Editorial: Special issue on non-classical mathematics

The 20th century witnessed not only incredible advances in the model theory and proof theory of classical logic, but also a corresponding advance of non-classical (or, as they are also called, non-standard) logics—i.e. the logics that either non-trivially extend, or actually compete with, classical logic of the Aristotelian, Stoic and Boolean tradition. Since the beginning of the 20th century, various shortcomings of classical logic motivated the development of numerous families of alternative logical systems. Perhaps the earliest manifestation of a non-classical account of logic was in L. E. J. Brouwer's intuitionistic approach to mathematics in the beginning of the 20th century, later formalized by A. Heyting in the form of *intuitionistic logic* and followed by several schools of constructive mathematics. The first formally defined non-classical logic was the three-valued logic of Łukasiewicz, followed by a host of *many-valued* logics of wildly differing properties, finding applications in epistemology, the philosophy of vagueness or computer science (to name just a few). A criticism of material implication gave birth to *modal logics*, first axiomatized by C. I. Lewis; later the modal apparatus was applied to various other modalities, such as deontic, doxastic or epistemic, and extended to multimodal systems capable of formalizing multi-agent and dynamic aspects of reasoning. The non-discriminating treatment of contradictions in classical logic inspired, in view of various logical paradoxes, the development of *paraconsistent logics*. Considerations about belief revision, default reasoning, abduction and similar reasoning methods motivated the study of non-monotonic consequence relations, or *non-monotonic logics*. Several families of logics of different origins have been recognized as branches of a broader family of *substructural logics*—among them, relevant logics as another reply to the criticism of material implication, linear logics as resource-sensitive calculi, fuzzy logics as the logics of gradual truth and variants of the Lambek calculus, originally formalizing categorial grammar. Furthermore, numerous possible distinctions regarding existence, denotation and quantification structured the field of non-standard *logics of quantification*, including partial and free logics, logics of generalized and branching quantifiers, as well as various intensional and higher-order formalisms.

While some of the aforementioned logics—such as intuitionistic logic or some paraconsistent or higher-order systems—have from the outset been intended to serve as a basis for non-classical mathematical reasoning, most others were originally developed as primarily propositional logics, aimed at capturing only the most fundamental reasoning patterns regarding propositional combinations of basic propositions, perhaps illustrating some philosophical point, or satisfying algebraic constraints of some kind or other. Nevertheless, even most of the latter systems have later been extended to first-order formalisms capable of supporting non-trivial mathematical theories. So, besides intuitionistic and constructive mathematics, which are without doubt the most advanced areas of non-classical mathematics, by the beginning of the 21st century there have been developed dozens of mathematical theories axiomatized in some non-classical logic, and so effectively using some non-classical mode of reasoning for the derivation of their theorems. To give an incomplete list of examples, there are books and papers on intuitionistic and constructive mathematics, inconsistent infinitesimal calculi and naïve set theories, relevant arithmetic, fuzzy arithmetic and fuzzy set theory, etc. with scores of researchers active in these areas.

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Closely related to mathematics axiomatized over non-classical logics, and so part of a broader family, are the fields of mathematics which, while using classical logic for reasoning, are developed within some non-traditional foundational theory (such as Quine's New Foundations, various predicative set theories, Vopěnka's and Hrbáček's non-standard set theories, etc.). Just like mathematical theories over non-classical logics, such theories offer a different perspective on certain mathematical objects—in some of them, for instance, the reals are countable—and enable methods alternative to those of traditional mathematics (e.g. non-standard analysis). Finally, also the metamathematical study of predicate non-classical logics themselves can be viewed as an indispensable prerequisite to, and so a part of, non-classical mathematics.

This special issue collects some works across several branches of non-classical mathematics. The purpose of such a 'cross-section' selection of papers related only by the use of non-classical logic for mathematical reasoning is twofold. First, to promote non-classical mathematics in a broader mathematical community: namely, to demonstrate the utility of non-classical reasoning on particular mathematical problems, to show the richness of results and the breadth of the field, to help both specialists and outsiders assess the merits of different non-classical approaches and to provide samples of current research in the area. Secondly, by gathering papers from different branches of non-classical mathematics in a single issue of a journal, to facilitate the exchange of ideas between the particular branches, which till now have often been disconnected and isolated. It can be observed, though, that different branches of non-classical mathematics, while varying significantly in the underlying principles and formalisms, often encounter similar problems (such as the split of classically equivalent definitions) or even share similar features (such as the non-well-foundedness of the set universe, the non-linearity of reals and so on). Consequently, the solutions adopted in one branch can possibly help to find the right solutions to analogous problems in other branches, despite the fundamental differences between their starting points.

These two intentions have inspired several recent activities in non-classical mathematics, of which this special issue is an integral part. Several of the papers published in this issue were presented at the conference on *Non-Classical Mathematics,* held on 18–22 June 2009 in Hejnice, Czech Republic. This event was followed by a special session on non-classical mathematics at the *Third World Congress on Universal Logic*, which—despite the disruption of air traffic by volcanic ashes that prevented a full half of contributors from participation—took place in Estoril, Portugal, on 22–25 April 2010. Non-classical mathematics also formed a significant part of the conference *Non-Classical Modal and Predicate Logics*, held in Guangzhou, China, on 5–9 December 2011. And needless to say, the research in non-classical mathematics has always been continuously presented by individual researchers at mainstream conferences as well as at specialized workshops organized by groups active in particular branches of non-classical mathematics or logic. Hopefully the interaction between researchers from different fields of non-classical mathematics, promoted by these activities and this special issue, will continue and grow.

As already mentioned, the special issue presents papers pertaining to several different fields of non-classical mathematics. A general pluralistic philosophy of mathematics is described in Graham Priest's paper *Mathematical pluralism*. Intuitionistic and constructive mathematics is represented by four papers: *A first constructive look at the comparison of projections* by Douglas Bridges and Luminiţa Vîţă; *Lipschitz functions in constructive reverse mathematics* by Iris Loeb; *Constructive version of Boolean algebra* by Francesco Ciraulo, Maria Emilia Maietti and Paola Toto; and *A generalised cut characterisation of the fullness axiom in CZF* by Laura Crosilla, Erik Palmgren and Peter Schuster. Many-valued mathematics is the topic of two papers: *Interpreting lattice-valued set theory in fuzzy set theory* by Petr Hájek and Zuzana Haniková and *On equality and natural numbers in Cantor–Lukasiewicz set theory* by Petr Hájek. (With the author's consent, the latter paper was

moved to this special issue from a regular issue of the *Logic Journal of the IGPL*: thanks are due to Dov Gabbay for handling the paper as editor.) The notion of identity in an inconsistent, relevant-flavoured framework is treated in the paper *Identity taken seriously: a non-classical approach* by Chris Mortensen. Finally, Peter Verdée's paper *Strong, universal and provably non-trivial set theory by means of adaptive logic* is, to our knowledge, the first paper on a non-monotonic, in particular inconsistency-adaptive mathematics.

A paper we hoped to be able to include in this special issue, but sadly would never have received, was one by Bob Meyer. As a prominent relevant logician who developed the fundamental theory of relevant arithmetics, he was invited to the *Non-Classical Mathematics* 2009 conference: he managed to send in an (unfinished) abstract for his planned lecture on *Ternary semantics, combinators, K2U* (available in the *Volume of Abstracts* at the conference website, http://www.cs.cas.cz/ncm/ncm2009.pdf), but passed away six weeks before the conference, on 6 May 2009.

LIBOR BĚHOUNEK
Institute of Computer Science,
Academy of Sciences of the Czech Republic,
Pod Vodárenskou věží 2,
182 07 Prague 8, Czech Republic.
Email: behounek@cs.cas.cz

GREG RESTALL
School of Philosophy,
Anthropology and Social Inquiry,
The University of Melbourne,
Parkville 3010, Australia.
Email: restall@unimelb.edu.au

GIOVANNI SAMBIN
Dipartimento di Matematica,
Università degli Studi di Padova,
Via Trieste 63, 35121 Padua, Italy.
Email: sambin@math.unipd.it

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