Paraconsistency Everywhere

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Abstract: Paraconsistent logics are, by definition, *inconsistency tolerant*: In a paraconsistent logic, inconsistencies need not entail everything. However, there is more than one way a body of information can be inconsistent. In this paper I distinguish contradictions from other inconsistencies, and I show that many different logics are in an important sense, "paraconsistent" in virtue of being *inconsistency tolerant* without thereby being *contradiction tolerant*. For example, even though no inconsistencies are tolerated by intuitionistic *propositional* logic, some inconsistencies are tolerated by intuitionistic *predicate* logic. In this way, intuitionistic predicate logic is, in a mild sense, paraconsistent. So too are orthologic and quantum propositional logic, and other formal systems. Given this fact, a widespread view—that traditional paraconsistent logics are especially repugnant because they countenance inconsistencies—is undercut. Many well-understood non-classical logics countenance inconsistencies as well.

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"Paraconsistent" means "beyond the consistent" [3, 15]. Paraconsistent logics tolerate *inconsistencies* in a way that traditional logics do not. In a paraconsistent logic, the inference of *explosion*

$$A, {\sim}A \vdash B$$

is rejected. This may be for any of a number of reasons [16]. For proponents of *relevance* [1, 2] the argument has gone awry when we infer an irrelevant B from the inconsistent premises. Those who argue that inconsistent theories may have some logical content but do not commit us to *everything*, have reason to think that these theories are closed under a relation of paraconsistent logical consequence [12, 18]. Another reason to adopt a paraconsistent logic is more extreme. You may take the *world* to be inconsistent [14], and a *true* theory incorporating this inconsistency must be governed by a paraconsistent logic.

However, not all inconsistencies are straightforward contradictions. As a simple example, consider the set $\{A \vee B, {\sim}A, {\sim}B\}$. It is as inconsistent as can be, yet it contains no contradictory pair of formulas. This set is inconsistent, and it is classically unsatisfiable without containing an explicit contradiction. Of course, we can note that to members of the set, ${\sim}A$ and ${\sim}B$ together entail ${\sim}(A \vee B)$ and *this* is the negation of a formula in this set. Therefore, we might say that the set contains an *implicit* contradiction, without containing an explicit one. The fact that some inconsistent sets are not themselves *explicit* contradictions motivates a closer look at the definition of paraconsistency. Let's specify what it is for a consequence relation to be paraconsistent in the following two ways:

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- A consequence relation $\vdash$ is *contradiction tolerant* if and only if for some formulas A and B, the contradictory set $\{A, {\sim}A\}$ does not entail B; that is, $A, {\sim}A \nvdash B$.

- A consequence relation $\vdash$ is *inconsistency tolerant* if and only if for some inconsistent set X and some formula B we have $X \nvdash B$.

A contradiction tolerant consequence relation is also inconsistency tolerant—as the set $\{A, {\sim}A\}$ is inconsistent. However, it is not at all obvious that the converse holds. Perhaps there are inconsistency tolerant consequence relations which are not contradiction tolerant. Such consequence relations are the focus of the rest of this paper.

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To judge whether or not a relation is inconsistency tolerant we must know when a set is inconsistent. This makes a judgement about inconsistency tolerance depend on a judgement about what could count as an inconsistency. This might appear to make the notion of inconsistency tolerance more problematic than contradiction tolerance, which has the virtue of being much more straightforward to check for. However, this appearance is misleading. While it *seems* straightforward to check for the presence of an an explicit contradiction in a given set, this requires at least *some* judgement. In particular, you must know what counts as a *negation* in the logic in question. For example, the classical modal logic S5 is contradiction tolerant, if we take $\Diamond\neg$ (combining possibility with the Boolean negation of classical logic) to be the negation in question. We might argue over whether or not $\Diamond\neg$ deserves to be called "negation," and this argument is similar to an argument over whether or not a set deserves to be thought of as inconsistent. Determinations of inconsistency tolerance require an account of consistency, and determinations of contradiction tolerance require an account of negation.

There are a number of different possibilities for characterising inconsistency. Let me consider some here.

- *Inconsistency as unsatisfiability*: A set X is *inconsistent* if and only if $X \vdash A$ for each A.[1]

- *Inconsistency as contradiction entailing*: A set X is *inconsistent* if and only if there is some A such that $X \vdash A$ and $X \vdash {\sim}A$.

FACT 1 *These two characterisations of inconsistency agree if the consequence relation $\vdash$ satisfies explosion, transitivity and the structural rule of contraction, and if negation is present.*

PROOF Suppose we have $X \vdash A$ and $X \vdash {\sim}A$. Then transitivity applied to $X \vdash A$ and $A, {\sim}A \vdash B$ gives $X, {\sim}A \vdash B$, and transitivity again, with $X \vdash {\sim}A$ gives $X, X \vdash B$. Contraction, then, supplies $X \vdash B$. B was arbitrary, so X is unsatisfiable: it entails every formula whatsoever. Conversely, if $X \vdash B$ for every B, then for any A, $X \vdash A$ and $X \vdash {\sim}A$ (at least if negation is in the language in question). □

[1] I call this feature *unsatisfiability* because it is suggestive of its reading in model theory. If $X \vdash A$ if and only if every model satisfying X also satisfies A, then $X \vdash A$ *for every* A when X is satisfiable in no model at all. Of course, X might also be satisfied in a model provided that this model satisfies *every* statement whatsoever, but any model such as this is of no use in determining the difference between valid and invalid argument forms, and so it can safely be ignored for these purposes.

In the context of a logic rejecting explosion, inconsistency as unsatisfiability is a much stronger requirement than contradiction entailment. In fact, in many paraconsistent logics (such as first degree entailment [1], or Priest's propositional logic LP [13]) *no* finite set of formulas is unsatisfiable, but many entail contradictions. So, these notions can come apart.

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With these distinctions at hand, we can now begin to consider what some have taken to be a decisive failing of paraconsistent logics. For some opponents of paraconsistency, paraconsistent logics are especially bad because they are inconsistency tolerant: they take as "possible" things which are genuinely impossible [19]. In the semantics for paraconsistent logics, valuations (or worlds, or set-ups or situations or what-have-you) allow inconsistencies to be true, and no sense can be supplied to this notion. The paraconsistentist countenances A and $\sim A$ being true (while at the very same time some other B is not true) but the critic cannot see what it is for A and $\sim A$ to both be true together.

Now, it is not my place to endorse this reasoning [17]. It begs the question against the paraconsistentist, if it is an argument at all and not merely an expression of an inability to understand. My point here is that this objection, if it is any good at all, applies equally to logics which are not paraconsistent in the traditional sense. That is, *inconsistency* tolerance is just as bad as *contradiction* tolerance. For the objection that there is no sense to be made of the joint truth of A and $\sim A$ applies just as well to any other inconsistent set. So, for the rest of this paper I will look at how this objection applies in two cases: intuitionistic predicate logic and orthologic, as these are both inconsistency tolerant logics. To fruitfully continue this discussion we ought to settle on a notion of inconsistency, and for simplicity, I will choose this notion:

- *Inconsistency as classical unsatisfiability*: A set X is *inconsistent* if and only if $X \vdash_K A$ for each A, where $\vdash_K$ is *classical* logical consequence.

This is not problematic, because the opponents of paraconsistency most often endorse classical consequence. We will see that their objections, if sustainable against paraconsistent logics, ought to apply much more generally to other non-classical logics too. But before discussing our two main examples, let us show that the notion of paraconsistency as tolerating *classical* inconsistency is not a completely trivial notion. It does not include *every* non-classical logic.

Non-Example 1 *Intuitionistic propositional logic is not tolerant of classical inconsistency. That is, if* $X \vdash_K A$ *for each* A*, then* $X \vdash_J A$ *for each* A *too, where* $\vdash_K$ *and* $\vdash_J$ *are classical and intuitionistic propositional consequence respectively.*

Proof Suppse $X \vdash_K A$ for each A. Then it follows that $X \vdash_K \perp$ where $\perp$ is some contradiction. By compactness, $X' \vdash_K \perp$ where X' is a finite subset of X. Take B to be the conjunction of X', and then $B \vdash_K \perp$. It follows that $\vdash_K \sim B$. Now if a formula is provable in classical propositional logic, its *double negation* is intuitionistically provable. Therefore $\vdash_J \sim\sim\sim B$. But $\sim\sim\sim B$ is intuitionistically equivalent to $\sim B$, so $\vdash_J \sim B$. As a result, $B \vdash_J \perp$ and hence $X' \vdash_J \perp$ and $X \vdash_J \perp$. That is, X is intuitionistically unsatisfiable. $\square$

So, even though intuitionistic propositional logic is strictly weaker than classical propositional logic, this weakness does not apply when it comes to proving *inconsistency*. If $X \vdash_K \perp$ then $X \vdash_J \perp$ too. However, this does not apply in the case of *predicate* logic, as we will see.

Example 1 *Intuitionistic predicate logic is tolerant of some classical inconsistencies. That is, there are sets* X *where* $X \vdash_K A$ *for each* A*, but* $X \nvdash_J A$ *for each* A *too, where* $\vdash_K$ *and* $\vdash_J$ *are now classical and intuitionistic predicate consequence respectively.*

Proof In order to give a concrete example, I will present the Kripke semantics for intuitionistic propositional logic. I will attempt to do with as little technicality as possible. Introductions for intuitionistic predicate logic are available elsewhere [6, 7, 8, 9]. For us, an interpretation for the language of intuitionistic predicate logic will consist of a domain C of *constructions*, partially ordered by a relation $\sqsubseteq$ of *inclusion*, such that for each $c \in C$, D_c is the *domain* of objects constructed by C. If $c \sqsubseteq c'$ (c' is a *stronger* construction than c) then we must have $D_c \subseteq D'_c$: anything constructed by c is also constructed by c'. A infinite sequence α assigning an element of D_c for each variable in the language is said to be an *assignment fit for* c. As is customary, we are interested in varying assignments one variable at a time. In our case, $\alpha(x{:=}d)$ is the assignment which agrees with α about the value of variable except for x, to which this new assignment gives the value d. The final element in an interpretation is the relation $\Vdash$ of *forcing* (or *constructing*, or *proving*) between a construction together with an assignment fit for that construction, and a formula (possibly containing free variables). So, an interpretation relation is a quadruple $\langle C, \sqsubseteq, D, \Vdash \rangle$. The assignment relation must satisfy these inductive clauses.

- $c, \alpha \Vdash A \wedge B$ if and only if $c \Vdash A$ and $c \Vdash B$.
- $c, \alpha \Vdash A \vee B$ if and only if $c \Vdash A$ or $c \Vdash B$
- $c, \alpha \Vdash A \supset B$ if and only if for any $c', \alpha \sqsupseteq c$, if $c', \alpha \Vdash A$ then $c', \alpha \Vdash B$.
- $c, \alpha \Vdash {\sim}A$ if and only if for any $c' \sqsupseteq c$, $c', \alpha \nVdash A$.
- $c, \alpha \Vdash \exists x A$ if and only if for some $d \in D_c$, $c, \alpha(x{:=}d) \Vdash A$.
- $c, \alpha \Vdash \forall x A$ if and only if for any $c' \sqsupseteq c$ and any $d \in D'_c$, $c', \alpha(x{:=}d) \Vdash A$.

An entailment $X \vdash A$ holds according to a particular interpretation $\langle C, \sqsubseteq, D, \Vdash \rangle$ if for every $c \in C$ and every α appropriate for c, if $c, \alpha \Vdash B$ for every $B \in X$ then $c, \alpha \Vdash A$. An entailment $X \vdash A$ holds in intuitionistic predicate logic if and only if it holds in every interpretation.

I will not tarry to discuss the significance of these clauses here: suffice to say that they are well motivated by the Brouwer, Heyting, Kolmogorov (BHK) interpretation of constructions, and the resulting logic is weaker than classical logic. However, to my knowledge, no-one has claimed that the logic is so weak as to interpret *impossibilities* which cannot be understood. The constructive account of the connectives makes sense, given constructive motivations. However, it is not difficult find classical inconsistencies tolerated in models for intuitionistic predicate logic.

Here a simple example of an inconsistency tolerance. We will examine an interpretation verifying that

$$\sim\forall x(Fx \vee {\sim}Fx) \nvdash B$$

The interpretation is straightforward. The set C of constructions is the infinite set $\{c_0, c_1, c_2, \ldots\}$, ordered with $c_i \sqsubseteq c_j$ if and only if $i \leq j$. Each construction has the same domain $D_{c_j} = \{0, 1, 2, \ldots\}$ at each construction. Finally, let's set $F(i)$ true at c_j if and only if $i \leq j$. (More precisely, $c_j, \alpha \Vdash Fx$ if and only if $\alpha(x) = i$ and $i \leq j$.) This means that at each stage c_j, F is true of the objects 0 up to j but not true of $j+1$, $j+2$, and the rest. So, for every point c_i, there is an object $i+1$ such that $c_i \nVdash F(i+1)$ but $c_i \nVdash {\sim}F(i+1)$. So, for each construction, $c_i \nVdash \forall x(Fx \vee {\sim}Fx)$. So, *nowhere* in the model is $\forall x(Fx \vee {\sim}Fx)$ true, and it follows that ${\sim}\forall x(Fx \vee {\sim}Fx)$ is true *everywhere*. But $\forall x(Fx \vee {\sim}Fx)$ is a classical *tautology*, and its negation ${\sim}\forall x(Fx \vee {\sim}Fx)$ is a classical *inconsistency*. Yet, we have found an interpretation in which it is true. Intuitionistic predicate logic tolerates this inconsistency. □

Models like this are of independent interest. The *smooth worlds* of constructive infinitesimal analysis rely essentially on these strong counterexamples of the law of the excluded middle [5]. It is essential to this program of analysis that classical inconsistencies like these be tolerated. (In fact, they are not only tolerated: they are true in the *intended models*.)

For someone committed to classical consequence, thinking that any *possibility* is closed under classical predicate consequence, the smooth worlds of intuitionistic analysis are genuinely *impossible*. They are *just* as impossible as the impossible worlds of the paraconsistent logician. They do not include outright contradictions, but they do include propositions which *cannot* be true, and are no more palatable than the inconsistencies of more traditional paraconsistent logics. If paraconsistent logics are to be rejected, then so intuitionistic predicate logic ought to be rejected alongside them.

Example 2 *Both lattice logic with orthonegation (or simply, orthologic) and quantum logic, which extends lattice logic with the orthomodular law (*$A \wedge ({\sim}A \vee (A \wedge B)) \vdash B$*) tolerate classical inconsistencies.*

Proof Lattice logic is a straightforward account of conjunction and disjunction which avoids the inference of *distribution*: $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$. In other respects, conjunction and disjunction behave normally. Conjunction is the greatest lower bound (with respect to the ordering of entailment) and disjunction is the least upper bound (on that same ordering). The most orthodox way to extend lattice logic with negation is to add an *orthonegation*. The resulting logic we will call *orthologic*. An operator ${\sim}$ is an *orthonegation* in a lattice logic when it satisfies the double negation rules

$$A \vdash {\sim\sim}A \qquad {\sim\sim}A \vdash A$$

and the *bound* rules

$$A \wedge {\sim}A \vdash B \qquad A \vdash B \vee {\sim}B$$

These are the most orthodox negation rules imaginable. Were we to add them to the logic of *distributive* lattices, the result would be classical propositional logic. The context of general lattices, however, provides more leeway. Let's consider a simple non-distributive lattice model for orthologic. The following diagram is a *Hasse* diagram for a six-element lattice. The lines in the digram represent the ordering of entailment: $\bot$ is the lowest element in the order. Next come a, b, c, d which are pairwise incomparable. The greatest element in the order is $\top$. Conjunction is defined as greatest lower bound, and disjunction as least upper bound. So, the conjunction of any two different elements from a, b, c, d will be $\bot$ and

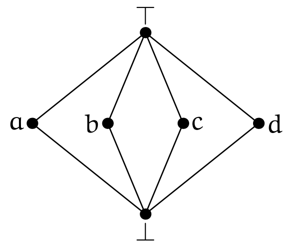

Figure 1: a non-distributive lattice

their disjunction will be $\top$. This lattice is not distributive, because $a \wedge (b \vee c)$ is the element $a \wedge \top$ which is a, while $(a \wedge b) \vee (a \wedge c)$ is $\bot \vee \bot$, which is $\bot$, and a does not entail $\bot$ because a comes strictly higher than $\bot$ in the ordering.

We can make this a model for an orthonegation by choosing the interpretation for $\sim$ carefully. We must take $\sim\bot = \top$ and $\sim\top = \bot$. The negations of a, b, c, d must also be values from a, b, c, d. The negation of a may be any from b, c, d (but it cannot be a, for the bound laws must be satisfied). Once we make the choice, the negation of this element must be a. So without loss of generality, take $\sim a$ to be c. Then the other negations are fixed: $\sim b$ must be d, for we must have $\sim\sim b = b$, and this rules out a or c for $\sim\sim b$. So, $\sim d$ must be b. (It follows that there are exactly three orthonegations on this lattice, corresponding to the three choices possible for $\sim a$.)

The *orthomodular law* $A \wedge (\sim A \vee (A \wedge B)) \vdash B$ (which holds in all lattices of subspaces of Hilbert spaces—which arise in the interpretations of *quantum logic*) holds on this lattice. So this lattice is a model of quantum logic too.

This lattice gives us the following counterexample, showing orthologic and quantum logic are both tolerant of classical inconsistency.

$$A \wedge (B \vee C) \wedge \sim((A \wedge B) \vee (A \wedge C)) \nvdash \bot$$

As discussed before, $a \wedge (b \vee c)$ takes the value a, while $(a \wedge b) \vee (a \wedge c)$ takes $\bot$, which means that $\sim((a \wedge b) \vee (a \wedge c))$ is $\top$. So, $a \wedge (b \vee c) \wedge \sim((a \wedge b) \vee (a \wedge c))$ is a, and a does not entail $\bot$ in this lattice.

I will end this discussion by recasting the counterexample in a *frame* model for quantum logic. These models (due to Goldblatt [10]) stand to orthologic and quantum logic as Kripke frames stand to intuitionstic logic.[2] A *compatibility frame*, for our purposes here, will be a nonempty set P of points, together with a symmetric and reflexive binary relation C to model negation. Conjunction and negation are modelled on a compatibility frame as you would expect.

- $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$
- $x \Vdash \sim A$ iff for each y where xCy, $y \nVdash A$

A conjunction is true at a point just when the conjuncts are true there. A negation is true at a point just when its negand is not true at any *compatible* points. Now,

[2] Bell gives a philosophical analysis of Goldblatt's semantics for orthologic, in which the two-place compatibility relation is interpreted as *proximity* [4].

this is not enough to model orthologic. For one thing, we have no guarantee that the double negation laws hold. For another, we have not said how we are to model disjunction. The naive interpretation, setting $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$ will do us no good, as it will validate distribution. Thankfully we can solve both problems in one go, as Goldblatt noticed. I will explain how by way of an example compatibility frame, with four points, $\{0, 1, 2, 3\}$, such that each point is compatible with all points other than its *opposite* (found by adding 2, modulo 4). In a diagram we can present C by addows, to get this: Now, consider a proposition true at 0 only—

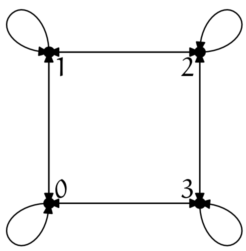

Figure 2: a compatibility frame

modelling propositions as sets of points the proposition is $\{0\}$. Consider where its negation $\sim\{0\}$ might be true. It is not true at 0, as 0 is compatible with itself. It is not true at 1 as 1 is compatible with 0. Neither is it true at 3, since 3 is compatible with 0. However, it is true at 2, since 2 is not compatible with 0. So, $\sim\{0\}$ is $\{2\}$—it is true at 2 only. The same reasoning shows that $\sim\{1\}$ is $\{3\}$, $\sim\{2\}$ is $\{0\}$ and $\sim\{3\}$ is $\{1\}$. Now consider $\sim\{0, 1\}$, the negation of a proposition true at both 0 and 1. This cannot be true at either 0 or 1 (as $0C0$ and $1C1$) but neither is it true at 2 or 3, for $2C1$ and $3C0$. So, $\sim\{0, 1\}$ is $\{\}$, the empty set. But this is the case for any proposition true at two or more points. For any point in our model is compatible with every point except one. It will always manage to be compatible with some member of a set with two or more elements. So the negation of every set with two or more elements is $\{\}$. As a result, the double negation laws fail with these propositions: as an example, note that $\sim\sim\{0, 1\} = \sim\{\} = \{0, 1, 2, 3\}$.

It is not difficult to show that for every proposition X on a compatibility frame, $X \subseteq \sim\sim X$ (symmetry of C does the work here). Propositions X for which $\sim\sim X = X$ are called *closed*. The closed propositions on our example frame are the *empty* proposition $\{\}$, each one element proposition—$\{0\}$, $\{1\}$, $\{2\}$ and $\{3\}$—and the *full* proposition $\{0, 1, 2, 3\}$. If we demand that sentences be interpreted on a compatibility frame only at *closed* propositions, then $\sim$ is an orthonegation.

It remains to define disjunction. It could be done indirectly, taking $A \vee B$ to be defined as $\sim(\sim A \wedge \sim B)$. Or we could define it directly in the following way:

- $x \Vdash A \vee B$ iff $x \in \sim\sim(\llbracket A \rrbracket \cup \llbracket B \rrbracket)$

That means that $A \vee B$ is true at x if x is a member of the *closure* of the set of points where A or B are true. So, the disjunction of $\{0\}$ and $\{1\}$ in our frame the closure of $\{0, 1\}$ which is the entire set $\{0, 1, 2, 3\}$.

Now note that in our frame we have exactly six closed propositions. This lattice of proposition is isomorphic to the six element lattice shown in Figure 1. An isomorphism maps $\{\}$ to $\bot$, $\{0, 1, 2, 3\}$ to $\top$, $\{0\}$ to a, $\{1\}$ to b, $\{2\}$ to c and $\{3\}$ to

d. Each of the logical connectives (conjunction, disjunction and negation) are preserved by this isomorphism. The frame provides a concrete model of the lattice we have already seen.

This frame also provides another way to view the classical inconstency tolerated in models of orthologic and quantum logic. In this frame no point allows a contradiction—the reflexivity of the compatibility relation sees to that—but the classical inconsistency $A \wedge (B \vee C) \wedge \sim((A \wedge B) \vee (A \wedge C))$ is tolerated. In the case where A, B and C are true at 0, 1 and 2 respectively, $A \wedge (B \vee C)$ is true at 0, because A is true at 0 and $B \vee C$ is true everywhere. However, $(A \wedge B) \vee (A \wedge C)$ is true nowhere, so its negation $\sim((A \wedge B) \vee (A \wedge C))$ is true everywhere. Why is this classical inconsistency tolerated here? It is not purely because negation is interpreted non-classically. Negation is as classical as one can hope for in a non-distributive lattice. The classical inconsistency is tolerated because of the interpretation of disjunction. $B \vee C$ is true at more than the places where either B is true or C is true. It is true everwhere. This allows $A \wedge (B \vee C)$ to be true somewhere, despite the fact that $A \wedge B$ and $A \wedge C$ are true nowhere. This allows the impossible to happen: not simply that $A \wedge (B \vee C)$ is true and $(A \wedge B) \vee (A \wedge C)$ isn't true: that would not be enough for inconsistency tolerance—recall the example of intuitionistic propositional logic which is not inconsistency tolerant at all. Some classical inferences fail, such as $\sim\sim A \vdash A$. Kripke frames may have points where $\sim\sim A$ is true and A is not. This is not enough for inconsistency tolerance, for we do not yet have a classical inconsistency true at these points. Similarly, the presence of $A \wedge (B \vee C)$ and the absence of $(A \wedge B) \vee (A \wedge C)$ is not enough to show inconsistency tolerance. What we need, and what we have here, is the presence of $A \wedge (B \vee c)$ and the *presence* of the negation $\sim((A \wedge B) \vee (A \wedge C))$. This provides us with a classical inconsistency, an example of something which *cannot happen* according to classical logic, but which is allowed in models of orthologic and quantum logic. $\Box$

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These examples have brought to light a general phenomenon of which the example of paraconsistent logics is just a single species. Many different non-classical logics (but not all of them) tolerate classical inconsistencies. If this toleration is taken to be a failing of paraconsistent logics, then the same must apply to intuitionistic predicate logic, orthologic and quantum logic. If, on the other hand, we can make sense of inconsistencies in these cases, the fact that inconsistencies are tolerated in paraconsistent logics is not a failing. Rather, it shows that these logics are in good company.

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