On t and u, and what they can do

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One way to respond to the paradoxes of self-reference is to steer clear of classical logic, and to embrace truth-value *gaps* (Brady 2006, Field 2008) or truth-value *gluts* (Beall 2009, Priest 1987). This means not only explicitly rejecting classical logic, it also means rejecting any vocabulary — like Boolean negation or the material conditional — which would allow a derivation of triviality (Denyer 1989, Priest 1990). You must be on your guard if you adopt a non-classical logic as a response to the paradoxes.

Here is another pair of items that such a non-classical logician should avoid: the two statements t and u. Admittedly, they are rather special statements: you can think of t as the conjunction of all truths. However, the statement t isn't literally an infinitely long conjunction. Rather, for a particular statement to do the job of t, it is to entail p if and only if p is true. The statement u is similar: you can think of u as the disjunction of all untruths. For a statement to do the job of u, it must be that p entails u if and only if p is untrue.

Since my targets are friends of truth-value *gaps* and truth-value *gluts*, I do not mean to say that u is the negation of t. If p falls into a truth-value *gap*, then neither p nor $\sim p$ is entailed by t, but since p is *untrue*, p entails u. So, something entails u that doesn't entail $\sim t$, namely p.[1] Similarly, if q is in a truth-value *glut*, then both q and $\sim q$ are true (so both are entailed by t) but it neither entail u. In this case, q is something that entails $\sim t$ but not u.[2]

Should we think that there *are* propositions like t and u? If you think of propositions as sets of situations, then t is the *intersection* of all the true propositions, and u is the *union* of all untrue propositions. On the 'possible world' semantics of the non-classical logics favoured by Brady,

[1] I presume here that since $\sim p$ doesn't entail t, neither does $\sim t$ entail p. This form of contraposition is not in question in the logics favoured by Beall, Brady, Field and Priest, though it is not intuitionistically valid.

[2] There is nothing in the definition of t or of u which rules out truth value gaps or gluts. Take a simple four-valued matrix of semantic values t, b, n and f, where t and b are *true* and b and f are *false*. Here, we can take any statement with semantic value b to be t, and anything with semantic value n to be u. This does not close any gaps or eliminate gluts. We merely isolate the strongest true proposition (here that's b) and the weakest untrue proposition (here that's n).

Field, Beall and Priest alike, these count as propositions. They have said nothing that can rule t and u out of court.[3]

Now let's see why t and u are problematic for friends of non-classical treatments of the paradoxes. Our starting point is the *conditional*. It is well known that we ought to avoid Boolean negation if we are to treat the paradoxes with non-classical logic. Russell's paradox bites if we can define a negation which is true if and only if the item negated untrue. It's less well known that we must avoid the material conditional. If we are to avoid *Curry's paradox*, we must not have a conditional which is true if and only if either the antecedent is untrue or the consequent is true (Meyer, Routley and Dunn 1979).

That is no bar to having some kind of conditional while avoiding the paradoxes. Beall, Brady, Field and Priest each take there to be some kind of conditional connective "→" satisfying *modus ponens*, *identity* ($p \to p$ is logically true) and strengthening and weakening properties. They each agree that an antecedent may be *strengthened* (if $p \to q$ is true and p' logically entails p, then $p' \to q$ is true too) and a consequent may be *weakened* (if $p \to q$ is true and q logically entails q', then $p \to q'$ is true too).[4] For Brady and Field and other friends of gaps, the material conditional $\sim p \vee q$ will not work as a conditional, as it does not satisfy *identity* (we don't always have $\sim p \vee p$, when p falls into a gap) and for Beall, Priest and other friends of gluts, the material conditional will not do as a conditional as it does not satisfy *modus ponens* (we don't always get from $\sim p \vee q$ and p to q when p stands in a glut). So, the favoured conditional for each of Beall, Brady, Field and Priest are not *material*. Exactly what properties they *do* endorse beyond *identity*, *modus ponens*, and the other properties mentioned above need not detain us here. Given any conditional like satisfying these conditions, t and u are a cause for concern. They allow us to define *another* conditional, which I will write with a double arrow "⇒" as follows.

$$p \Rightarrow q \quad \text{is} \quad (p \wedge t) \to (q \vee u)$$

Now consider: if p and $p \Rightarrow q$ both hold, what can we say about q? We can reason as follows:

$$\dfrac{\dfrac{\dfrac{p \qquad \overline{\;t\;}}{p \wedge t} \qquad (p \wedge t) \to (q \vee u)}{q \vee u}}{q \qquad \underline{\;u\;}}$$

From p (our assumption) and t (which is given to us as true, hence the line above, indicating that it does not need to be assumed) we derive $p \wedge t$. From the assumption $(p \wedge t) \to (q \vee u)$, by *modus ponens* we derive $q \vee u$. Here, we split into two cases, q and u. The second case is ruled

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[3] This is *unlike* the case where someone might attempt to define a proposition T true at the actual world @ alone. A friend of gluts is well within their rights to say that this is *not* a proposition. If a situation like the actual world could be extended by a situation that is *even more* inconsistent, making true everything true at @ but adding more, then T would be true *there* as well as at @, so {@} need not define a proposition. No such reasoning rules out t or u.

[4] They do not agree on the other logical properties of the conditional, but that won't matter here.

out, since u is untrue, and only the first case q remains. Since $q \vee u$ holds, and we have only included in u things that are *untrue*, the only way the disjunction gets to be true is in virtue of q.

So, the conditional "$\Rightarrow$" is truth preserving. If p and $p \Rightarrow q$ are true, so is q. In other words, when $p \Rightarrow q$ is true, either p is untrue, or q is true. But when does $p \Rightarrow q$ fail to be true?

If p is *untrue*, then p entails u. Identity gives us $u \rightarrow u$. Since p entails u we may strengthen the antecedent to $p \rightarrow u$. Since $p \wedge t$ entails p, strengthening again gives $(p \wedge t) \rightarrow u$. Finally, since u entails $q \vee u$, we may weaken the consequent to get $(p \wedge t) \rightarrow (q \vee u)$. In other words, if p is untrue, $p \Rightarrow q$ is true.

Similarly, if q is true, then t entails q. Identity gives us $t \rightarrow t$. Strengthening the antecedent gives $(p \wedge t) \rightarrow t$, and weakening the consequent once to q and again to $q \vee u$ gives us $(p \wedge t) \rightarrow (q \vee u)$. In other words, if q is true, so is $p \Rightarrow q$.

So, we have shown that if p is untrue, or if q is true, $p \Rightarrow q$ is true. But the previous argument showed that if $p \Rightarrow q$ is true, then p is untrue or q is true. In other words, $p \Rightarrow q$ is true if and only if either p is untrue or q is true.

This does *not* mean that $p \Rightarrow q$ is the material conditional. For although $p \Rightarrow q$ is true if and only if either p is untrue or q is true, that 'if and only if' is contingent. Recall: $p \Rightarrow q$ is defined to be $(p \wedge t) \rightarrow (q \vee u)$, and while t is true and u is not, they are only contingently so. Suppose things had gone differently, and t had not been true. There is no guarantee that if p is untrue in that circumstance, that $p \Rightarrow q$ is true there too, for there is no guarantee in that circumstance that p entails u, for u is the disjunction of things *actually* untrue, not those things untrue *elsewhere*.

Consider a more familiar case. Take classical material conditional, and define $-p$ as $p \supset m$, where m is "I'm a monkey's uncle." Given that I'm not a monkey's uncle, m is false, and $-p$ is true if and only if p is false. However, that is contingent. Were we in a circumstance where m were true, then $-p$ would be true, whatever the status of p. We have $-p$ true if and only if p is not, but only contingently so.

Now we see why t and u bring trouble. The conditional $\Rightarrow$ is enough like a material conditional to bring us Curry's paradox. Since $p \Rightarrow q$ is true if and only if either p is untrue or q is true, the usual reasoning for Curry's paradox applies. The paradox returns.

Perhaps Curry's paradox is not your style. That does not matter. Where a material conditional is, there is a negation. Let us define $\neg p$ as $(p \wedge t) \rightarrow u$. We simply leave out the consequent. Our reasoning with $\Rightarrow$ applies in this case: it tells us that $\neg p$ is true if and only if p is untrue. The law of the excluded middle and the law of non-contradiction hold (but only contingently) with respect to ¬-negation, and so, the *liar* paradox expressed with *this* negation trivialises. The paradox returns.

(In fact, in this case, we didn't need to use t at all. If we defined the 'negation' of p as $p \to u$, the paradoxical reasoning would still work. However, we would not have the result in quite the same generality, as we would not have recovered the material conditional from the intensional conditional '→'.)

So, if you wish to use a non-classical logic as a solution to the paradoxes, you must must avoid t and u, because of what they can do.[5]

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