \wedge \Box\mathsf{B}$. It proceeds via the initial derivations from $\Box\mathsf{A}$ to $\bullet\mathsf{A}$ (and from $\Box\mathsf{B}$ to $\bullet\mathsf{B}$), which tells us that if $\Box\mathsf{A}$ holds *here* then A holds *there*, that is at points accessible from here (and similarly for $\Box\mathsf{B}$ and B). This fact is then weakened to $\Box\mathsf{A} \circ \Box\mathsf{B} \Rightarrow \bullet\mathsf{A}$ and similarly, $\Box\mathsf{A} \circ \Box\mathsf{B} \Rightarrow \bullet\mathsf{B}$, and these are then rearranged to the equivalent sequents $\bullet(\Box\mathsf{A} \circ \Box\mathsf{B}) \Rightarrow \mathsf{A}$, and similarly, $\bullet(\Box\mathsf{A} \circ \Box\mathsf{B}) \Rightarrow \mathsf{B}$, which allow for the

[4]Note that from now on the operator $\Diamond$ will not be taken as primitive but as defined in the following standard way: $\Box A = \neg\Diamond\neg A$, not because it couldn't be primitive, but for compactnesss of presentation.

[5]In Belnap's original work on Display Logic, the modal operators are treated with another family of display connectives $\circ'$, $*'$, I' which have an intensional interpretation along with the extensional interpretation of the original family. This is more complex than we need to treat normal modal logic, for which a single structure connective suffices.

A and B to be conjoined. The resulting sequent is rearranged into $\Box A \circ \Box B \Rightarrow \bullet(A \wedge B)$, which tells us that if $\Box A$ and $\Box B$ are true here, then at any accessible point *there*, $A \wedge B$ holds. This can be reformulated as a claim that $\Box(A \wedge B)$ is true *here*, as the structural connective $\bullet$ is rewritten as the object language connective $\Box$.

The identity axioms, the basic structural rules, the logical rules $\neg L$, $\neg R$, $\wedge L$ and $\wedge R$, the modal rules $\Box L$ and $\Box R$, plus the classical structural rules of weakening, contraction and cut, form the Display proof system for the basic normal modal logic K. In order to obtain display calculi for other modal systems one adds to the calculus K structural rules governing the structural-connectives I, $*$, $\circ$, $\bullet$. In the calculus K and its extensions the cut-rule is shown to be eliminable by way of an elegant and highly general cut-elimination argument in the style of Curry [4]. The system provides a cut-free sequent calculus for a range of logics, including classical, intuitionistic, and substructural logics, as well as modal logics.

Our topic is not the finer details of Display Logic, but rather, the way that display sequents encode modal information and represent modal deduction. As we will see, we can think of sequents featuring $\bullet$, $\circ$, $*$ and I as describing conditions on modal models. This connection is completely general. We can show that any derivable sequent is satisfied in any Kripke model — once you define what it *is* for a sequent to be valid on a model. This definition generalises the well-known validity condition for classical sequents. For a classical sequent $X \Rightarrow Y$, we require that any world where each formula in X is true, some formula in Y is true too. We generalise this to apply to Display sequents, and we will see how these sequents describe truth in Kripke frames. Given a frame $\langle W, R\rangle$ — consisting of a set W of worlds and a binary accessibility relation R on W — and a relation $\vDash$ of truth at worlds, we may define for each structure X the conditions $P_w(X)$ (the structure X in *positive* position is true at world w) and $N_w(X)$ (the structure X in *negative* position is true at world w). It is defined inductively:

	A	$*X$	$X \circ Y$	$\bullet X$
N_w	$w \vDash A$	$\neg P_w(X)$	$N_w(X) \wedge N_w(Y)$	$(\exists v)(vRw \wedge N_v(X))$
P_w	$w \vDash A$	$\neg N_w(X)$	$P_w(X) \vee P_w(Y)$	$(\forall v)(wRv \supset P_v(X))$

Then a sequent $X \Rightarrow Y$ is said to be valid on a model if and only if, according to that model $(\forall w)(N_w(X) \supset P_w(Y))$. It is a straightforward induction on the construction of a derivation that all derivable sequents are valid. For example, the sequent $X \Rightarrow \bullet Y$ is valid on a model if and only if on that model we have

$$(\forall w)(N_w(X) \supset (\forall v)(wRv \supset P_v(Y)))$$

a straightforward quantifier shift converts this to

$$(\forall v)((\exists w)(wRv \wedge N_w(X)) \supset P_v(Y))$$

which is $(\forall v)(N_v(\bullet X) \supset P_v(Y))$, the condition arising from the display equivalent sequent $\bullet X \Rightarrow Y$. These two sequents describe the same conditions on models, from the point of view of different points on the frame. We can think of a display sequent as giving a *local* or *internal* perspective on a frame. When we say $X \Rightarrow \bullet Y$, we are saying that if X holds *here*, then at all later points, Y holds. When we say $\bullet X \Rightarrow Y$, we are saying that if X holds at some earlier point, then Y holds here. Provided that *here* is arbitrary, this is exactly the same fact about frames, described in two different ways.

Despite these pleasing features, display logic has not been widely used.[6] Part of this may be explained in terms of the unique features of display calculi: systems for modal logics are not merely *expansions* of classical Gentzen-style sequent systems, as proofs in the boolean fragment use the exotic machinery of $*$, and $\circ$ instead of the familiar sequent structure $X \Rightarrow Y$ where X and Y are multisets (or lists) of formulas. This new structure does not simplify derivations: it complicates them with what seem to be inessential and bureaucratic choreography which does nothing to expose the essential deductive steps in a derivation. The essential work of the display property seems to be to ensure that every position in a sequent is uniform, in that it is available for a *cut* or for a connective rule. A formula in a sequent may be displayed, and a displayed position is the site for a cut or for a connective step. The trouble to which we must go to display a formula in order to process it seems to indicate that we do not have the most perspicuous or concise mode of formulating modal deduction. It looks like display logic can be simplified, which may motivate different vocabularies for sequents, so it is to other systems, in which this kind of redundancy does not feature, that we now turn.

3 Labelled Sequents

Labelled modal calculi are a different solution to the problem of finding a sequent calculus for the main systems of modal logic. The core idea is in this case quite simple: the whole relational structure of a Kripke model is internalised explicitly in the proof system. This means that we are not going to work, as in the display case, with new meta-linguistic connectives between formulas, but instead, we are going to change the language itself, i.e. we will deal with *labelled formulas* of the form $x : A$ and *relational statements* of the form xRy. $x : A$ stands for: "the formula A is true at the world x" while xRy stands for: "the world x is linked by the relation $\mathcal{R}$ to the world y." So, a labelled modal sequent is still an object of the form $M \Rightarrow N$ but now M and N are composed by formulas of the kind described above.

The structural rules of the labelled modal calculi are the usual classical rules of weakening, contractions and cut, transposed into this new setting with labelled formulas and relational statements. Note that the weakening rules as well as the contraction rules will have a double form: one that deals with the first kind of formulas $x : A$, and one that deals with the second kind of formulas xRy. For example, we are going to have both:

$$\frac{M \Rightarrow N}{M \Rightarrow N, x : A} \qquad \frac{M \Rightarrow N}{M \Rightarrow N, xRy}$$

Even the axioms have two forms: $x : p \Rightarrow x : p$ and $xRy \Rightarrow xRy$. As for the logical rules, these are just the standard rules, adapted to include formulas of

[6] This is not to disparage the work done in the area [6, 8, 17, 18, 25]. However, there is no doubt that the work in this area has been driven by a small number of researchers.

the form $x : A$, as well as relational statements xRy So for example we will have:

$$\frac{M \Rightarrow N, x : A \quad M \Rightarrow N, x : B}{M \Rightarrow N, x : A \wedge B}$$

In order to obtain the calculus for the system K, we of course need to introduce the modal rules. These rules reproduce at the proof-theoretical level the forcing relation of Kripke semantics, i.e. $i \vDash \Box A$ if, and only if, $(\forall j)(iRj \supset j \vDash A)$. More precisely the right-left direction of the above equivalence is rendered by the rule:

$$\frac{xRy, M \Rightarrow N, y : A}{M \Rightarrow N, x : \Box A}\ \Box R$$

where in the premise the y does not appear in M nor in N. The left–right direction is rendered by the rule:

$$\frac{xRy, y : A, M \Rightarrow N}{x : \Box A, xRy, M \Rightarrow N}\ \Box L$$

Here is an example derivation in the system.

$$\dfrac{\dfrac{\dfrac{y : A \Rightarrow y : A}{xRy, x : \Box A \Rightarrow y : A}\ \Box L}{xRy, x : \Box A, x : \Box B \Rightarrow y : A}\ \mathit{weaken} \qquad \dfrac{\dfrac{y : B \Rightarrow y : B}{xRy, x : \Box B, \Rightarrow y : B}\ \Box L}{xRy, x : \Box A, x : \Box B \Rightarrow y : B}\ \mathit{weaken}}{\dfrac{\dfrac{xRy, x : \Box A, x : \Box B \Rightarrow y : A \wedge B}{x : \Box A, x : \Box B \Rightarrow x : \Box(A \wedge B)}\ \Box R}{x : \Box A \wedge \Box B \Rightarrow x : \Box(A \wedge B)}\ \wedge L}\ \wedge R$$

This derives the labelled sequent analogue of our sequent $\Box A \wedge \Box B \Rightarrow \Box(A \wedge B)$, for which we now must add labels in order to treat with our new rules. Notice that the derivation is *shorter* than the corresponding display derivation of $\Box A \wedge \Box B \Rightarrow \Box(A \wedge B)$, as nothing here corresponds to the display equivalences which simply change our perspective point in a frame. Here, the labelled sequent

$$xRy, x : \Box A, x : \Box B \Rightarrow y : A \wedge B$$

corresponds to the *two* display sequents

$$\bullet(\Box A \circ \Box B) \Rightarrow A \wedge B \qquad \Box A \circ \Box B \Rightarrow \bullet(A \wedge B)$$

and there is no need for an inference step to move between two sequents as they are collapsed into the one sequent.

The axioms, the structural rules of weakening, contraction and cut, the logical rules $\neg L$, $\neg R$, $\wedge L$ and $\wedge R$, the modal rules $\Box L$ and $\Box R$ form the Labelled Sequent system for the basic normal modal logic K. In order to obtain labelled calculi for other modal systems one adds to the calculus for K, logical rules governing formulas of the form xRy, plus the rules that result from the application of the closure condition on these rules. The closure condition is merely a technical device for proving that the contraction rules are eliminable in all the extensions of the calculus K. However, it has untoward consequences. As has been noticed [13], the presence of the closure condition leads to a lack of modularity. In the calculus K and its extensions the cut-rule is shown to be eliminable, nevertheless the calculi do not satisfy the subformula property.

When we examine the rules for this calculus, it seems clear that we have constructed proof theories which *explicitly* describe Kripke models for modal logic, rather than giving an independent way to reason modally without describing models. One way to see that we have gone beyond what is expressible in the modal language is to see that we may introduce an axiom for which imposes the condition that a frame be *irreflexive* (that no points in the frame access themselves by the accessibility relation). If we add the following sequent as an *axiom*

$$\mathsf{xRx} \Rightarrow$$

This allows us to formulate conditions directly on the *frame* without going through particular relations between formulas. There is no formula (or sequent) that corresponds to the frame being irreflexive, so labelled sequents provide a means to describe frames directly, beyond the resources of the language of propositional modal logic. In this way, we have gone beyond what can be expressed in display sequents, where the extra structure of the sequent gave us merely a way to describe what could already be described in the vocabulary of the boolean and modal connectives.[7] Labelled formulas and relational statements give us resources to state conditions significantly beyond the vocabulary of propositional modal logic.

Another example of this phenomenon is this: we have many different ways to state exactly the same fact on a frame, without a difference in formulas, but only in labels. The axiomatic sequent $\mathsf{A} \Rightarrow \mathsf{A}$ may be encoded as $\mathsf{x} : \mathsf{A} \Rightarrow \mathsf{x} : \mathsf{A}$ or as $\mathsf{y} : \mathsf{A} \Rightarrow \mathsf{y} : \mathsf{A}$ or with any other world variable. There is no difference in the meaning of each of these sequents. In display logic, these are represented as the same sequent. Similarly, a labelled sequent of the form $\mathsf{x} : \mathsf{A} \Rightarrow \mathsf{y} : \mathsf{B}$ is part of the vocabulary of the labelled language, but it need never arise in the derivation of a modal formula, for the only way different world variables can arise (in a well-behaved system, at least) is if some family of accessibility relation facts connects them. Here, we have the general statement that if A holds at x then B holds at y, where there is no connection given between x and y.[8] World variables that are needed in a derivation of a modal claim are those that arise out of the modal rule $\Box\mathsf{R}$ (and $\Diamond\mathsf{L}$) and these are connected, by R, to world variables already in the sequent. The labelled vocabulary also allows us to introduce sequents significantly beyond what we need in modal deduction. So, let us continue our search for a simpler vocabulary for the structure of modal derivations.

[7]Well, more precisely, at the modal connectives $\Box$, $\Diamond$ and their *duals* $\Box^{-1}$ and $\Diamond^{-1}$ which use the *converse* of the accessibility relation in a modal frame. $\bullet A$ in positive position is interpreted as $\Box A$, and in negative position, it is interpreted as $\Diamond^{-1}A$. The equivalence between $\Diamond^{-1}A \Rightarrow B$ and $A \Rightarrow \Box B$ drives the display equivalence between $\bullet A \Rightarrow B$ and $A \Rightarrow \bullet B$. This is still the modal vocabulary: nothing in the vocabulary of the boolean connectives and these modal operators allows us to state the condition that a frame is irreflexive.

[8]Of course, this could be *derived* by weakening a derivation of $\Rightarrow \mathsf{y} : \mathsf{B}$ or of $\mathsf{x} : \mathsf{A} \Rightarrow$, or in other ways, but this is not the issue at hand. The salient fact is that we never *need* to deal with sequents of this form when deriving a claim of the from $\Rightarrow \mathsf{z} : \mathsf{C}$.

4 Tree Hypersequents

The display method is, like display logic, a syntactic method. It does not make any use of semantic parameters beyond the language of formulas. The labelled method, on the contrary, is a semantic method since it imports in its language the whole structure of Kripke semantics in an explicit and significant way. The tree-hypersequent method, which we introduce in this section, is a syntactic method, since it does not deal with variables nor with relational atoms, but nonetheless, it can be understood in semantic terms.

The basic idea of the tree-hypersequent calculi is to reproduce in the framework of the sequent calculus the structure of the tree-frames of Kripke semantics. In order to do this operation, we need to make several steps. The first one consists in looking at a classical sequent as a world of a Kripke tree-frame. But a tree-frame can be composed of n different worlds. Therefore the second move will consist in considering n sequents a time, as in the hypersequent case. In a tree-frame the worlds are not randomly mixed up but they are combined with an accessibility relation to form a tree. The third move will then be the one of introducing in the meta-language of the sequent calculus two new symbols, the slash (/) and the semicolon (;). The slash will represent the accessibility relation in a tree-frame: if we have $M \Rightarrow N/S \Rightarrow T$, this should be read as "the world-sequent $M \Rightarrow N$ is linked by the accessibility relation $\mathcal{R}$ to the world-sequent $S \Rightarrow T$." The semi-colon will serve to reproduce the fact that in a tree-frame of Kripke semantics n different worlds $y_1, \dots, y_n$ can all be related to a world x. If for example we have $M \Rightarrow N/S_1 \Rightarrow T_1; S_2 \Rightarrow T_2$, this should be read as "the world-sequent $M \Rightarrow N$ is linked by the accessibility relation $\mathcal{R}$ to two different world-sequents: the world-sequent $S_1 \Rightarrow T_1$ and the world-sequent $S_2 \Rightarrow T_2$." This is the way to intuitively understand and introduce tree-hypersequents. The inductive definition and interpretation are the following:

Syntactic Notation. We shall use Γ, Δ, ... to denote sequents (SEQ), G, H, ... to denote tree-hypersequents (THS), and $\underline{X}$, $\underline{Y}$, ... to denote finite multisets of tree-hypersequents (MTHS).

Tree-hypersequents and their Interpretation: The notion of tree-hypersequent is inductively defined in the following way:

- if $\Gamma \in$ SEQ, then $\Gamma \in$ THS,
- if $\Gamma \in$ SEQ and $\underline{X} \in$ MTHS, then $\Gamma/\underline{X} \in$ THS.

Given this definition, we have that an example of a tree-hypersequent is an object of the following form:

$$\Delta_1/(\Gamma_2/\Gamma_3);(\Gamma_4/(\Gamma_5/\Gamma_6);\Gamma_7)$$

The intended interpretation of a tree-hypersequent is:

- $(M \Rightarrow N)^\tau := \bigwedge M \supset \bigvee N$
- $(\Gamma/G_1; \cdots ; G_n)^\tau := \Gamma^\tau \vee \Box G_1^\tau \vee \cdots \vee \Box G_n^\tau$

In order to display the rules of the tree-hypersequent calculi, we will use the notation $G[*]$ to refer to a tree-hypersequent G together with one hole $[*]$; metaphorically the hole should be understood as a zoom by means of which we focus attention on a particular point $*$ of G. Such an object becomes a real tree-hypersequent whenever the symbol $*$ is *appropriately* replaced by: (i) a sequent Γ; in this case we will write $\mathsf{G}[\Gamma]$ to denote the tree-hypersequent G together with a specific occurrence of a sequent Γ in it; (ii) two sequents, Γ/Σ, one after another and separated by a slash; in this case we will write $\mathsf{G}[\Gamma/\Sigma]$ to denote the tree-hypersequent G together with a specific occurrence of a sequent Γ immediately followed by a specific occurrence of a sequent Σ; (iii) a tree-hypersequent H; in this case we will write $\mathsf{G}[\mathsf{H}]$ to denote the tree-hypersequent G together with a specific occurrence of a tree-hypersequent H in it.

Tree-hypersequents are more complex objects than simple sequents so that one may wonder how axioms and rules work in this framework. Let us start by the axioms that have the form:

$$\mathsf{G}[p, M \Rightarrow N, p]$$

The idea which is behind the axioms is the following. Consider a tree-hypersequent G. If in G there is an occurrence of a sequent which is a classical axiom, then the whole tree-hypersequent becomes an axiom of the tree-hypersequent calculi.

As for the logical rules for negation and conjunction, these are just the classical ones; they can be applied to any sequent occurring in a tree-hypersequent G by leaving the other sequents untouched. We consider the example of the rule $\wedge\mathsf{L}$:

$$\frac{\mathsf{G}[A, B, M \Rightarrow N]}{\mathsf{G}[A \wedge B, M \Rightarrow N]}\ \wedge\mathsf{L}$$

The rule should be read as follows. Consider a tree-hypersequent G. If at whatever point of this tree-hypersequent G there is an occurrence of the sequent $A, B, M \Rightarrow N$, then we can apply to this sequent the classical rule that introduces the conjunction on the left of a sequent and leave the rest of the tree-hypersequent G unchanged. The same of course holds for the other logical rules $\wedge\mathsf{R}$, $\neg\mathsf{R}$ and $\neg\mathsf{L}$.

Let us pass to the two modal rules. These rules are the following:

$$\frac{\mathsf{G}[M \Rightarrow N/A, S \Rightarrow T]}{\mathsf{G}[\Box A, M \Rightarrow N/S \Rightarrow T]}\ \Box\mathsf{L} \qquad \frac{\mathsf{G}[M \Rightarrow N/ \Rightarrow A; \underline{X}]}{\mathsf{G}[M \Rightarrow N, \Box A/\underline{X}]}\ \Box\mathsf{R}$$

Let us start by analysing the rule $\Box\mathsf{L}$. If we focus on the two sequents displayed in brackets, this rule tells us that we can introduce the formula $\Box A$ at the left of a sequent Γ, if the formula A occurs at the left of a sequent which is linked to the sequent Γ by a slash. In more semantic terms this rule tells us (read bottom-up) that if a formula $\Box A$ is true at the world x, then the formula A is true at every world y such that xRy.

As for the rule $\Box\mathsf{R}$, if we focus again on the tree-hypersequent displayed in the brackets, we have that this rule tells us that we can introduce the formula $\Box A$ at the right side of a sequent Γ, if the formula A occurs at the right side of a sequent which: (i) is linked to the sequent Γ by a slash, (ii) does not contain any other formula than A, (iii) is not followed by any other sequent Δ. In more semantic terms this rule tells us (read bottom-up) that if a formula $\Box A$ is false at a world x, then there exists a world y such that xRy where A is false.

In the tree-hypersequent calculi there is a family of structural rules. These rules can be divided in two groups. In the first group we have the classical structural rules of weakening, contractions and cut that can be applied, just as the logical rules, to any sequent occurring in a tree-hypersequent G by leaving the other sequents untouched. In the second group we have the external structural rules of external weakening and merge. These rules operate on the structure of the tree-hypersequent and are the following:

$$\frac{\mathsf{G}[\Gamma/\underline{\mathsf{X}}]}{\mathsf{G}[\Gamma/\Sigma;\underline{\mathsf{X}}]}\,EW \qquad \frac{\mathsf{G}[\Gamma/(\mathsf{M}\Rightarrow\mathsf{N}/\underline{\mathsf{X}});(\mathsf{P}\Rightarrow\mathsf{Q}/\underline{\mathsf{X}}');\underline{\mathsf{Y}}]}{\mathsf{G}[\Gamma/(\mathsf{M},\mathsf{P}\Rightarrow\mathsf{N},\mathsf{Q}/\underline{\mathsf{X}};\underline{\mathsf{X}}');\underline{\mathsf{Y}}]}\,merge$$

The axioms, the classical and external structural rules, the logical rules $\neg\mathsf{L}$, $\neg\mathsf{R}$, $\wedge\mathsf{L}$ and $\wedge\mathsf{R}$, the modal rules $\Box\mathsf{L}$ and $\Box\mathsf{R}$ form the system K. To illustrate, ere is our derivation of the distribution of conjunction over necessity, which we have seen twice before:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow/\mathsf{A}\Rightarrow\mathsf{A}}{\Box\mathsf{A}\Rightarrow/\Rightarrow\mathsf{A}}\,\Box\mathsf{L}}{\Box\mathsf{A},\Box\mathsf{B}\Rightarrow/\Rightarrow\mathsf{A}}\,weaken \quad \dfrac{\dfrac{\Rightarrow/\mathsf{B}\Rightarrow\mathsf{B}}{\Box\mathsf{B}\Rightarrow/\Rightarrow\mathsf{B}}\,\Box\mathsf{L}}{\Box\mathsf{A},\Box\mathsf{B}\Rightarrow/\Rightarrow\mathsf{B}}\,weaken}{\dfrac{\Box\mathsf{A},\Box\mathsf{B}\Rightarrow/\Rightarrow\mathsf{A}\wedge\mathsf{B}}{\Box\mathsf{A},\Box\mathsf{B}\Rightarrow\Box(\mathsf{A}\wedge\mathsf{B})}\,\Box\mathsf{R}}\,\wedge\mathsf{R}}{\Box\mathsf{A}\wedge\Box\mathsf{B}\Rightarrow\Box(\mathsf{A}\wedge\mathsf{B})}\,\wedge\mathsf{L}$$

Notice that this derivation has exactly the same structure of rules as the Labelled Sequent derivation, but now the sequents have none of the paraphernalia of labels and facts concerning accessibility. The labels are gone. The sequents are purely logical in just the same way that display sequents are logical. However, tree-hypersequents are *global* rather than *local* constraints on frames. There is no perspective shift as found in display rules. The two display sequents

$$\bullet(\Box\mathsf{A}\circ\Box\mathsf{B})\Rightarrow\mathsf{A}\wedge\mathsf{B} \qquad \Box\mathsf{A}\circ\Box\mathsf{B}\Rightarrow\bullet(\mathsf{A}\wedge\mathsf{B})$$

correspond to the one tree-hypersequent

$$\Box\mathsf{A},\Box\mathsf{B}\Rightarrow/\Rightarrow\mathsf{A}\wedge\mathsf{B}$$

which, if we are to think of it in terms of constraints on a Kripke frame, takes a *God's-eye perspective* on the frame and does not choose a local point of evaluation. So, we have a system which combines good features of Display Logic—there are no extraneous labels, and no way to state things beyond the vocabulary of formulas—and Labelled Sequents—there are no Display equivalences giving us many different ways to restate the same fact about frames. The derivation of the distribution of necessity over conjunction is as *short* in tree-hypersequents as it is in labelled sequents, but it is as purely *logical* as it is in display logic.

In order to obtain tree-hypersequent calculi for other modal systems one adds to the calculus K pairs of rules. Each pair is composed by a logical rule, that regulates formulas of the form $\Box\mathsf{A}$ on the left side of the sequent, and a structural rule, that govern the structure of the tree-hypersequent. In the calculus K and its extensions the cut-rule is shown to be eliminable. In some cases, the structure of the tree-hypersequents simplifies radically: in the case of the modal logic $\mathsf{S5}$, we can replace trees by multisets of hypersequents, and a very simple cut-free hypersequent calculus is the result [20]. If we allow as hypersequents richer structures including two *different* accessibility relations, we can model the two-dimensional modal logic of Davies and Humberstone, treating metaphysical and epistemic necessities, linked together with an actuality operator @ [5, 22].

5 Consequences

What consequences do these formal considerations have for our understanding of modal concepts? There are a few ways that different philosophical conclusions can be drawn from these formal frameworks.

LOGICALITY: Introducing two necessities $\Box$ and $\Box'$ satisfying the same rules in a tree hypersequent calculus is enough to show that $\Box A$ and $\Box' A$ are interderivable. This result does not hold in the case of Hilbert axiomatisations or classical sequent presentations for modal logics logics [21]. This fact is due to the tree hypersequent structure: both $\Box$ and $\Box'$ gain their logical properties in terms of the same structure, and it is through this that they can be shown to be interderivable. The derivations are straightforward.

$$\dfrac{\dfrac{\Rightarrow / A \Rightarrow A}{\Box' A \Rightarrow / \Rightarrow A}\,\mathsf{L}\Box'}{\Box' A \Rightarrow \Box A}\,\mathsf{R}\Box \qquad \dfrac{\dfrac{\Rightarrow / A \Rightarrow A}{\Box A \Rightarrow / \Rightarrow A}\,\mathsf{L}\Box}{\Box A \Rightarrow \Box' A}\,\mathsf{R}\Box'$$

This raises the question of whether or not the paraphernalia of the proof theory is the kind of thing to which we may appeal in fixing meaning. Is it? Well, whether we can appeal to some structural item depends on what that structural item *is*, and for that we need to say something more.

Consider the kinds of discourse shifts found in modal reasoning. Given the reading of sequent derivations in terms of assertion and denial [19] and its extension to the case of modal talk [20, 22]. It's clear that we can shift 'locations' when we reason with modal concepts. This is the core idea in all modal different kinds of semantics where we relate $\Box A$ to A. We always consider A *elsewhere*. The same holds in modal deduction. For example, if we assume as a premise $\Diamond(A \vee B)$ and we wish to deduce $\Diamond A \vee \Diamond B$, in natural language we could reason as follows:

> Suppose $\Diamond(A \vee B)$. So, in some circumstance, $A \vee B$. There are two cases: Case (i) A, and Case (ii) B. Take case (i) first. Then in this circumstance, A and so, back where we started, $\Diamond A$, and hence $\Diamond A \vee \Diamond B$. On the other hand, we might have case (ii). There we have B, so back in the original circumstance, $\Diamond B$, and hence, $\Diamond A \vee \Diamond B$. So, in either case, we have $\Diamond A \vee \Diamond B$, which is what we wanted.

The shifts in the discourse are flagged by the markers 'in some circumstance' and 'back in the original circumstance.' From the point of view of the model theory of modal logic, we can treat these expressions as quantifying over items which can be referred to, treated in some ontology and generally be used as the raw materials for metaphysical speculation. The work in proof theory we have seen shows that this need not be the only approach. We can treat these markers as simply separating assertions into different 'zones,' marking that an assertion of A in one zone is not to be taken to clash with a denial of A in another. The natural language reasoning here corresponds tightly to the hypersequent derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow \Diamond A \;/\; A \Rightarrow}\,\mathsf{R}\Diamond}{\Rightarrow \Diamond A, \Diamond B \;/\; A \Rightarrow}\,\mathit{weaken} \qquad \dfrac{\dfrac{B \Rightarrow B}{\Rightarrow \Diamond B \;/\; B \Rightarrow}\,\mathsf{R}\Diamond}{\Rightarrow \Diamond A, \Diamond B \;/\; B \Rightarrow}\,\mathit{weaken}}{\Rightarrow \Diamond A, \Diamond B \;/\; A \vee B \Rightarrow}\,\mathsf{L}\vee}{\Rightarrow \Diamond A \vee \Diamond B \;/\; A \vee B \Rightarrow}\,\mathsf{L}\Diamond}{\Diamond(A \vee B) \Rightarrow \Diamond A \vee \Diamond B}\,\mathsf{R}\vee
$$

The discourse shifts in natural language are modelled in the formal structure by the zones in each tree hypersequent: these can be thought of as keeping track of the 'score' or 'status' at different steps of the deduction. The markers in the discourse, flagging 'in this circumstance' need not be treated as substantial referential items any more than the other markers flagging 'Case (i)' or 'Case (ii)' need to be thought of as referring to anything substantial. The fact that the structure of a tree hypersequent mimics the kinds of dependency relations among zones in a discourse, and that we are reasonably competent in tracking such zones then gives us an answer to the puzzling question of why we can disagree about modal matters (about what is possible and what is necessary) without necessarily thinking that we have to disagree about what we *mean* when we take something to be possible or to be necessary. Here is why this is a puzzle, and why the proof theory for modal logics can provide a solution.

Let's suppose that the meaning of modal statements is to be given in terms some model theory of modal phenomena, for example, that we think that the logic of necessity is the modal logic S4, and that the truth conditions of modal statements are to be given by a Kripke model satisfying the usual S4 conditions. You and I could agree on all of this, without agreeing on the meaning for modal statements. '$\Box$' in your vocabulary could be an S4 necessity and '$\Box'$' in my vocabulary could also be an S4 necessity, both constrained by the rules of S4, and we could nonetheless be talking past one another. This is easy enough to see, since there are modal models in which there are *two* independent S4-accessibility relations, and two different S4 necessities, $\Box$ and $\Box'$. More must be done to make sure that $\Box$ and $\Box'$ agree in meaning. In a modal model it is clear what must be done: we must coordinate in the set of *worlds* to be quantified over in the interpretation of necessity, and the accessibility relation governing the behaviour of the modal operator. But doesn't answer the issue at hand, it is restate what is required of a solution. Unless we can give an independent account of how we have access to these worlds, we have no other way to ensure that we are quantify over the same worlds when we use our modal vocabulary. Finding such an independent account is a difficult task. It may not be insuperable, but it is a task for any semantics that takes possible worlds (or points in a model structure more generally) to be playing an explanatory role.

The same kind of task is not required in a modal proof theory. We have a simple explanation of why you and I might recognise that $\Box$ in your vocabulary coordinates in meaning with $\Box'$ in mine, because we can both appeal to the shared, public nature of the zones in our discourse as we modally reason together. These shifts in discourse are moves that we make together when we modally reason, and agreement on those shifts is enough to coordinate agreement in modal concepts defined in terms of those shifts. It is important to be clear on what this means: we need only that what you take to be a modal shift, I take to be a modal shift too (and *vice versa*), not that we agree on the content of what is at issue when we take these modal shifts. Take an analogy with conditional reasoning. If I agree with you about when we are *supposing* an hypothesis and *discharging* what is supposed, then we can reason using conditional statements using the standard conditional introduction and elimination rules, even though we might well disagree about the truth of conditional statements. Agreement on the structure of condition deductions then shifts disagreement about conditional statements into disagreements about about subderivations and other things. We can agree on how to treat a conditional statement without agreeing on its truth. Similarly, if we can recognise what it is to substitute a singular term into a quantified statement, and when a singular term is free of assumptions on it, then we can use the standard quantifier introduction and elimination rules to coordinate our use of the quantifiers, even though we may well disagree on whether there is some object satisfying some condition. The syntax of the proof theory for modal logics points to a resource we can use to explain our coordination in our modal thought and talk, and this syntax points to a phenomenon (the shifts made by supposing and discharging different 'circumstances') that are just as public as other structural features of our vocabulary. These public features of our talk and thought provide another means to coordinate our concepts.

Given this scope for agreement on modal vocabulary, it must be recognised that matters are not so simple or straightforward as all that. One of the difficulties in our use of modal concepts is that discourse shifts for our modal concepts occur in more than one way. In fact, it looks as if there are many different kind of modal shifts, sharing an underlying structure, but differing in matters of detail and of application. In this last section we will consider just two, *metaphysical* modality and *epistemic* modality, and we will see how tree hypersequents can model these just as well as the more familiar Kripke model theory for two-dimensional modal logic [5, 22].

METAPHYSICAL MODALITY. It is quite plausible that $\mathsf{a} = \mathsf{b}, \Box \mathsf{Fa} \Rightarrow \Box \mathsf{Fb}$ holds, where F is a one-place predicate and a and b are names. If a and b name the same object, and a must be F, then b must be F too, where we conceive of this 'must' as a kind of metaphysical, and non-epistemic necessity—or so it has seemed to many. If we think of this necessity as the limit case of subjunctive alternativeness, the limit case of what could happen were things to turn out differently, then many have thought that identities are necessary in just this sense. Why is this the case? The account suggested by tree-hypersequent proof theory is that we should look at the interaction between identity and the *structure* of sequents, rather than to posit a direct connection between the logic of identity and of necessity. Consider the hypersequent:

$$\mathsf{a} = \mathsf{b} \Rightarrow \ \ / \ \mathsf{Fa} \Rightarrow \mathsf{Fb}$$

To take this to be *valid* is to rule out as incoherent a position in which we assert $\mathsf{a} = \mathsf{b}$ in some context and assert Fa but deny Fb one of its *subjunctive* *alternative* contexts. Is this plausible? If we take subjunctive alternatives to be the kinds of alternatives we consider when *planning* (or *regretting*), then it is very plausible indeed. If I have granted that Hesperus is identical to Phosphorous, then it would be very odd to consider some future (or past) travel plans according to which I am going (or could have gone) to Hesperus but not to Phosphorous. We use our information about identities when planning what we do or consider what could have been.

On the other hand, when we consider alternatives, these are treated as *non-actual* alternatives. I can grant some possibility for consideration while not taking it to be *actual.* If we single out one context as the *actual* one (subscript the sequent arrow of this part of the hypersequent with an @ to mark this off) then the behaviour of the actuality operator tells us that, for example

$$\mathsf{p} \Rightarrow_{@} \;\; / \Rightarrow @\mathsf{p}$$

is a valid sequent. It is incoherent for me to assert p in the *actual* context while deny $@\mathsf{p}$ in some successor context. However, the denial of $\Box(\mathsf{p} \supset @\mathsf{p})$ is completely coherent. There is no problem in denying p in the *actual* context while asserting it in some subjunctive alternative. This is to grant an alternative context in which p is taken to be true but $@\mathsf{p}$ denied—in which $\mathsf{p} \supset @\mathsf{p}$ is denied, and so, $\Box(\mathsf{p} \supset @\mathsf{p})$ may be denied in the actual context. In other words, for metaphysical necessity $\Box$, governed by subjunctive discourse shifts, we have the following two principles:

$$\mathsf{a} = \mathsf{b}, \Box\mathsf{Fa} \Rightarrow \Box\mathsf{Fb} \qquad\qquad \not\Rightarrow \Box(\mathsf{p} \supset @\mathsf{p})$$

EPISTEMIC MODALITY: On the other hand, if we think of a shift not as considering what might be or might have been, given alternative courses of action, but what might well be *for all we know*, then this gives rise to an epistemic modality, which we will denote with a 'K'.[9] Under this reading of the modal operator, it is very plausible that $\Rightarrow \mathsf{K}(\mathsf{p} \supset @\mathsf{p})$ (I can know that if p is the case then it's actually the case), but it is equally plausible that $\mathsf{a} = \mathsf{b}, \mathsf{KFa} \Rightarrow \mathsf{KFb}$ fails to hold. After all, even if Hesperus is Phosphorous, it does not follow that if I know that Hesperus is Venus that I know that Phosphorous is Venus. I may not know that Hesperus is Phosphorous. Epistemic modalities differ from metaphysical modalities. However, we do not need to explain this in terms of different kinds of possible worlds. We can explain this in terms of norms governing different kinds of shifts in discourse. In this case, the kind of context shift salient for the interpretation of the epistemic operator K does not always allow identity claims to cross the barrier. Identity does not, in general, mandate this hypersequent

$$\mathsf{a} = \mathsf{b} \Rightarrow \;\; /\!\!/ \;\; \mathsf{Fa} \Rightarrow \mathsf{Fb}$$

for we can grant that $\mathsf{a} = \mathsf{b}$ holds, while also consider as coherent an *epistemic* alternative, that this claim is mistaken. We may coherently deign to grant Fa but deny Fb, as an epistemic alternative to our original position. There is no *inconsistency* in considering the denial of an identity that we have granted. The discourse position in which $\mathsf{a} = \mathsf{b}$ is asserted at some point, and at an epistemic

[9] Whether this expresses what is knowable or what is a consequence of what is known, it doesn't matter much here.

alternative of that point we grant $\mathsf{F}a$ but deny $\mathsf{F}b$ is completely coherent. This is part

However, if we consider an epistemic alternative, there is a sense in which we consider that alternative *as actual*: instead of considering an alternative course of action (whether future or past), we are considering that our view of the world may be mistaken, or at the very least, we are 'trying out' some other view of the world, even if we do not think that our views could be wrong in any strong sense. In this case, the alternative is considered to be actually the case, and so, the actuality operator @ is treated differently in epistemic alternatives than in subjunctive alternatives. In this case, $\mathsf{K}(p \supset @p)$ is plausibly a tautology in the logic of epistemic modalities:

$$\dfrac{\dfrac{\dfrac{\Rightarrow /\!\!/ \; p \Rightarrow_{@} p}{\Rightarrow /\!\!/ \; p \Rightarrow_{@} @p}\,@\mathsf{R}}{\Rightarrow /\!\!/ \Rightarrow_{@} p \supset @p}\,\supset\mathsf{R}}{\Rightarrow \mathsf{K}(p \supset @p)}\,\mathsf{KR}$$

The rule KR for introducing K on the right considers arbitrary *epistemic* alternatives to the original zone, but all of these are zones-considered-as-actual, allowing for the deduction of $@p$ from p. More details of how the logic works can be found elsewhere [22].[10] It is enough to say that this provides a cut-free hypersequent calculus for the logic of necessity, a priori knowability and actuality from Humberstone and Davies' two-dimensional modal logic [5], which provides a well-understood reading of epistemic and metaphysical necessities and the relationships between them. The results here explain why we can enjoy using this reading without taking the appeal to possible worlds in the model theory of this logic to be doing any explanatory work. That work can be done by the structures in the proof theory, which track zone shifts in discourse.

We have seen a use for our sequent calculi. They expose the structure of modal deduction, and show it involves shifts in discourse. Once we understand this to be the case, we can see the similarities and differences between metaphysical and epistemic modalities to be grounded in this structure. Instead of thinking that Kripke models underlie the parallels between (this kind of) K and $\Box$, which raises the thorny issue of the nature of the points related by these two accessibility relations,[11] we can say, instead, that the underlying similarities are due to the fact the rules for making claims with both K and $\Box$ are governed by contextual shifts in very similar ways. The differences are grounded in the differences in norms governing those shifts, and connecting those shifts to other parts of our conceptual apparatus, such as identity and actuality. The tree-hypersequent proof theory gives us a framework in which the logical behaviour of these operators can be exposed and precisely treated, but also in which also has rich connections to other areas of investigation [22].

[10] As to a discussion of what this might mean, and why it might be that we not only have a capacity to consider both subjunctive (metaphysical) and indicative (epistemic) alternatives, the reader is encouraged to read Mark Lance and Heath White's "Stereoscopic Visions: Persons, Freedom, and Two Spaces of Material Inference" [10].

[11] The problem is not just an issue of determining the difference between metaphysical possible worlds and epistemic scenarios, but in all the descendents thereof. Given the two operators K and $\Box$, multiple nestings are possible. Suppose $xR_{\Box}y$ (so y is a metaphysical alternative of x) and $yR_{\mathsf{K}}z$ (and z is an epistemic alternative of y) and $zR_{\Box}w$ (w is a modal alternative of z). What kind of object is w?

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