PROOF THEORY & PHILOSOPHY

Greg Restall

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This manuscript has been a very long time coming. Many people have contributed to my understanding of these topics over the last fifteen years. My debts are plentiful.

[Acknowledgements of specific people to come.]

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WHERE TO BEGIN

INTRODUCTION

I should like to outline an image which is connected with the most profound intuitions which I always experience in the face of logistic. That image will perhaps shed more light on the true background of that discipline, at least in my case, than all discursive description could. Now, whenever I work even on the least significant logistic problem, for instance, when I search for the shortest axiom of the implicational propositional calculus I always have the impression that I am facing a powerful, most coherent and most resistant structure. I sense that structure as if it were a concrete, tangible object, made of the hardest metal, a hundred times stronger than steel and concrete. I cannot change anything in it; I do not create anything of my own will, but by strenuous work I discover in it ever new details and arrive at unshakable and eternal truths. Where is and what is that ideal structure? A believer would say that it is in God and is His thought.
— Jan Łukasiewicz

This is a draft of a monograph on proof theory and philosophy. The focus will be a detailed examination of the different ways to understand *proof*, and how understanding the norms governing logical vocabulary can give us insight into questions in the philosophy of language, epistemology and metaphysics. Along the way, we will also take a few glances around to the other side of logical consequence, the kinds of *counterexamples* to be found when an deduction fails to be valid.

The book is designed to serve a number of different purposes, and it can be used in a number of different ways. In writing the book I have two distinct aims in mind.

GENTLY INTRODUCING KEY IDEAS IN PROOF THEORY FOR PHILOSOPHERS: There are a number of very good books that introduce proof theory: for example, Bostock's *Intermediate Logic* [18], Tennant's *Natural Logic* [114], Troelstra and Schwichtenberg's *Basic Proof Theory* [117], and von Plato and Negri's *Structural Proof Theory* [78] are all excellent books, with their own virtues. However, they all introduce the core ideas of proof theory in what can only be described as a rather complicated fashion. The core *technical* results of proof theory (normalisation for natural deduction and cut elimination for sequent systems) are *relatively* simple ideas at their heart, but the expositions of these ideas in the available literature are quite difficult and detailed. This is through no fault of the existing literature. It is due to a choice. In each book, a proof system for the whole of classical or intuitionistic logic is introduced, and then, formal properties are demonstrated about such a system. Each proof system has different rules for each of the connectives, and this makes the proof-theoretical results such as normalisation and cut elimination case-ridden and lengthy. (The standard techniques are complicated inductions with different cases for each connective: the more connectives and rules, the more cases.)

In this book, the exposition will be rather different. Instead of taking a proof system as given and proving results about *it*, we will first look at the core ideas (normalisation for natural deduction, and cut elimination for sequent systems) and work with them in their simplest and purest manifestation. In Section 1.3 we will see a two-page normalisation proof. In Section 2.2 we will see a two-page cut-elimination proof. In each case, the aim is to understand the key concepts behind the central results. Then, we show how these results can be generalised to a much more abstract setting, in which they can be applied to a wide range of logical systems, and once we have established these general results, we apply them to specific systems of interest, including first order predicate logic, propositional modal and temporal logics, and quantified modal logics.

EXPLORING THE CONNECTIONS BETWEEN PROOF THEORY AND PHILOSOPHY: The central part of the book (Chapters 4 to 6) answer a central question in philosophical proof theory: When do inference rules define a logical concept? The first part of the book (Chapters 1 to 3) introduces the tools and techniques needed to both *understand* and to *address* the question. The central part of the book formulates the problem and offers a distinctive solution to it. A very particular kind of inference rule (a rule we will describe as a *defining rule*) defines a concept satisfying some very natural conditions—and there are good reasons to think of concepts satisfying these conditions as properly *logical* concepts. Then the remainder of the book (from Chapter 7) explores consequences and applications of these ideas for particular issues in logic, language, epistemology and metaphysics. Alone the way, we will explore the connections between proof theories and theories of meaning. What does this account of proof tell us about how we might *apply* the formal work of logical theorising? All accounts of meaning have something to say about the role of inference. For some, it is what things *mean* that tells you what inferences are appropriate. For others, it is what inferences are appropriate that helps constitute what particular words *mean*. For everyone, there is an intimate connection between inference and semantics.

The precise defintion is spelled out, along with its consequences, in Chapter 6.

I have in mind the distinction between *representationalist* and *inferentialist* theories of meaning. For a polemical and provocative account of the distinction, see Robert Brandom's *Articulating Reasons* [19].

The book includes marginal notes that expand on and comment on the central text. Feel free to read or ignore them as you wish, and to add your own comments. Each chapter (other than this one) contains definitions, examples, theorems, lemmas, and proofs. Each of these (other than the proofs) are numbered consecutively, first with the chapter number, and then with the number of the item within the chapter. Proofs end with a little box at the right margin, like this: ■

The manuscript is divided into three parts, each of which is divided into chapters. The first part, *Tools*, covers the basic concepts, arguments and results which we will use throughout the book. These chapters can be used as a gentle introduction to proof theory for anyone who is interested in the field, perhaps supplemented by (or supplementing) one or more of the texts mentioned earlier in this chapter. The second part, *The Core Argument*, introduces Prior's puzzle concerning inference rules and definitions, and presents and defends a distinct answer to that question. The answer takes the form of an *argument*, to the effect that a particular kind of rule—what I call a *defining rule*—can be used to introduce a logical concept into a discourse, and shows that this concept in an important sense both *free to add*, and *sharply delineated*. The third part, *Insights* then draws out the consequences of this argument to different kinds of logical concepts (the connectives, quantifiers, identity, and modal operators) and for different issues in the philosophy of language, epistemology, metaphysics and the philosophy of mathematics.

A slogan: A logical concept is one that can be introduced by means of a *defining rule*.

In addition to these three major parts, the book contains a small introduction designed to set the scene (this chapter) and a coda, which points forward to issues to be explored in the future.

Some chapters in the *Tools* section contain exercises to complete. Logic is never learned without hard work, so if you want to *learn* the material, work through the exercises: especially the basic and intermediate exercises, which should be taken as a guide to mastery of the techniques we discuss. The advanced exercises are more difficult, and should be dipped into as desired, in order to truly gain expertise in these tools and techniques. The *project* questions are examples of current research topics.

The book has an accompanying website: `http://consequently.org/writing/ptp`. From here you can look for an updated version of the book, leave comments, read the comments others have left, check for solutions to exercises and supply your own. Please visit the website and give your feedback. Visitors to the website have already helped me make this volume much better than it would have been were it written in isolation. It is a delight to work on logic within such a community, spread near and far.

MOTIVATION

Why? My first and overriding reason to be interested in proof theory is the beauty and simplicity of the subject. It is one of the central strands of the discipline of logic, along with its partner, model theory. Since the flowering of the field with the work of Gentzen, many beautiful definitions, techniques and results are to be found in this field, and they deserve a wider audience. In this book I aim to provide an introduction to proof theory that allows the reader with only a minimal background in logic to start with the flavour of the central results, and then understand techniques in their own right.

It is one thing to be interested in proof theory in its own right, or as a part of a broader interest in logic. It's another thing entirely to think that proof theory has a role in philosophy. Why would a *philosopher* be interested in the theory of proofs? Here are just three examples of concerns in philosophy where proof theory finds a place.

EXAMPLE 1: MEANING. Suppose you want to know when someone is using "or" in the same sense that you do. When does "or" in their vocabulary have the same significance as "or" in yours? One answer could be given in terms of *truth-conditions*. The significance of "or" can be given in a rule like this one:

Perhaps you've heard of the difference between 'inclusive' and 'exclusive' disjunction. And maybe you're worried that 'or' can be used in many ways, each meaning something different.

$$\ulcorner p \text{ or } q \urcorner \text{ is true if and only if } p \text{ is true or } q \text{ is true.}$$

Perhaps you have seen this information presented in a truth-table.

p	q	p or q
0	0	0
0	1	1
1	0	1
1	1	1

Clearly, this table can be used to distinguish between some uses of disjunctive vocabulary from others. We can use it to rule out *exclusive* disjunction. If we take ⌜p or q⌝ to be *false* when we take p and q to be both true, then we are using "or" in a manner that is at odds with the truth table.

However, what can we say of someone who is ignorant of the truth or falsity of p and of q? What does the truth table tell us about ⌜p or q⌝ in that case? It seems that the application of the truth table to our *practice* is less-than-straightforward.

It is for reasons like this that people have considered an alternate explanation of a logical connective such as "or." Perhaps we can say that someone is using "or" in the way that you do if you are disposed to make the following deductions to reason *to* a disjunction

$$\frac{p}{p \text{ or } q} \qquad \frac{q}{p \text{ or } q}$$

and to reason *from* a disjunction

$$\frac{p \text{ or } q \quad \begin{matrix}[p]\\ \vdots \\ r\end{matrix} \quad \begin{matrix}[q]\\ \vdots \\ r\end{matrix}}{r}$$

That is, you are prepared to infer *to* a disjunction on the basis of either disjunct; and you are prepared to reason by cases *from* a disjunction. Is there any more you need to do to fix the use of "or"? That is, if you and I both use "or" in a manner consonant with these rules, then is there any way that our usages can differ with respect to *meaning*?

Clearly, this is not the end of the story. Any proponent of a *proof*-first explanation of the meaning of a word such as "or" will need to say something about what it is to accept an inference rule, and what sorts of inference rules suffice to define a concept such as disjunction (or negation, or universal quantification, and so on). When does a definition work? What are the sorts of things that can be defined using inference rules? What are the sorts of rules that may be used to define these concepts? We will consider these issues in Chapter 6.

EXAMPLE 2: GENERALITY. It is a commonplace that it is impossible or very difficult to *prove* a nonexistence claim. After all, if there is *no* object with property F, then *every* object fails to have property F. How can we demonstrate that every object in the entire universe has some property? Surely we cannot survey each object in the universe one-by-one. Furthermore, even if we come to believe that object a has property F for each object a that happens to exist, it does not follow that we ought to believe that *every* object has that property. The universal judgement tells us more than the truth of each particular instance of that judgement, for given all of the objects $a_1, a_2, \ldots$, it certainly seems *possible* that a_1 has property F, that a_2 has property F and so on, without *everything* having property F since it seems possible that there might be some *new* object which does not *actually* exist. If you care to talk of 'facts' then we can express the matter by saying that the fact that everything is F cannot amount to just the fact that a_1 is F and the fact that a_2 is F, etc., it must also include the fact that $a_1, a_2, \ldots$ are all of the objects. There seems to be some irreducible *universality* in universal judgements.

If this was all that we could say about universality, then it would be very difficult to come to universal conclusions. However, we seem to manage to derive universal conclusions regularly. Consider mathematics: it is not difficult to prove that *every* whole number is either even or odd. We can do this without examining every number individually. Just how do we do this?

It is a fact that we *do* accomplish this, for we are able to come to universal conclusions as a matter of course. In the course of this book we will see how such a thing is possible. Our facility at reasoning with *quantifiers*, such as 'for every' and 'for some,' is intimately tied up with the structures of the claims we can make, and how the formation of judgements from *names* and *predicates* gives us a foothold which may be exploited in reasoning. When we understand the nature of proofs involving quantifiers, this will give us insight into how we can gain *general* information about our world.

It is one thing to know that $2 + 3 = 3 + 2$. It is quite another to conclude that for *every* pair of natural numbers n and m that $n+m = m+n$. Yet we do this sort of thing quite regularly.

EXAMPLE 3: MODALITY. A third example is similar. Philosophical discussion is full of talk of *possibility* and *necessity*. What is the significance of this talk? What is its logical structure? One way to give an account of the logical structure of possibility and necessity talk is to analyse it in terms of possible worlds. To say that it is possible that Australia win the World Cup is to say that there is some possible world in which Australia wins the World Cup. Talk of possible worlds helps clarify the logical structure of possibility and necessity. It is possible that either Australia or New Zealand win the World Cup only if there's a possible world in which either Australia or New Zealand win the World Cup. In other words, either there's a possible world in which Australia wins, or a possible world in which New Zealand wins, and hence, it is either possible that Australia wins the World Cup or that New Zealand wins. We have reasoned from the possibility of a disjunction to the disjunction of the corresponding possibilities. Such an inference seems correct. Is talk of possible worlds required to explain this kind of derivation, or is there some other account of the logical structure of possibility and necessity? If we agree with Arthur Prior that we understand possible worlds be-

"… possible worlds, in the sense of possible states of affairs are not *really* individuals (just as numbers are not *really* individuals). To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i. e. if something else were the case … We understand 'truth in states of affairs' because we understand 'necessarily'; not *vice versa*." — Arthur Prior, *Worlds, Times and Selves* [90].

cause we understand the concepts of possibility and necessity, then it's incumbent on us to give some explanation of how we come to understand those concepts—and how they come to have the structure that makes talk of possible worlds appropriate. I will argue in this book that when we attend to the structure of proofs involving modal notions, we will see how this use helps determine the concepts of necessity and possibility, and this *thereby* gives us an understanding the notion of a possible world. We don't first understand modal concepts by invoking possible worlds—we can invoke possible worlds when we first understand modal concepts, and the logic of modal concepts can be best understood when we understand what modal reasoning is *for* and how we do it.

EXAMPLE 4: A NEW ANGLE ON OLD IDEAS Lastly, one reason for studying proof theory is the perspective it brings on familiar themes. There is a venerable and well-trodden road between truth, models and logical consequence. Truth is well-understood, models (truth tables for propositional logic, or Tarski's models for first-order predicate logic, Kripke models for modal logic, or whatever else) are taken to be models of truth, and logical consequence is understood as the preservation of truth in all models. Then, some proof system is designed as a way to give a tractable account of that logical consequence relation. Nothing in this book will count as an argument *against* taking that road from truth, through logical consequence, to proof. However, we will travel that road in the other direction. By starting with proofs we will retrace those steps in reverse, to construct *models* from a prior understanding of proof, and then with an approach to *truth* once we have a notion of a model in hand. This is a very different way to chart the connection between proof theory and model theory. At the very least, tackling this terrain from that angle will allow us to take a different perspective on some familiar ground, and will give us the facility to offer new answers to some perennial questions about meaning, metaphysics and epistemology. Perhaps, when we see matters from this new perspective, the insights will be of lasting value.

However, the notion of truth is beset by paradox, and this should at least serve as a warning sign. Using the notion of truth as a starting point to define core features of logic may not provide the most stable foundation. It is at least worth exploring different approaches.

These are four examples of the kinds of issues that we will consider in the light of proof theory in the pages ahead. To broach these topics, we need to learn some proof theory, so let's dive in.

PART I

Tools

NATURAL DEDUCTION 1

We start with modest ambitions. In this section we focus on one way of understanding proof—natural deduction, in the style of Gentzen [43]—and we will consider just one kind of judgement: *conditionals*.

Gerhard Gentzen, German Logician: Born 1909, student of David Hilbert at Göttingen, died in 1945 in World War II. http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Gentzen.html

1.1 | CONDITIONALS

Conditional judgements have this shape

If . . . then . . .

where we can fill in both "..." with other judgements. Conditional judgements are a useful starting point for thinking about logic and proof, because conditionals play a central role in our thinking and reasoning, in reflection and in dialogue. If we move beyond judgements about what is the case to reflect on how our judgements hang together and stand with regard to one another, it is very natural to form *conditional* judgements. You may not want to claim that the Number 58 tram is about to arrive, but you may at be in a position to judge that *if the timetable is correct*, the Number 58 tram is about to arrive. This is a conditional judgement, with the antecedent "the timetable is correct," and consequent "the Number 58 tram is about to arrive."

"The aim of philosophy, abstractly formulated, is to understand how things in the broadest possible sense of the term hang together in the broadest possible sense of the term." — Wilfrid Sellars [106]

In the study of *formal* logic, we focus on the *form* or *structure* of judgements. One aspect of this involves being precise and attending to those structures and shapes in some detail. We will start this by defining a grammar for conditional judgements. Any grammar has to start somewhere, and we will start with labels for atomic judgements—those judgements which aren't themselves conditionals, but which can be used to build conditionals. We'll use the letters p, q and r for these atoms, and if they're not enough, we'll use numerical subscripts to make more—that way, we never run out.

$$p,\ q,\ r,\quad p_0,\ p_1,\ p_2,\ldots\quad q_0,\ q_1,\ q_2,\ldots\quad r_0, r_1,\ r_2,\ldots$$

Each of these formulas is an ATOM. Whenever we have two formulas A and B, whether A and B are ATOMs or not, we will say that $(A \to B)$ is also a formula. In other words, given two judgements, we can (at least, in theory) form the conditional judgement with the first as the antecedent and the second as consequent. Succinctly, this *grammar* can be represented as follows:

This is BNF, or "Backus Naur Form," first used in the specification of formal computer programming languages such as ALGOL. http://cui.unige.ch/db-research/Enseignement/analyseinfo/AboutBNF.html

$$\text{FORMULA} ::= \text{ATOM} \mid (\text{FORMULA} \to \text{FORMULA})$$

That is, a FORMULA is either an ATOM, or is found by placing an arrow (written like this '$\to$') between two FORMULAs, and surrounding the result with parentheses.

So, the next line contains four different formulas

$$p_3 \quad (q \to r) \quad ((p_1 \to (q_1 \to r_1)) \to (q_1 \to (p_1 \to r_1))) \quad (p \to (q \to (r \to (p_1 \to (q_1 \to r_1)))))$$

but these are not formulas:

$$t \qquad p \to q \to r \qquad p \to p$$

The first, t, fails to be a formula since is not in our set ATOM of atomic formulas (so it doesn't enter the collection of formulas by way of being an atom) and it does not contain an arrow (so it doesn't enter the collection through the clause for complex formulas). The second, $p \to q \to r$ does not enter the collection because it is short of a few parentheses. The only expressions that enter *our* language are those that bring a pair of parentheses along with every arrow: "$p \to q \to r$" has two arrows but no parentheses, so it does not qualify. You can see why it *should* be excluded because the expression is ambiguous. Does it express the conditional judgement to the effect that if p then if q then r, or is it the judgement that if it's true that if p then q, then it's also true that r? In other words, it is ambiguous between these two formulas:

You can do without parentheses if you use 'prefix' notation for the conditional: 'Cpq' instead of '$p \to q$'. The conditional are then $CpCqr$ and $CCpqr$. This is *Polish notation*.

$$(p \to (q \to r)) \quad ((p \to q) \to r)$$

We really need to distinguish these two judgements, so we make sure our formulas contain parentheses. Our last example of an offending non-formula, $p \to p$, does not offend nearly so much. It is not ambiguous. It merely offends against the letter of the law laid down, and not its spirit. I will feel free to use expressions such as "$p \to p$" or "$(p \to q) \to (q \to r)$" which are missing their outer parentheses, even though they are, strictly speaking, not FORMULAS.

If you like, you can think of them as including their outer parentheses very *faintly*, even more faintly than this: $((p \to q) \to (q \to r))$.

Given a formula containing at least one arrow, such as $(p \to q) \to (q \to r)$, it is important to be able to isolate its main connective (the last arrow introduced as it was constructed). In this case, it is the middle arrow. The formula to the left of the arrow (in this case $p \to q$) is said to be the *antecedent* of the conditional, and the formula to the right is the *consequent* (here, $q \to r$).

We can think of formulas generated in this way in at least two different ways. We can think of them as the sentences in a very simple language. This language is either something completely separate from our natural languages, or it is a fragment of a natural language, consisting only of atomic expressions and the expressions you can construct using a conditional construction like "if . . . then . . .".

On the other hand, you can think of formulas as not constituting a language in themselves, but as constructions used to display the *form* of expressions in a language. Both of these interpretations of this syntax are open to us, and everything in this chapter (and in much of the rest of the book) is written with both interpretations in mind. Formal languages can be used to describe the forms of different languages, and they can be thought to be languages in their own right.

The issue of interpreting the formal language raises another question: What is the relationship between languages (formal or informal) and the judgements expressed in those languages? This question is not unlike the question concerning the relationship between a name and the bearer of that name, or a term and the thing (if anything) denoted by that term. The numeral '2' is not to be identified with number 2, and the formula $p \to q$ (or a sentence with that shape) is not the same as the conditional judgement expressed by that formula. Talk of judgements is itself ambiguous between the act of judging (my act of judging that the Number 58 tram is coming soon is not the same act as your act of judging this), and the *content* of any such act. When it comes to interpreting and applying the formal language of logic, it is important to reflect on not only the languages that you and I might speak (or write, or use in computer programs, etc.) but also attend to the *content* expressed when we use such languages [118].

The term "$1 + 1$" to be identified with the numeral "2", though both denote the same number. One term contains the numeral "1" and the other doesn't.

»» ««

Often, we will want to talk quite generally about all formulas with a given shape. We do this very often, when it comes to logic, because we are interested in the forms of valid arguments. The structural or formal features of arguments apply generally, to more than just a particular argument. (If we know that an argument is valid in virtue of its possessing some particular form, then other arguments with that form are valid as well.) So, these formal or structural principles must apply *generally*. Our formal language goes some way to help us express this, but it will turn out that we will not want to talk merely about specific formulas in our language, such as $(p_3 \to q_7) \to r_{26}$. We will, instead, want to say things like

> A *modus ponens* inference is the inference from a conditional formula and the antecedent of that conditional, to its consequent.

This can get very complicated very quickly. It is not easy to understand

> *Given a conditional formula whose consequent is also a conditional, the conditional formula whose antecedent is the antecedent of the consequent of the original conditional, and whose consequent is a conditional whose antecedent is the antecedent of the original conditional and whose consequent is the consequent of the conditional inside the first conditional follows from the original conditional.*

Instead of that mouthful, we will use *variables* to talk generally about formulas in much the same way that mathematicians use variables to talk generally about numbers and other such things. We will use capital letters, such as

$$A, B, C, D, \ldots$$

as variables ranging over the FORMULAS. So, instead of the long paragraph above, it suffices to say

Number theory books don't often include lots of *numerals*. Instead, they're filled with *variables* like 'x' and 'y.' This isn't because these books aren't about numbers. They are, but they don't merely list *particular* facts about numbers. They talk about *general* features of numbers, and hence the use of variables.

$$\textit{From } A \to (B \to C) \textit{ you can infer } B \to (A \to C).$$

which seems much more perspicuous and memorable. The letters A, B and C aren't any *particular* formulas. They each can stand in for any formula at all.

Now we have the raw formal materials to address the question of deduction using conditional judgements. How may we characterise proofs reasoning using conditionals? That is the topic of the next section.

1.2 | PROOFS FOR CONDITIONALS

Start with some of reasoning using conditional judgements. One example might be reasoning of this form:

> *Suppose* $A \to (B \to C)$. *Suppose* A. *It follows that* $B \to C$. *Suppose* B. *It follows that* C.

This kind of reasoning has two important features. We make *suppositions*. We also infer *from* these suppositions. From $A \to (B \to C)$ and A we inferred $B \to C$. From this new information, together with the supposition that B, we inferred a new conclusion, C.

One way to represent the structure of this piece of reasoning is in this *tree diagram* shown here

$$\cfrac{\cfrac{A \to (B \to C) \qquad A}{B \to C} \qquad B}{C}$$

The *leaves* of the tree are the formulas $A \to (B \to C)$, A and B. They are the assumptions upon which the deduction rests. The other formulas in the tree are *deduced* from formulas occurring above them in the tree. The formula $B \to C$ is written immediately below a line, above which are the formulas from which we deduced it. So, $B \to C$ didn't have to be supposed. It *follows from* the leaves $A \to (B \to C)$ and A. Then the *root* of the tree (the formula at the bottom), C, follows from that formula $B \to C$ and the other leaf B. The ordering of the formulas bears witness to the relationships of inference between those formulas in our process of reasoning.

The two steps in our example proof use the same kind of reasoning. The inference from a conditional, and from its antecedent to its consequent. This step is called *modus ponens*. It's easy to see that using *modus ponens* we always move from more complicated formulas to less complicated formulas. However, sometimes we wish to infer the conditional $A \to B$ on the basis of our information about A and about B. And it seems that sometimes this is legitimate. Suppose we want to know about the connection between A and C in a context in which we are happy to grant both $A \to (B \to C)$ and B. What kind of connection is there (if any) between A and C? It would seem that it would be appropriate to infer $A \to C$, since we can derive C if we are willing to grant

"*Modus ponens*" is short for "*modus ponendo ponens*," which means "the mode of affirming by affirming." You get to the affirmation of B by way of the affirmation of A (and the other premise, $A \to B$). It may be contrasted with *Modus tollendo tollens*, the mode of denying by denying: from $A \to B$ and *not* B to *not* A.

A as an assumption. In other words, we have the means to conclude C from A, using the other resources we have already granted. But what does the conditional judgement $A \to C$ say? That if A, then C. So we can make that explicit and conclude $A \to C$ from that reasoning. We can represent the structure of chain of reasoning in the following way:

$$\cfrac{\cfrac{\cfrac{A \to (B \to C) \quad [A]^{(1)}}{B \to C} \qquad B}{C}}{A \to C}\,[1]$$

This proof can be read as follows: At the step marked with [1], we make the inference to the *conditional* conclusion, on the basis of the reasoning up until that point. Since we can conclude C *using* A as an assumption, we can make the further conclusion $A \to C$. At this stage of the reasoning, A is no longer active as an assumption: we *discharge* it. It is still a leaf of the tree (there is no node of the tree above it), but it is no longer an active assumption in our reasoning. So, at this stage we bracket it, and annotate the brackets with a label, indicating the point in the demonstration at which the assumption is discharged. Our proof now has two assumptions, $A \to (B \to C)$ and B, and one conclusion, $A \to C$.

$$\frac{A \to B \qquad A}{B}\,{\scriptstyle \to E} \qquad\qquad \frac{\begin{array}{c}[A]^{(i)}\\ \vdots\\ B\end{array}}{A \to B}\,{\scriptstyle \to I, i}$$

Figure 1.1: natural deduction rules for conditionals

We have motivated two rules for proofs with conditionals. These rules are displayed in Figure 1.1. The first rule, *modus ponens*, or *conditional elimination* [$\to E$] allows us to step from a conditional and its antecedent to the consequent of the conditional. We call the conditional premise $A \to B$ the *major* premise of the [$\to E$] inference, and the antecedent A the *minor* premise of that inference. When we apply the inference [$\to E$], we combine two proofs: the proof of $A \to B$ and the proof of A. The new proof has as assumptions any assumptions made in the proof of $A \to B$ and also any assumptions made in the proof of A. The conclusion is B.

The major premise in a connective rule features that connective.

The second rule, *conditional introduction* [$\to I$], allows us to use a proof from A to B as a proof of $A \to B$. The assumption of A is discharged in this step. The proof of $A \to B$ has as its assumptions all of the assumptions used in the proof of B except for the instances of A that we discharge in this step. Its conclusion is $A \to B$.

Now we come to the first formal definition, giving an account of what counts as a proof in this natural deduction system for the language of conditionals.

DEFINITION 1.1 [PROOFS FOR CONDITIONALS] A proof is a *tree* consisting of formulas, some of which may be *bracketed*. The formula at the root of a proof is said to be its CONCLUSION. The unbracketed formulas at the leaves of the tree are the PREMISES of the proof.

» Any FORMULA A is a proof, with premise A and conclusion A. The formula A is not bracketed.

» If π_l is a proof, with conclusion $A \to B$ and π_r is a proof, with conclusion A, then these proofs may be combined, into the following proof,

$$\dfrac{\begin{matrix}\vdots\,\pi_l & \quad & \vdots\,\pi_r \\ A \to B & \quad & A\end{matrix}}{B}\,{\scriptstyle \to E}$$

which has conclusion B, and which has premises consisting of the premises of π_l together with the premises of π_r.

How do you choose the number for the label (i) on the discharged formula? Find the largest number labelling a discharge in the proof π, and then choose the next one.

» If π is a proof with conclusion B, then the following tree

$$\dfrac{\begin{matrix}[A]^{(i)} \\ \vdots\,\pi \\ B\end{matrix}}{A \to B}\,{\scriptstyle \to I, i}$$

is a proof with conclusion $A \to B$. Its premises are the premises of the original proof π, except for the premise A which is now discharged. We indicate this discharge by *bracketing* it.

» Nothing else is a proof.

This is a recursive definition, in just the same manner as the recursive definition of the class FORMULA. We define atomic proofs (in this case, consisting of a single formula), and then show how new (larger) proofs can be built out of smaller proofs.

$$\dfrac{\dfrac{\dfrac{[B \to C]^{(2)} \quad \dfrac{A \to B \quad [A]^{(1)}}{B}\,{\scriptstyle \to E}}{C}\,{\scriptstyle \to E}}{A \to C}\,{\scriptstyle \to I,1}}{(B \to C) \to (A \to C)}\,{\scriptstyle \to I,2}$$

SUFFIXING (DEDUCTION)

$$\dfrac{\dfrac{\dfrac{[A \to B]^{(1)} \quad [A]^{(2)}}{B}\,{\scriptstyle \to E}}{(A \to B) \to B}\,{\scriptstyle \to I,1}}{A \to ((A \to B) \to B)}\,{\scriptstyle \to I,2}$$

ASSERTION (FORMULA)

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \to B]^{(3)} \quad \dfrac{[C \to A]^{(2)} \quad [C]^{(1)}}{A}\,{\scriptstyle \to E}}{B}\,{\scriptstyle \to E}}{C \to B}\,{\scriptstyle \to I,1}}{(C \to A) \to (C \to B)}\,{\scriptstyle \to I,2}}{(A \to B) \to ((C \to A) \to (C \to B))}\,{\scriptstyle \to I,3}$$

PREFIXING (FORMULA)

Figure 1.2: THREE IMPLICATIONAL PROOFS

Figure 1.2 gives three proofs constructed using our rules. The first is a proof from $A \to B$ to $(B \to C) \to (A \to C)$. This is the inference of *suffixing*. (We "suffix" both A and B with $\to C$.) The other proofs conclude in formulas justified on the basis of *no* undischarged assumptions. It is worth your time to read through these proofs to make sure that you understand the way each proof is constructed. A good way to understand the shape of these proofs is to try writing them out from top-to-bottom, identifying the basic proofs you start with, and only adding the discharging brackets at the stage of the proof where the discharge occurs.

You can try a number of different strategies when making proofs for yourself without copying existing ones. For example, you might like to try your hand at constructing a proof to the conclusion that $B \to (A \to C)$ from the assumption $A \to (B \to C)$. Here are two strategies you could use to piece a proof together.

TOP–DOWN: You start with the assumptions and see what you can do with them. In this case, with $A \to (B \to C)$ you can, clearly, get $B \to C$, if you are prepared to assume A. And then, with the assumption of B we can deduce C. Now it is clear that we can get $B \to (A \to C)$ if we discharge our assumptions, A first, and then B.

BOTTOM–UP: Start with the conclusion, and find what you could use to prove it. Notice that to prove $B \to (A \to C)$ you could prove $A \to C$ using B as an assumption. Then to prove $A \to C$ you could prove C using A as an assumption. So, our goal is now to prove C using A, B and $A \to (B \to C)$ as assumptions. But this is an easy pair of applications of $[\to E]$.

In either case, this is the proof we construct:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (B \to C) \quad [A]^{(1)}}{B \to C}{\scriptstyle \to E} \quad [B]^{(2)}}{C}{\scriptstyle \to E}}{A \to C}{\scriptstyle \to I,1}}{B \to (A \to C)}{\scriptstyle \to I,2}$$

» «

Before exploring some more of the formal and structural properties of this kind of proof, let's pause for a moment to consider in more detail we might interpret the components of these proof structures. It is one thing to specify a formal structure as representing a network of connections between judgements. It is another to have a view of what kinds of connections between judgements are modelled in such a structure. To be specific: What kind of act is making an assumption? What kind of act is discharging that assumption that has been made? There are different things that you can say about this, but one way to understand the making of assumptions in a proof is that when you *suppose* A in a proof, you (under the scope of that assumption) treat the judgement A as if it had been asserted. You enter "A" in the "asserted" scoresheet, and treat it as if it had been asserted, for the purposes of reasoning, without actually undergoing the commitments. This enables us to infer *from* that commitment without actually having to undertake the commitment. We can attend to the inferential transition between A and B independently of actually asserting A. Doing so gives us a way to distinguish different senses of inferring B from A. The strong sense is the sense in which we have already granted A: To infer B from A in that context tells us that A and B both hold — or it commits us to the even strong claim, B *because* A. This can be distinguished from the weak sense of inferring B from A *hypothetically*, which tells us merely that *if* A *then* B. This incurs no commitment to B (or to A), but gives us a way to make explicit the inferential commitment we incur. With interpretation of supposition in mind, we can interpret proof structures as follows.

- An identity proof A represents the act of supposing A. Its *conclusion* is A, the very content that is supposed. Its only *active supposition* is A.
- Given a proof of the conclusions $A \to B$ (in which the suppositions in X are active) and a proof of A (in which the suppositions in Y are active), we have two corresponding (possibly complex) acts, the act of inferring $A \to B$ from X and the act of inferring A from Y. The proof given by extending those two proofs by an [$\to E$] step, to conclude B represents the complex act of (a) inferring $A \to B$ from X, (b) inferring A from Y, and then (c) deducing B from $A \to B$ and A. The active suppositions of this complex deduction are given in X and Y.
- Given a proof of B, in which the suppositions in X and A are active, this corresponds to the act of deducing B from X together with A. We interpret the proof of $A \to B$ from X, found by discharging A from the active assumptions, as representing the complex act of (a) first deducing B from X and A, and then (b) concluding $A \to B$, from the deduction from A to B. Now the conclusion is $A \to B$ and the active suppositions are those in X.

» «

I have been intentionally unspecific when it comes how formulas are discharged formulas in proofs. In the examples in Figure 1.2 you will notice that at each step when a discharge occurs, one and only one formula is discharged. By this I do not mean that at each [$\to I$] step a formula A is discharged and a different formula B is not. I mean that in the proofs we have seen so far, at each [$\to I$] step, a single *instance* of the formula is discharged. Not all proofs are like this. Consider this proof from the assumption $A \to (A \to B)$ to the conclusion $A \to B$. At the final step of this proof, two instances of the assumption A are discharged at once.

$$\dfrac{\dfrac{\dfrac{A \to (A \to B) \quad [A]^{(1)}}{A \to B}{\scriptstyle \to E} \qquad [A]^{(1)}}{B}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to I,1}$$

For this to count as a proof, we must read the rule [$\to I$] as licensing the discharge of *one or more instances* of a formula in the inference to the conditional. Once we think of the rule in this way, one further generalisation comes to mind: If we think of an [$\to I$] move as discharging a *collection* of instances of our assumption, someone of a generalising spirit will ask if that collection can be empty. Can we discharge an assumption that *isn't there*? If we can, then *this* counts as a proof:

"Yesterday upon the stair, I met a man who wasn't there. He wasn't there again today. I wish that man would go away." — Hughes Mearns

$$\frac{A}{B \to A}\ {\scriptstyle \to I,1}$$

Here, we assume A, and then, we infer $B \to A$ discharging *all* of the active assumptions of B in the proof at this point. The collection of active assumptions of B is, of course, empty. No matter, they are all discharged, and we have our conclusion: $B \to A$.

You might think that this is silly: how can you discharge a nonexistent assumption? Nonetheless, discharging assumptions that are not there plays a role. To give you a taste of why, notice that the inference from A to $B \to A$ is *valid* if we read "$\to$" as the material conditional of standard two-valued classical propositional logic. In a pluralist spirit we will investigate different policies for discharging formulas.

For more in a "pluralist spirit" see my work with Jc Beall [7, 8, 97].

definition 1.2 [discharge policy] A discharge policy may either allow or disallow *duplicate* discharge (more than one instance of a formula at once) or *vacuous* discharge (*zero* instances of a formula in a discharge step). Here are the names for the four discharge policies:

		vacuous	
		yes	no
duplicates	yes	*Standard*	*Relevant*
	no	*"Affine"*	*Linear*

I am not happy with the label "affine," but that's what the literature has given us. Does anyone have any better ideas for this? "Standard" is not "classical" because it suffices for intuitionistic logic in this context, not classical logic. It's not "intuitionistic" because "intuitionistic" is difficult to pronounce, and it is not *distinctively* intuitionistic. As we shall see later, it's the shape of proof and not the discharge policy that gives us intuitionistic implicational logic.

The "standard" discharge policy is to allow both vacuous and duplicate discharge.

There are reasons to explore each of the different policies. As I indicated above, you might think vacuous discharge does not make much sense. However, we can say more than that: it seems downright *mistaken* if we are to understand a judgement of the form $A \to B$ to record the claim that B may be inferred *from* A. If A is not used in the inference to B, then we hardly have reason to think that B follows from A in this sense. So, if you are after a conditional which is *relevant* in this way, you would be interested in discharge policies that ban vacuous discharge [1, 2, 93].

There are also reasons to ban duplicate discharge: Victor Pambuccian has found an interesting example of doing without duplicate discharge in early 20th Century geometry [80]. He traces cases where geometers took care to keep track of the number of times a postulate was used in a proof. So, they draw a distinction between $A \to (A \to B)$ and $A \to B$. The judgement that $A \to (A \to B)$ records the fact that B can be deduced from two uses of A. $A \to B$ records that B can be deduced from A used only once. More recently, work in *fuzzy logic* [11, 52, 72] motivates keeping track of the number of times premises are used. If a conditional $A \to B$ fails to be true to the degree that A is truer than B, then $A \to (A \to B)$ may be truer than $A \to B$.

Consider the claim I'll call (α) — *If* (α) *is true, then I am a monkey's uncle.*

Finally, for some [6, 74, 88, 94], Curry's Paradox motivates banning indiscriminate duplicate discharge. If we have a claim A which both implies $A \to B$ and is implied by it then we can reason as follows:

$$
\begin{prooftree}
\AxiomC{$[A]^{(1)}$}
\RightLabel{$\dagger$}
\UnaryInfC{$A \to B$}
\AxiomC{$[A]^{(1)}$}
\RightLabel{$\to E$}
\BinaryInfC{B}
\RightLabel{$\to I,1$}
\UnaryInfC{$A \to B$}
\AxiomC{$[A]^{(2)}$}
\RightLabel{$\dagger$}
\UnaryInfC{$A \to B$}
\AxiomC{$[A]^{(2)}$}
\RightLabel{$\to E$}
\BinaryInfC{B}
\RightLabel{$\to I,2$}
\UnaryInfC{$A \to B$}
\RightLabel{$\dagger$}
\UnaryInfC{A}
\RightLabel{$\to E$}
\BinaryInfC{B}
\end{prooftree}
$$

Where we have used '†' to mark the steps where we have gone from A to $A \to B$ or back. Notice that this is a proof of B from *no premises at all*! So, if we have a claim A which is equivalent to $A \to B$, and if we allow vacuous discharge, then we can derive B.

DEFINITION 1.3 [KINDS OF PROOFS] A proof in which every discharge is *linear* is a *linear proof*. Similarly, a proof in which every discharge is *relevant* is a *relevant proof*, a proof in which every discharge is *affine* is an *affine proof*. If a proof has some duplicate discharge and some vacuous discharge, it is at least a *standard proof*.

We will *generalise* the notion of an argument later, in a number of directions. But this notion of argument is suited to the kind of proof we are considering here.

Proofs underwrite *arguments*. If we have a proof from a collection X of assumptions to a conclusion A, then the argument $X \therefore A$ is *valid* by the light of the rules we have used. So, in this section, we will think of *arguments* as structures involving a collection of assumptions and a single conclusion. But what kind of thing is that collection X? It isn't a *set*, because the number of premises makes a difference: (The example here involves linear discharge policies. We will see later that even when we allow for duplicate discharge, there is a sense in which the number of occurrences of a formula in the premises might still matter.) There is a linear proof from $A \to (A \to B), A, A$ to B:

$$
\begin{prooftree}
\AxiomC{$A \to (A \to B)$}
\AxiomC{A}
\RightLabel{$\to E$}
\BinaryInfC{$A \to B$}
\AxiomC{A}
\RightLabel{$\to E$}
\BinaryInfC{B}
\end{prooftree}
$$

We shall see later that there is *no* linear proof from $A \to (A \to B), A$ to B. (If we ban duplicate discharge, then the number of assumptions in a proof matters.) The *collection* appropriate for our analysis at this stage is what is called a *multiset*, because we want to pay attention to the number of times we make an assumption in an argument.

DEFINITION 1.4 [MULTISET] Given a class X of objects (such as the class FORMULA), a *multiset* M of objects from X is a special kind of collection of elements of X. For each x in X, there is a natural number $o_M(x)$, the number of OCCURRENCES of the object x in the multiset M. The number

$o_M(x)$ is sometimes said to be the DEGREE to which x is a member of M. The multiset M is FINITE if $o_M(x) > 0$ for only finitely many objects x. The multiset M is identical to the multiset M' if and only if $o_M(x) = o_{M'}(x)$ for every x in X.

Multisets may be presented in lists, in much the same way that sets can. For example, $[1, 2, 2]$ is the finite multiset containing 1 only once and 2 twice. $[1, 2, 2] = [2, 1, 2]$, but $[1, 2, 2] \neq [1, 1, 2]$. We shall only consider finite multisets of *formulas*, and not multisets that contain other multisets as members. This means that we can do without the brackets and write our multisets as lists. We will write "A, B, B, C" for the finite multiset containing B twice and A and C once. The empty multiset, to which everything is a member to degree zero, is $[\,]$.

If you like, you could define a multiset of formulas to *be* the occurrence function $o_M(x)$ function $o_M : \text{FORMULA} \to \omega$. Then $o_1 = o_2$ when $o_1(A) = o_2(A)$ for each formula A. $o(A)$ is the number of times A is in the multiset o.

DEFINITION 1.5 [COMPARING MULTISETS] When M and M' are multisets and $o_M(x) \leq o_{M'}(x)$ for each x in X, we say that M is a SUB-MULTISET of M', and M' is a SUPER-MULTISET of M.

The GROUND of the multiset M is the set of all objects that are members of M to a non-zero degree. So, for example, the ground of the multiset A, B, B, C is the set $\{A, B, C\}$.

We use finite multisets as a part of a discriminating analysis of proofs and arguments. (An even more discriminating analysis will consider premises to be structured in *lists*, according to which A, B differs from B, A. You can examine this in Exercise 24 on page 46.) We have no need to consider *infinite* multisets in this section, as multisets represent the premise collections in arguments, and it is quite natural to consider only arguments with finitely many premises, since proofs, as we have defined them feature only finitely many assumptions. So, we will consider arguments in the following way.

DEFINITION 1.6 [ARGUMENT] An ARGUMENT $X \therefore A$ is a structure consisting of a finite multiset X of formulas as its *premises*, and a single formula A as its *conclusion*. The premise multiset X may be empty. An argument $X \therefore A$ is *standardly valid* if and only if there is some proof with undischarged assumptions forming the multiset X, and with the conclusion A. It is *relevantly valid* if and only if there is a relevant proof from the multiset X of premises to A, and so on.

John Slaney has joked that the empty multiset $[\,]$ should be distinguished from the empty set $\emptyset$, since *nothing* is a member of $\emptyset$, but *everything* is a member of $[\,]$ *zero times*.

Here are some features of validity.

LEMMA 1.7 [VALIDITY FACTS] *Let v-validity be any of linear, relevant, affine or standard validity.*

1. $A \therefore A$ *is v-valid.*
2. $X, A \therefore B$ *is v-valid if and only if* $X \therefore A \to B$ *is v-valid.*
3. *If* $X, A \therefore B$ *and* $Y \therefore A$ *are both v-valid, so is* $X, Y \therefore B$.
4. *If* $X \therefore B$ *is affine or standardly valid, so is* $X, A \therefore B$.
5. *If* $X, A, A \therefore B$ *is relevantly or standardly valid, so is* $X, A \therefore B$.

Proof: (1) is given by the proof consisting of A as premise and conclusion.

For (2), take a proof π from X, A to B, and in a single step $\to I$, discharge the (single instance of) A to construct the proof of $A \to B$ from X. Conversely, if you have a proof from X to $A \to B$, add a (single) premise A and apply $\to E$ to derive B. In both cases here, if the original proofs satisfy a constraint (vacuous or multiple discharge) so do the new proofs.

For (3), take a proof from X, A to B, but replace the instance of assumption of A indicated in the premises, and replace this with the *proof* from Y to A. The result is a proof, from X, Y to B as desired. This proof satisfies the constraints satisfied by both of the original proofs.

For (4), if we have a proof π from X to B, we extend it as follows

$$\dfrac{\dfrac{\begin{array}{c} X \\ \vdots\,\pi \\ B \end{array}}{A \to B}{\scriptstyle \to I} \qquad A}{B}{\scriptstyle \to E}$$

to construct a proof from to B involving the new premise A, as well as the original premises X. The $\to I$ step requires a vacuous discharge.

Finally (5): if we have a proof π from X, A, A to B (that is, a proof with X and *two* instances of A as premises to derive the conclusion B) we discharge the two instances of A to derive $A \to B$ and then reinstate a single instance of A to as a premise to derive B again.

$$\dfrac{\dfrac{\begin{array}{c} X, [A, A]^{(i)} \\ \vdots\,\pi \\ B \end{array}}{A \to B}{\scriptstyle \to I, i} \qquad A}{B}{\scriptstyle \to E}$$

∎

Now, having established these facts, we might focus all our attention on the distinction between those arguments that are valid and those that are not, to attend to facts about validity such as those we have just proved. That would be to ignore the distinctive features of proof theory. We care not only *that* an argument is proved, but *how* it is proved. For each of these facts about validity, we showed not only the bare existential fact (for example, if there is a proof from X, A to B, then there is a proof from X to $A \to B$) but the stronger and more specific fact (if there is a proof from X, A to B then from this proof we construct the proof from X to $A \to B$ in a uniform way). This is the power of proof theory. We focus on proofs, not only as a certificate for the validity of an argument, but as a structure worth attention in its own right.

» «

It is often a straightforward matter to show that an argument is valid. Find a proof from the premises to the conclusion, and you are done. It seems more difficult to show that an argument is not valid. According to the literal reading of this definition, if an argument is not valid there is no proof from the premises to the conclusion. So, the direct way to show that an argument is invalid is to show that it has no proof from the premises to the conclusion. There are infinitely many proofs. It would take forever to through all of the proofs and check that none of them are proofs from X to A in order to convince yourself that the argument from X to A is not valid. To show that the argument is not valid, that there is no proof from X to A, some subtlety is called for. We will end this section by looking at how we might summon up the skill we need.

One subtlety would be to change the terms of discussion entirely, and introduce a totally new concept. If you could show that all valid arguments have some special property – and one that is easy to detect when present and when absent – then you could show that an argument is invalid by showing it lacks that special property. How this might manage to work depends on the special property. We shall look at one of these properties in Chapter 3 when we show that all valid arguments *preserve truth* in all *models*. Then to show that an argument is invalid, you could provide a model in which truth is *not* preserved from the premises to the conclusion. If all valid arguments are truth-in-a-model-preserving, then such a model would count as a counterexample to the validity of your argument.

In this chapter, on the other hand, we will not go beyond the conceptual bounds of proof theory itself. We will find instead a way to show that an argument is invalid, using an analysis of the structure of proofs. The collection of *all* proofs is too large to survey. From premises X and conclusion A, the collection of *direct* proofs – those that go straight from X to A without any detours down byways or highways – should be more tractable. If we could show that there are not many *direct* proofs from a given collection of premises to a conclusion, then we might be able to exploit this fact to show that for a given set of premises and a conclusion there are *no* direct proofs from X to A. If, in addition, you were to how that any proof from a premise set to a conclusion could somehow be converted into a direct proof from the same premises to that conclusion, then you would have shown that there is no proof from X to A.

Happily, this technique works. To show how it works we need to understand what it is for a proof to be have no detours. These proofs which head straight from the premises to the conclusion without detours are so important that they have their own name. They are called *normal*.

I *think* that the terminology 'normal' comes from Prawitz [86], though the idea comes from Gentzen.

1.3 | NORMAL PROOFS

It is best to introduce normal proofs by contrasting them with *non*-normal proofs. Non-normal proofs are not difficult to find. Suppose you want to show that the following argument is valid

$$p \to q \therefore p \to ((q \to r) \to r)$$

On page 8 You might note first that we have already seen an argument which takes us from $p \to q$ to $(q \to r) \to (p \to r)$. This is SUFFIXING.

$$
\begin{prooftree}
\AxiomC{$[q \to r]^{(2)}$}
\AxiomC{$p \to q$}
\AxiomC{$[p]^{(1)}$}
\RightLabel{$\to E$}
\BinaryInfC{q}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,1$}
\UnaryInfC{$p \to r$}
\RightLabel{$\to I,2$}
\UnaryInfC{$(q \to r) \to (p \to r)$}
\end{prooftree}
$$

So, we have $p \to q \therefore (q \to r) \to (p \to r)$. But we also have the general principle *permuting* antecedents: $A \to (B \to C) \therefore B \to (A \to C)$.

$$
\begin{prooftree}
\AxiomC{$A \to (B \to C)$}
\AxiomC{$[A]^{(3)}$}
\RightLabel{$\to E$}
\BinaryInfC{$B \to C$}
\AxiomC{$[B]^{(4)}$}
\RightLabel{$\to E$}
\BinaryInfC{C}
\RightLabel{$\to I,3$}
\UnaryInfC{$A \to C$}
\RightLabel{$\to I,4$}
\UnaryInfC{$B \to (A \to C)$}
\end{prooftree}
$$

We can apply this in the case where $A = (q \to r)$, $B = p$ and $C = r$ to get $(q \to r) \to (p \to r) \therefore p \to ((q \to r) \to r)$. We then chain reasoning together, to get us from $p \to q$ to $p \to ((q \to r) \to r)$, which we wanted. But take a look at the whole proof:

$$
\begin{prooftree}
\AxiomC{$[q \to r]^{(2)}$}
\AxiomC{$p \to q$}
\AxiomC{$[p]^{(1)}$}
\RightLabel{$\to E$}
\BinaryInfC{q}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,1$}
\UnaryInfC{$p \to r$}
\RightLabel{$\to I,2$}
\UnaryInfC{$(q \to r) \to (p \to r)$}
\AxiomC{$[q \to r]^{(3)}$}
\RightLabel{$\to E$}
\BinaryInfC{$p \to r$}
\AxiomC{$[p]^{(4)}$}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,3$}
\UnaryInfC{$(q \to r) \to r$}
\RightLabel{$\to I,4$}
\UnaryInfC{$p \to ((q \to r) \to r)$}
\end{prooftree}
$$

This proof is circuitous. It gets us from our premise $(p \to q)$ to our conclusion $(p \to ((q \to r) \to r))$, but it does it in a roundabout way. We break down the conditionals $p \to q$, $q \to r$ to construct $(q \to r) \to (p \to r)$ halfway through the proof, only to break that down again (deducing r on its own, for a second time) to build the required conclusion. This is most dramatic around the intermediate conclusion $p \to ((q \to r) \to r)$ which is built up *from* $p \to r$ only to be used to justify $p \to r$ at the next step. We may eliminate this redundancy by cutting out the intermediate formula $p \to ((q \to r) \to r)$ like this:

$$
\begin{prooftree}
\AxiomC{$[q \to r]^{(3)}$}
\AxiomC{$p \to q$}
\AxiomC{$[p]^{(1)}$}
\RightLabel{$\to E$}
\BinaryInfC{q}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,1$}
\UnaryInfC{$p \to r$}
\AxiomC{$[p]^{(4)}$}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,3$}
\UnaryInfC{$(q \to r) \to r$}
\RightLabel{$\to I,4$}
\UnaryInfC{$p \to ((q \to r) \to r)$}
\end{prooftree}
$$

The resulting proof is a lot simpler already. But now the $p \to r$ is constructed from r only to be broken up immediately to return r. We can delete the redundant $p \to r$ in the same way.

$$
\begin{prooftree}
\AxiomC{$[q \to r]^{(3)}$}
\AxiomC{$p \to q$}
\AxiomC{$[p]^{(4)}$}
\RightLabel{$\to E$}
\BinaryInfC{q}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,3$}
\UnaryInfC{$(q \to r) \to r$}
\RightLabel{$\to I,4$}
\UnaryInfC{$p \to ((q \to r) \to r)$}
\end{prooftree}
$$

This proof takes us directly from its premise to its conclusion, through no extraneous formulas. Every formula used in this proof is either found in the premise, or in the conclusion. This wasn't true in the original, roundabout proof. We say this new proof is *normal*, the original proof was not.

This is a general phenomenon. Take a proof ending in [$\to E$]: it goes from A to B by way of a sub-proof π_1, and then we discharge A to conclude $A \to B$. Imagine that at the very next step, we use a different proof – say π_2 – with conclusion A to deduce B by means of an implication elimination. This proof contains a redundant step. Instead of taking the detour through the formula $A \to B$, we could use the proof π_1 of B, but instead of taking A as an *assumption*, we could use the proof of A we have at hand, namely π_2. The before-and-after comparison is this:

$$
\text{BEFORE:}\quad
\begin{prooftree}
\AxiomC{$[A]^{(i)}$}
\noLine
\UnaryInfC{$\vdots\ \pi_1$}
\noLine
\UnaryInfC{B}
\RightLabel{$\to I,i$}
\UnaryInfC{$A \to B$}
\AxiomC{$\vdots\ \pi_2$}
\noLine
\UnaryInfC{A}
\RightLabel{$\to E$}
\BinaryInfC{B}
\end{prooftree}
\qquad
\text{AFTER:}\quad
\begin{array}{c}
\vdots\ \pi_2 \\
A \\
\vdots\ \pi_1 \\
B
\end{array}
$$

The result is a proof of B from the same premises as our original proof. The premises are the premises of π_1 (other than the instances of A that were discharged in the other proof) together with the premises of π_2.

This new proof does not go through the formula $A \to B$, so it is, in a sense, simpler than the original.

Well … there are some subtleties with counting, as usual with proofs. If the discharge of A was vacuous, then we have nowhere to plug in the new proof π_2, so π_2, and its premises, don't appear in the final proof. On the other hand, if a number of duplicates of A were discharged, then the new proof will contain that many copies of π_2, and hence, that many copies of the premises of π_2.

Let's make this discussion more concrete, by considering an example where π_1 has two instances of A in the premise list. The original proof containing the introduction and then elimination of $A \to B$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (A \to B) \quad [A]^{(1)}}{A \to B}{\scriptstyle \to E} \quad [A]^{(1)}}{B}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to I,1} \qquad \dfrac{(A \to A) \to A \quad \dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2}}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}$$

We can cut out the [$\to I/\to E$] pair (we call such pairs INDIRECT PAIRS) using the technique described above, we place a copy of the inference to A at *both* places that the A is discharged (with label 1). The result is this proof, which does not make that detour.

$$\dfrac{\dfrac{A \to (A \to B) \qquad \dfrac{(A \to A) \to A \quad \dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2}}{A}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to E} \qquad \dfrac{(A \to A) \to A \quad \dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2}}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}$$

which is a proof from the same premises ($A \to (A \to B)$ and $(A \to A) \to A$) to the same conclusion B, except for multiplicity. In this proof the premise $(A \to A) \to A$ is used twice instead of once. (Notice too that the label '2' is used twice. We could relabel one subproof to $A \to A$ to use a different label, but there is no ambiguity here because the two proofs to $A \to A$ do not overlap. Our convention for labelling is merely that at the time we get to an [$\to I$] label, the numerical tag is unique in the proof *above* that step.)

We have motivated the concept of normality. Here is the definition:

DEFINITION 1.8 [NORMAL PROOF] A proof is NORMAL if and only if the concluding formula $A \to B$ introduced in an [$\to I$] step is not then immediately used as the major premise of an [$\to E$] step.

DEFINITION 1.9 [INDIRECT PAIR; DETOUR FORMULA] If a formula $A \to B$ introduced in an [$\to I$] step in a proof is also the major premise of a following [$\to E$] step in that proof, then we shall call this pair of inferences an INDIRECT PAIR and we will call the instance $A \to B$ in the middle of this indirect pair a DETOUR FORMULA in that proof.

So, a normal proof is one without any indirect pairs. It has no detour formulas.

Normality is not only important for proving that an argument is invalid by showing that it has no normal proofs. The claim that every valid argument has a normal proof could well be *vital*. If we think of the rules for conditionals as somehow *defining* the connective, then proving something by means of a roundabout [$\to$*I*/$\to$*E*] step that you *cannot* prove without it would seem to be illicit. If the conditional is *defined* by way of its rules then it seems that the things one can prove *from* a conditional ought to be merely the things one can prove from whatever it was you used to *introduce* the conditional. If we could prove more from a conditional $A \to B$ than one could prove on the basis on the information used to *introduce* the conditional, then we are conjuring new arguments out of thin air.

For this reason, many have thought that being able to convert non-normal proofs to normal proofs is not only desirable, it is critical if the proof system is to be properly logical. We will not continue in this philosophical vein here. We will take up this topic in a later section, after we understand the behaviour of normal proofs a little better. Let us return to the study of normal proofs.

Normal proofs are, intuitively at least, proofs without a kind of redundancy. It turns out that avoiding this kind of redundancy in a proof means that you must avoid another kind of redundancy too. A normal proof from X to A may use only a very restricted repertoire of formulas. It will contain only the *subformulas* of X and A.

DEFINITION 1.10 [SUBFORMULAS AND PARSE TREES] The PARSE TREE for an atom is that atom itself. The PARSE TREE for a conditional $A \to B$ is the tree containing $A \to B$ at the root, connected to the parse tree for A and the parse tree for B. The SUBFORMULAS of a formula A are those formulas found in A's parse tree. We let $sf(A)$ be the set of all subformulas of A, so $sf(p) = \{p\}$, and $sf(A \to B) = \{A \to B\} \cup sf(A) \cup sf(B)$. To generalise, when X is a multiset of formulas, we will write $sf(X)$ for the set of subformulas of each formula in X.

Here is the parse tree for $(p \to q) \to ((q \to r) \to p)$:

$$\dfrac{\dfrac{p \quad q}{p \to q} \qquad \dfrac{\dfrac{q \quad r}{q \to r} \quad p}{(q \to r) \to p}}{(p \to q) \to ((q \to r) \to p)}$$

So, $sf((p \to q) \to ((q \to r) \to p)) = \{(p \to q) \to ((q \to r) \to p), p \to q, p, q, (q \to r) \to p, q \to r, r\}$.

We may prove the following theorem.

THEOREM 1.11 [THE SUBFORMULA THEOREM] *If π is a normal proof from the premises X to the conclusion A, then π contains only formulas in* $\mathrm{sf}(X, A)$.

Notice that this is *not* the case for non-normal proofs. Consider the following circuitous proof from A to A.

$$\dfrac{\dfrac{[A]^{(1)}}{A \to A}{\scriptstyle \to I,1} \quad A}{A}{\scriptstyle \to E}$$

Here $A \to A$ is in the proof, but it is not a subformula of the premise (A) or the conclusion (also A).

The subformula property for normal proofs goes some way to reassure us that a normal proof is *direct*. A normal proof from X to A cannot stray so far away from the premises and the conclusion so as to incorporate material outside X and A. This fact goes some way to defend the notion that validity is *analytic* in a strong sense. The validity of an argument is grounded a proof where the consitituents of that proof are found by *analysing* the premises and the conclusion into their constituents. Here is how the subformula theorem is proved.

Proof: We look carefully at how proofs are constructed. If π is a normal proof, then it is constructed in exactly the same way as all proofs are, but the fact that the proof is normal gives us some useful information. By the definition of proofs, π either is a lone assumption, or π ends in an application of [$\to$*I*], or it ends in an application of [$\to$*E*]. Assumptions are the basic building blocks of proofs. We will show that assumption-only proofs have the subformula property, and then, also show on the assumption that the proofs we have on had have the subformula property, then the normal proofs we construct from them also have the property. Then it will follow that all normal proofs have the subformula property, because all of the normal proofs can be generated in this way.

Notice that the subproofs of normal proofs are normal. If a subproof of a proof contains an indirect pair, then so does the larger proof.

ASSUMPTION: A sole assumption, considered as a proof, satisfies the subformula property. The assumption A is the only constituent of the proof and it is both a premise and the conclusion.

INTRODUCTION: In the case of [$\to$*I*], π is constructed from another normal proof π' from X to B, with the new step added on (and with the discharge of a number – possibly zero – of assumptions). π is a proof from X' to $A \to B$, where X' is X with the deletion of some number of instances of A. Since π' is normal, we may assume that every formula in π' is in $\mathrm{sf}(X, B)$. Notice that $\mathrm{sf}(X', A \to B)$ contains every element of $\mathrm{sf}(X, B)$, since X differs only from X' by the deletion of some instances of A. So, every formula in π (namely, those formulas in π', together with $A \to B$) is in $\mathrm{sf}(X', A \to B)$ as desired.

ELIMINATION: In the case of [$\to E$], π is constructed out of *two* normal proofs: one (call it π_1) to the conclusion of a conditional $A \to B$ from premises X, and the other (call it π_2) to the conclusion of the antecedent of that conditional A from premises Y. Both π_1 and π_2 are normal, so we may assume that each formula in π_1 is in $sf(X, A \to B)$ and each formula in π_2 is in $sf(Y, A)$. We wish to show that every formula in π is in $sf(X, Y, B)$. This seems difficult ($A \to B$ is in the proof—where can it be found inside X, Y or B?), but we also have some more information: π_1 cannot end in the *introduction* of the conditional $A \to B$. So, π_1 is either the assumption $A \to B$ itself (in which case $Y = A \to B$, and clearly in this case each formula in π *is* in $sf(X, A \to B, B)$) or π_1 ends in a [$\to E$] step. But if π_1 ends in an [$\to E$] step, the major premise of that inference is a formula of the form $C \to (A \to B)$. So π_1 contains the formula $C \to (A \to B)$, so *whatever* list Y is, $C \to (A \to B) \in sf(Y, A)$, and so, $A \to B \in sf(Y)$. In this case too, every formula in π is in $sf(X, Y, B)$, as desired.

This completes the proof of our theorem. Every normal proof is constructed from assumptions by introduction and elimination steps in this way. The subformula property is preserved through each step of the construction. ∎

Normal proofs are handy to work with. Even though an argument might have very many proofs, it will have many fewer normal proofs. We can exploit this fact when searching for proofs.

EXAMPLE 1.12 [NO NORMAL PROOFS] There is no normal proof from p to q. There is no normal relevant proof from $p \to r$ to $p \to (q \to r)$.

Proof: Normal proofs from p to q (if there are any) contain only formulas in $sf(p, q)$: that is, they contain only p and q. That means they contain no [$\to I$] or [$\to E$] steps, since they contain no conditionals at all. It follows that any such proof must consist solely of an assumption. As a result, the proof cannot have a premise p that differs from the conclusion q. There is no normal proof from p to q.

For the second example, if there is a normal proof of $p \to (q \to r)$, from $p \to r$, it must end in an $\to I$ step, from a normal (relevant) proof from $p \to r$ and p to $q \to r$. Similarly, this proof must also end in an $\to I$ step, from a normal (relevant) proof from $p \to r$, p and q to r. Now, what normal relevant proofs can be found from $p \to r$, p and q to r? There are none. Any such proof would have to use q as a premise somewhere, but since it is normal, it contains only subformulas of $p \to r$, p, q and r—namely those formulas themselves. There is no formula involving q other than q itself on that list, so there is nowhere for q to go. It cannot be used, so it will not be a premise in the proof. There is no normal relevant proof from the premises $p \to r$, p and q to the conclusion r. ∎

These facts are interesting enough. It would be more productive, however, to show that there is no proof at all from p to q, and no relevant proof from $p \to r$ to $p \to (q \to r)$. We can do this if we have some way of showing that if we have a proof for some argument, we have a normal proof for that argument.

So, we now work our way towards the following theorem:

THEOREM 1.13 [NORMALISATION THEOREM] *A proof π from X to A reduces in some number of steps to a normal proof π' from X' to A.*

If π is linear, so is π', and $X = X'$. If π is affine, so is π', and X' is a sub-multiset of X. If π is relevant, then so is π', and X' covers the same ground as X, and is a super-multiset of X. If π is standard, then so is π', and X' covers no more ground than X.

$[1, 2, 2, 3]$ covers the same ground as–and is a super-multiset of–$[1, 2, 3]$. And $[2, 2, 3, 3]$ covers no more ground than $[1, 2, 3]$.

Notice how the premise multiset of the normal proof is related to the premise multiset of the original proof. If we allow duplicate discharge, then the premise multiset may contain formulas to a greater degree than in the original proof, but the normal proof will not contain any premises that weren't in the original proof. If we allow vacuous discharge, then the normal proof might contain fewer premises than the original proof.

The normalisation theorem mentions the notion of *reduction*, so let us first define it.

DEFINITION 1.14 [REDUCTION] A proof π *reduces* to π' (shorthand: $\pi \rightsquigarrow \pi'$) if some indirect pair in π is eliminated, to result in π'.

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}[A]^{(i)}\\ \vdots\ \pi_1\\ B\end{array}}{A \to B}\,{\scriptstyle \to I,i} \qquad \begin{array}{c}\vdots\ \pi_2\\ A\end{array}}{B}\,{\scriptstyle \to E} \\ \vdots \\ C \end{array} \qquad \rightsquigarrow \qquad \begin{array}{c}\vdots\ \pi_2\\ A\\ \vdots\ \pi_1\\ B\\ \vdots\\ C\end{array}$$

If there is no π' such that $\pi \rightsquigarrow \pi'$, then π' is normal. If $\pi_0 \rightsquigarrow \pi_2 \rightsquigarrow \cdots \rightsquigarrow \pi_n$ we write "$\pi_0 \rightsquigarrow_* \pi_n$" and we say that π_0 reduces to π_n in a number of steps. We aim to show that for any proof π, there is some normal π^* such that $\pi \rightsquigarrow_* \pi^*$.

We allow that $\pi \rightsquigarrow_* \pi$. A proof 'reduces' to itself in *zero* steps.

The only difficult part in proving the normalisation theorem is showing that the process reduction can terminate in a normal proof. In the case where we do not allow duplicate discharge, there is no difficulty at all.

Proof [Theorem 1.13*: linear and affine cases*]: If π is a linear proof, or is an affine proof, then whenever you pick an indirect pair and normalise it, the result is a shorter proof. At most one copy of the proof π_2 for A is inserted into the proof π_1. (Perhaps no substitution is made in the case of an affine proof, if a vacuous discharge was made.) Proofs have some finite size, so this process cannot go on indefinitely. Keep deleting indirect pairs until there are no pairs left to delete. The result is a normal proof to the conclusion A. The premises X remain undisturbed, except in the affine case, where we may have lost premises along the way. (An assumption from π_2 might disappear if we did not need to make the substitution.) In this case, the premise multiset X' from the normal proof is a *sub*-multiset of X, as desired. ∎

If we allow duplicate discharge, however, we cannot be sure that in normalising we go from a larger to a smaller proof. The example on page 18 goes from a proof with 11 formulas to another proof with 11 formulas. In some cases a reduction step can take us from a smaller proof to a properly larger proof. Sometimes, the result is *much* larger. So size alone is no guarantee that the process terminates.

To gain some understanding of the general process of transforming a non-normal proof into a normal one, we must find some other measure that decreases as normalisation progresses. If this measure has a least value then we can be sure that the process will stop. The appropriate measure in this case will not be too difficult to find. Let's look at a part of the process of normalisation: the complexity of the formula that is normalised.

Well, the process stops if the measures are ordered appropriately—so that there's no *infinitely descending chain*.

definition 1.15 [complexity] A formula's *complexity* is the number of connectives in that formula. In this case, it is the number of instances of '$\to$' in the formula.

The crucial features of complexity are that each formula has a finite complexity, and that the proper subformulas of a formula each have a lower complexity than the original formula. This means that complexity is a good measure for an induction, like the size of a proof.

Now, suppose we have a proof containing just one indirect pair, introducing and eliminating $A \to B$, and suppose that otherwise, π_1 (the proof of B from A) and π_2 (the proof of A) are normal.

$$\text{BEFORE:}\quad \dfrac{\dfrac{\begin{matrix}[A]^{(i)}\\ \vdots\ \pi_1\\ B\end{matrix}}{A \to B}{\scriptstyle \to I,i} \qquad \begin{matrix}\vdots\ \pi_2\\ A\end{matrix}}{B}{\scriptstyle \to E} \qquad\qquad \text{AFTER:}\quad \begin{matrix}\vdots\ \pi_2\\ A\\ \vdots\ \pi_1\\ B\end{matrix}$$

This the new proof need not be necessarily normal, even though π_1 and π_2 are. The new proof is non-normal if π_2 ends in the introduction of A and π_1 starts off with the elimination of A. Notice, however, that the non-normality of the new proof is, somehow, *smaller*. There is no non-normality with respect to $A \to B$ or any other formula as complex as that. The potential non-normality is with respect to a subformula A. This result would still hold if the proofs π_1 and π_2 weren't normal themselves, but when they might have [$\to I/\!\to E$] pairs for formulas less complex than $A \to B$. If $A \to B$ is the most complex detour formula in the original proof, then the new proof has a *smaller* most complex detour formula.

DEFINITION 1.16 [NON-NORMALITY] The *non-normality measure* of a proof is a sequence $\langle c_1, c_2, \ldots, c_n\rangle$ of numbers such that c_i is the number of indirect pairs of formulas of complexity i. The sequence for a proof stops at the last non-zero value. Sequences are ordered with their last number as most significant. That is, $\langle c_1, \ldots, c_n\rangle > \langle d_1, \ldots, d_m\rangle$ if and only if $n > m$, or if $n = m$, when $c_n > d_n$, or if $c_n = d_n$, when $\langle c_1, \ldots, c_{n-1}\rangle > \langle d_1, \ldots, d_{n-1}\rangle$.

Non-normality measures satisfy the finite descending chain condition. Starting at any particular measure, you cannot find an infinite descending chain of measures below it. Of course, there are infinitely many measures smaller than $\langle 0, 1\rangle$ (in this case, $\langle 0\rangle, \langle 1\rangle, \langle 2\rangle, \ldots$). However, to form a *descending sequence* from $\langle 0, 1\rangle$ you must choose one of these as your next measure. Say you choose $\langle 500\rangle$. From that, you have only finitely many (500, in this case) steps until $\langle\rangle$ and the sequence stops. This generalises. From the sequence $\langle c_1, \ldots, c_n\rangle$, you lower c_n until it gets to zero. Then you look at the index for $n-1$, which might have grown enormously. Nonetheless, it is some finite number, and now you must reduce this value. And so on, until you reach the last quantity, and from there, the empty sequence $\langle\rangle$. Here is an example sequence using this ordering $\langle 3, 2, 30\rangle > \langle 2, 8, 23\rangle > \langle 1, 47, 15\rangle > \langle 138, 478\rangle > \cdots > \langle 1, 3088\rangle > \langle 314159\rangle > \cdots > \langle 1\rangle > \langle\rangle$.

LEMMA 1.17 [NON-NORMALITY REDUCTION] *Any a proof with an indirect pair reduces in one step to some proof with a lower measure of non-normality.*

Proof: Choose a detour formula in π of greatest complexity (say n), such that its proof contains no other detour formulas of complexity n. Normalise that proof. The result is a proof π' with fewer detour formulas of complexity n (and perhaps many more of $n-1$, etc.). So, it has a lower non-normality measure. ∎

Now we have a proof of our normalisation theorem.

Proof [of Theorem 1.13: *for the relevant and standard cases*]: Start with π, a proof that isn't normal, and use Lemma 1.17 to choose a proof π' with a lower measure of non-normality. If π' is normal, we're done. If it isn't, continue the process. There is no infinite descending chain of non-normality measures, so this process will stop at some point, and the result is a normal proof. ∎

Every proof may be transformed into a normal proof. If there is a linear proof from X to A then there is a normal linear proof from X to A. Linear proofs are satisfying and strict in this manner. If we allow vacuous discharge or duplicate discharge, matters are not so straightforward. For example, there is a non-normal standard proof from p, q to p:

$$\dfrac{\dfrac{p}{q \to p}{\scriptstyle \to I,1} \qquad q}{p}{\scriptstyle \to E}$$

but there is no normal standard proof from exactly these premises to the same conclusion, since any normal proof from atomic premises to an atomic conclusion must be an assumption alone. We have a normal proof from p to p (it is very short!), but there is no normal proof from p to p that involves q as an extra premise.

Similarly, there is a relevant proof from $p \to (p \to q), p$ to q, but it is non-normal.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \to (p \to q) \qquad [p]^{(1)}}{p \to q}{\scriptstyle \to E} \qquad [p]^{(1)}}{q}{\scriptstyle \to E}}{p \to q}{\scriptstyle \to I,1} \qquad p}{q}{\scriptstyle \to E}$$

There is no normal relevant proof from $p \to (p \to q), p$ to q. Any normal relevant proof from $p \to (p \to q)$ and p to q must use [$\to E$] to deduce $p \to q$, and then the only other possible move is either [$\to I$] (in which case we return to $p \to (p \to q)$ none the wiser) or we perform another [$\to E$] with another assumption p to deduce q, and we are done. Alas, we have claimed two undischarged assumptions of p. In the non-linear cases, the transformation from a non-normal to a normal proof does damage to the number of times a premise is used.

1.4 | STRONG NORMALISATION AND TERMS

It is very tempting to view normalisation as a process of reducing a proof down to its essence, of unwinding detours, and making explicit the essential logical connections made in the proof between the premises and the conclusion. The result of normalising a proof π from X to A shows the connections made from X to A in that proof π, without the need to bring in the extraneous information in any detours that may have been used in π. Another analogy is that the complex non-normal proof is *evaluated* into its normal form, in the same way that a numerical term like $5 + (2 \times (7 + 3))$ is evaluated into *its* normal form, the numeral 25.

This passage is the hardest part of Chapter 1. Feel free to skip over the proofs of theorems in this section, until page 38 on first reading.

If this is the case, then the process of normalisation should give us two distinct "answers" for the underlying structure of the one proof. Can two different reduction sequences for a single proof result in *different* normal proofs? To investigate this, we need to pay attention to the different processes of reduction we can take when reducing a proof. To do that, we'll introduce a new notion of reduction:

DEFINITION 1.18 [PARALLEL REDUCTION] A proof π *parallel reduces* to π' if some number of indirect pairs in π are eliminated in parallel. We write "$\pi \mathrel{\substack{\rightsquigarrow\\\rightsquigarrow}} \pi'$."

For example, consider the proof with the following two detour formulas marked:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (A \to B) \quad [A]^{(1)}}{A \to B}{\scriptstyle \to E} \quad [A]^{(1)}}{B}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to I,1} \qquad \dfrac{\dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2} \quad A}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}$$

To process them we can take them in any order. Eliminating the $A \to B$, we have

$$\dfrac{\dfrac{A \to (A \to B) \quad \dfrac{\dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2} \quad A}{A}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to E} \qquad \dfrac{\dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2} \quad A}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}$$

which now has two copies of the $A \to A$ to be reduced. However, these copies do not overlap in scope (they cannot, as they are duplicated in the place of assumptions discharged in an eliminated $\to I$ rule) so they can be processed together. The result is the proof

$$\dfrac{\dfrac{A \to (A \to B) \quad A}{A \to B}{\scriptstyle \to E} \quad A}{B}{\scriptstyle \to E}$$

You can check that if you had processed the formulas to be eliminated in the other order, the result would have been the same.

Can you see why this is called the diamond property?

LEMMA 1.19 [DIAMOND PROPERTY FOR $\mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}}$] *If $\pi \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi_1$ and $\pi \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi_2$ then there is some proof π' where $\pi_1 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi'$ and $\pi_2 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi'$.*

Proof: Take the detour formulas in the proof π that are eliminated in either the move to π_1 or the move to π_2. 'Colour' them in π, and transform the proof to π_1. Some of the coloured formulas may remain. Do the same in the move from π to π_2. The result are two proofs π_1 and π_2 in which some formulas may be coloured. The proof π' is found by parallel reducing either collection of formulas in π_1 or π_2. ■

They may have multiplied, if they occurred in a proof part duplicated in the reduction step. But some may have vanished, too, if they were in a part of the proof that disappeared during reduction.

THEOREM 1.20 [ONLY ONE NORMAL FORM] *Given any proof π, if $\pi \rightsquigarrow_* \pi'$ then if $\pi \rightsquigarrow_* \pi''$, it must be that $\pi' = \pi''$. That is, any sequence of reduction steps from π that terminates in a normal form must terminate in a unique normal form.*

Proof: Suppose that $\pi \rightsquigarrow_* \pi'$, and $\pi \rightsquigarrow_* \pi''$. It follows that we have two reduction sequences

$$\pi \overset{\rightsquigarrow}{\rightsquigarrow} \pi'_1 \overset{\rightsquigarrow}{\rightsquigarrow} \pi'_2 \overset{\rightsquigarrow}{\rightsquigarrow} \cdots \overset{\rightsquigarrow}{\rightsquigarrow} \pi'_n \overset{\rightsquigarrow}{\rightsquigarrow} \pi'$$

$$\pi \overset{\rightsquigarrow}{\rightsquigarrow} \pi''_1 \overset{\rightsquigarrow}{\rightsquigarrow} \pi''_2 \overset{\rightsquigarrow}{\rightsquigarrow} \cdots \overset{\rightsquigarrow}{\rightsquigarrow} \pi''_m \overset{\rightsquigarrow}{\rightsquigarrow} \pi''$$

By the diamond property, we have a $\pi_{1,1}$ where $\pi'_1 \overset{\rightsquigarrow}{\rightsquigarrow} \pi_{1,1}$ and $\pi''_1 \overset{\rightsquigarrow}{\rightsquigarrow} \pi_{1,1}$. Then $\pi''_1 \overset{\rightsquigarrow}{\rightsquigarrow} \pi_{1,1}$ and $\pi''_1 \overset{\rightsquigarrow}{\rightsquigarrow} \pi''_2$ so by the diamond property there is some $\pi_{2,1}$ where $\pi''_2 \overset{\rightsquigarrow}{\rightsquigarrow} \pi_{2,1}$ and $\pi_{1,1} \overset{\rightsquigarrow}{\rightsquigarrow} \pi_{2,1}$. Continue in this vein, guided by the picture below:

$$\begin{array}{ccccccccc}
\pi & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi'_1 & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi'_2 & \overset{\rightsquigarrow}{\rightsquigarrow} & \cdots & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi'_n \\
\wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr \\
\pi''_1 & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi_{1,1} & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi_{1,2} & \overset{\rightsquigarrow}{\rightsquigarrow} & \cdots & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi_{1,n} \\
\wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr \\
\pi''_2 & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi_{2,1} & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi_{2,2} & \overset{\rightsquigarrow}{\rightsquigarrow} & \cdots & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi_{2,n} \\
\wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr \\
\pi''_m & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi_{m,1} & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi_{m,2} & \overset{\rightsquigarrow}{\rightsquigarrow} & \cdots & \overset{\rightsquigarrow}{\rightsquigarrow} & \pi^*
\end{array}$$

to find the desired proof π^*. So, if π'_n and π''_n are *normal* they must be identical. ∎

So, sequences of reductions from π cannot terminate in two different proofs. A normal form for a proof is unique.

This result goes a lot of the way towards justifying the idea that normalisation corresponds to evaluating the underlying essence of a proof. The normal form is well defined an unique. But this leaves us with a remaining question. We have seen that for each proof π there is some process to evaluate its normal form π^*, and further, the proof of the previous theorem shows us that *any* finite sequence of reductions from π can be extended to eventually reach π^*. Does it follow that any process of reductions from π terminates in its normal form π^*? That is: are our proofs *strongly normalising*?

DEFINITION 1.21 [STRONGLY NORMALISING] A proof π is strongly normalising (under a reduction relation $\rightsquigarrow$) if and only if there is no infinite reduction sequence starting from π.

This does not follow from weak normalisation (there is some reduction to a normal form) and the diamond property, which gives us unique normal form theorem. This is straightforward to see, because a "reduction" process which allows us to run reduction steps backwards as well as forwards if the proof is not already normal, would still allow for weak normalisation, would still have the diamond property and would have a unique normal form. But it would not be strongly normalising. (We could go on forever reducing one detour only to put it back, forever.) Is there any guarantee that our reduction process process will always terminate?

A naive approach would be to define some measure on proofs which always reduces under any reduction step. This seems hopeless, because anything like the measure we have already defined can increase, rather than decrease, under reductions. (Take a proof with a small detour formula $A \to B$ where the assumption A is discharged a number of times in the proof of the major premise $A \to B$, and in which there are larger detour formulas in the proof of the minor premise A. This proof is duplicated in the reduction, and the measure of the new proof could rise significantly, as we have eliminated a small detour formula at the cost of many large detour formulas.)

We will prove that every proof is strongly normalising under the relation $\rightsquigarrow$ of deleting detour formulas. To assist in talking about this, we need to make a few more definitions. First, the *reduction tree*.

DEFINITION 1.22 [REDUCTION TREE] The reduction tree (under $\rightsquigarrow$) of a proof π is the tree whose branches are the reduction sequences on the relation $\rightsquigarrow$. So, from the root π we reach any proof accessible in one $\rightsquigarrow$ step from π. From each π' where $\pi \rightsquigarrow \pi'$, we branch similarly. Each node has only finitely many successors as there are only finitely many detour formulas in a proof. For each proof π, $\nu(\pi)$ is the size of its reduction tree.

LEMMA 1.23 [THE SIZE OF REDUCTION TREES] *A strongly normalising proof has a finite reduction tree. It follows that not only is every reduction path finite, but there is a longest reduction path.*

It's true that every finitely branching tree with finite branches is finite. But is it *obvious* that it's true?

Proof: This is a corollary of König's Lemma, which states that every tree in which the number of immediate descendants of a node is finite (it is finitely *branching*), and in which every branch is finitely long, is itself *finite*. Since the reduction tree for a strongly normalising proof is finitely branching, and each branch has a finite length, it follows that any strongly normalising proof not only has only finite reduction paths, it also has a *longest* reduction path. ∎

Now to prove that every proof is strongly normalising. To do this, we define a new property that proofs can have: of being **red**. It will turn out that all **red** proofs are strongly normalising. It will also turn out that all proofs are **red**.

The term '**red**' should bring to mind 'reducible.' This formulation of strong normalisation is originally due to William Tait [113]. I am following the presentation of Jean-Yves Girard [46, 48].

DEFINITION 1.24 [**red** PROOFS] We define a new predicate '**red**' applying to proofs in the following way.

» A proof of an atomic formula is **red** if and only if it is strongly normalising.

» A proof π of an implication formula $A \to B$ is **red** if and only if whenever π' is a **red** proof of A, then the proof

$$\dfrac{\begin{array}{c}\vdots\,\pi \\ A \to B\end{array} \quad \begin{array}{c}\vdots\,\pi' \\ A\end{array}}{B}$$

is a **red** proof of type B.

We will have cause to talk often of the proof found by extending a proof π of $A \to B$ and a proof π' of A to form the proof of B by adding an $\to E$ step. We will write '$(\pi\,\pi')$' to denote this proof. If you like, you can think of it as the application of the proof π to the proof π'.

Now, our aim will be twofold: to show that every **red** proof is strongly normalising, and to show that every proof is **red**. We start by proving the following crucial lemma:

LEMMA 1.25 [PROPERTIES OF **red** PROOFS] *For any proof π, the following three conditions hold:*

C1 *If π is* **red** *then π is strongly normalisable.*

C2 *If π is* **red** *and π reduces to π' in one step, then π' is* **red** *too.*

C3 *If π is a proof not ending in $\to I$, and whenever we eliminate one indirect pair in π we have a* **red** *proof, then π is* **red** *too.*

Proof: We prove this result by induction on the formula proved by π. We start with proofs of atomic formulas.

C1 Any **red** proof of an atomic formula is strongly normalising, by the definition of '**red**'.

C2 If π is strongly normalising, then so is any proof to which π reduces.

C3 π does not end in $\to I$ as it is a proof of an atomic formula. If whenever $\pi \Rightarrow_1 \pi'$ and π' is **red**, since π' is a proof of an atomic formula, it is strongly normalising. Since *any* reduction path through π must travel through one such proof π', each such path through π terminates. So, π is **red**.

Now we prove the results for a proof π of $A \to B$, under the assumption that C1, C2 and C3 they hold for proofs of A and proofs of B. We can then conclude that they hold of *all* proofs, by induction on the complexity of the formula proved.

C1 If π is a **red** proof of $A \to B$, consider the proof

$$\sigma : \quad \dfrac{\begin{array}{c}\vdots\,\pi \\ A \to B\end{array} \quad A}{B}$$

The assumption A is a normal proof of its conclusion A not ending in $\rightarrow I$, so C3 applies and it is **red**. So, by the definition of **red** proofs of implication formulas, σ is a **red** proof of B. Condition C1 tells us that **red** proofs of B are strongly normalising, so any reduction sequence for σ must terminate. It follows that any reduction sequence for π must terminate too, since if we had a non-terminating reduction sequence for π, we could apply the same reductions to the proof σ. But since σ is strongly normalising, this cannot happen. It follows that π is strongly normalising too.

C2 Suppose that π reduces in one step to a proof π'. Given that π is **red**, we wish to show that π' is **red** too. Since π' is a proof of $A \rightarrow B$, we want to show that for any **red** proof π'' of A, the proof $(\pi'\,\pi'')$ is **red**. But this proof is **red** since the **red** proof $(\pi\,\pi'')$ reduces to $(\pi'\,\pi'')$ in one step (by reducing π to π'), and C2 applies to proofs of B.

C3 Suppose that π does not end in $[\rightarrow I]$, and suppose that all of the proofs reached from π in one step are **red**. Let σ be a **red** proof of A. We wish to show that the proof $(\pi\,\sigma)$ is **red**. By C1 for the formula A, we know that σ is strongly normalising. So, we may reason by induction on the length of the longest reduction path for σ. If σ is normal (with path of length 0), then $(\pi\,\sigma)$ reduces in one step only to $(\pi'\,\sigma)$, with π' one step from π. But π' is **red** so $(\pi'\,\sigma)$ is too.

On the other hand, suppose σ is not yet normal, but the result holds for all σ' with shorter reduction paths than σ. So, suppose τ reduces to $(\pi\,\sigma')$ with σ' one step from σ. σ' is **red** by the induction hypothesis C2 for A, and σ' has a shorter reduction path, so the induction hypothesis for σ' tells us that $(\pi\,\sigma')$ is **red**.

There is no other possibility for reduction as π does not end in $\rightarrow I$, so reductions must occur wholly in π or wholly in σ, and not in the last step of $(\pi\,\sigma)$.

This completes the proof by induction. The conditions C1, C2 and C3 hold of every proof. ■

Now we prove one more crucial lemma.

LEMMA 1.26 [**red** PROOFS ENDING IN $[\rightarrow I]$] *If for each* **red** *proof σ of A, the proof*

$$\pi(\sigma):\quad \begin{array}{c} \vdots\,\sigma \\ A \\ \vdots\,\pi \\ B \end{array}$$

is **red**, *then so is the proof*

$$\tau:\quad \dfrac{\begin{array}{c} [A] \\ \vdots\,\pi \\ B \end{array}}{A \rightarrow B}\,{\scriptstyle \rightarrow I}$$

Proof: We show that the $(\tau\,\sigma)$ is **red** whenever σ is **red**. This will suffice to show that the proof τ is **red**, by the definition of the predicate '**red**' for proofs of $A \rightarrow B$. We will show that every proof resulting from $(\tau\,\sigma)$ in one step is **red**, and we will reason by induction on the sum of the sizes of the reduction trees of π and σ. There are three cases:

» $(\tau\,\sigma) \rightsquigarrow \pi(\sigma)$. In this case, $\pi(\sigma)$ is **red** by the hypothesis of the proof.

» $(\tau\,\sigma) \rightsquigarrow (\tau'\,\sigma)$. In this case the sum of the size of the reduction trees of τ' and σ is smaller, and we may appeal to the induction hypothesis.

» $(\tau\,\sigma) \rightsquigarrow (\tau\,\sigma')$. In this case the sum of the size of the reduction trees is τ and σ' smaller, and we may appeal to the induction hypothesis. ■

We are set to prove our major theorem:

THEOREM 1.27 [ALL PROOFS ARE **red**] *Every proof π is* **red**.

To do this, we'll approach it by induction, as follows:

LEMMA 1.28 [**red** PROOFS BY INDUCTION] *For each proof π with assumptions $A_1, \ldots, A_n$, and for any* **red** *proofs $\sigma_1, \ldots, \sigma_n$ of the formulas $A_1, \ldots, A_n$ respectively, the proof $\pi(\sigma_1, \ldots, \sigma_n)$ in which each assumption A_i is replaced by the proof σ_i is* **red**.

Proof: We prove this by induction on the construction of the proof.

» If π is an assumption A_1, the claim is a tautology (if σ_1 is **red**, then σ_1 is **red**).

» If π ends in [$\rightarrow E$], and is $(\pi_1\,\pi_2)$, then by the induction hypothesis $\pi_1(\sigma_1, \ldots, \sigma_n)$ and $\pi_2(\sigma_1, \ldots, \sigma_n)$ are **red**. Since $\pi_1(\sigma_1, \ldots, \sigma_n)$ has type $A \rightarrow B$ the definition of **red**ness tells us that when ever it is applied to a **red** proof the result is also **red**. Therefore, the proof $(\pi_1(\sigma_1, \ldots, \sigma_n)\,\pi_2(\sigma_1, \ldots, \sigma_n))$ is **red**, but this proof is simply $\pi(\sigma_1, \ldots, \sigma_n)$.

» If π ends in an application of [$\rightarrow I$], then this case is dealt with by Lemma 1.26: if π is a proof of $A \rightarrow B$ ending in $\rightarrow E$, then we may assume that π', the proof of B from A inside π is **red**, so by Lemma 1.26, the result π is **red** too.

It follows that *every* proof is **red**. ■

It follows also that every proof is strongly normalising, since all **red** proofs are strongly normalising.

» «

It is very tempting to think of proofs as *processes* or *functions* that convert the information presented in the premises into the information in the conclusion. This is doubly tempting when you look at the notation for implication. In $\to E$ we apply something which converts A to B (a function from A to B?) to something which delivers you A (from premises) into something which delivers you B. In $\to I$ if we can produce B (when supplied with A, at least in the presence of other resources—the other premises) then we can (in the context of the other resources at least) convert As into Bs at will.

Let's make this talk a little more precise, by making *explicit* this kind of *function*-talk. It will give us a new vocabulary to talk of proofs.

We start with simple notation to talk about functions. The idea is straightforward. Consider numbers, and addition. If you have a number, you can add 2 to it, and the result is another number. If you like, if x is a number then

$$x + 2$$

is another number. Now, suppose we don't want to talk about a particular number, like $5+2$ or $7+2$ or $x+2$ for any choice of x, but we want to talk about the *operation* or of adding two. There is a sense in which just writing "$x + 2$" should be enough to tell someone what we mean. It is relatively clear that we are treating the "x" as a marker for the input of the function, and "$x + 2$" is the output. The *function* is the output as it varies for different values of the input. Sometimes leaving the variables there is not so useful. Consider the subtraction

$$x - y$$

You can think of this as the function that takes the input value x and takes away y. Or you can think of it as the function that takes the input value y and subtracts it from x. or you can think of it as the function that takes two input values x and y, and takes the second away from the first. Which do we mean? When we apply this function to the input value 5, what is the result? For this reason, we have a way of making explicit the different distinctions: it is the λ-notation, due to Alonzo Church [23]. The function that takes the input value x and returns $x + 2$ is denoted

$$\lambda x.(x + 2)$$

The function taking the input value y and subtracts it from x is

$$\lambda y.(x - y)$$

The function that takes *two* inputs and subtracts the second from the first is

$$\lambda x.\lambda y.(x - y)$$

Notice how this function works. If you feed it the input 5, you get the output $\lambda y.(5 - y)$. We can write *application* of a function to its input by way of juxtaposition. The result is that

$$(\lambda x.\lambda y.(x - y)\, 5)$$

evaluates to the result $\lambda y.(5 - y)$. This is the function that subtracts y from 5. When you feed *this* function the input 2 (i.e., you evaluate $(\lambda y.(5 - y)\ 2)$) the result is $5 - 2$ — in other words, 3. So, functions can have other functions as outputs.

Now, suppose you have a function f that takes two inputs y and z, and we wish to consider what happens when you apply f to a pair where the first value is the repeated as the second value. (If f is $\lambda x.\lambda y.(x - y)$ and the input value is a number, then the result should be 0.) We can do this by applying f to the value x twice, to get $((f\,x)\,x)$. But this is not a function, it is the result of applying f to x and x. If you consider this as a function of x you get

$$\lambda x.((f\,x)\,x)$$

This is the function that takes x and feeds it *twice* into f. But just as functions can create other functions as *outputs*, there is no reason not to make functions take other functions as *inputs*. The process here was competely general — we knew nothing specific about f — so the function

$$\lambda y.\lambda x.((y\,x)\,x)$$

takes an input y, and returns the function $\lambda x.((y\,x)\,x)$. This function takes an input x, and then applies y to x and then applies the result to x again. When you feed it a function, it returns the *diagonal* of that function.

Draw the function as a table of values for each pair of inputs, and you will see why this is called the '*diagonal*.'

Now, sometimes this construction does not work. Suppose we feed our diagonal function $\lambda y.\lambda x.((y\,x)\,x)$ an input that is not a function, or that is a function that does not expect two inputs? (That is, it is not a function that returns another function.) In that case, we may not get a sensible output. One response is to bite the bullet and say that everything is a function, and that we can apply anything to anything else. We won't take that approach here, as something becomes very interesting if we consider what happens if we consider variables (the x and y in the expression $\lambda y.\lambda x.((y\,x)\,x)$) to be *typed*. We could consider y to only take inputs which are functions of the right kind. That is, y is a function that expects values of some kind (let's say, of type A), and when given a value, returns a function. In fact, the function it returns has to be a function that expects values of the very same kind (also type A). The *result* is an object (perhaps a function) of some kind or other (say, type B). In other words, we can say that the variable y takes values of type $A \to (A \to B)$. Then we expect the variable x to take values of type A. We'll write these facts as follows:

This is the *untyped* λ-calculus.

$$y : A \to (A \to B) \qquad x : A$$

Now, we may put these two things together, to say derive the type of the result of applying the function y to the input value x.

$$\frac{y : A \to (A \to B) \quad x : A}{(y\,x) : A \to B}$$

Applying the result to x again, we get

$$\dfrac{\dfrac{y : A \to (A \to B) \quad x : A}{(y\,x) : A \to B} \qquad x : A}{((y\,x)\,x) : B}$$

Then when we abstract away the particular choice of the input value x, we have this

$$\dfrac{\dfrac{\dfrac{y : A \to (A \to B) \quad [x : A]}{(y\,x) : A \to B} \qquad [x : A]}{((y\,x)\,x) : B}}{\lambda x.((y\,x)\,x) : A \to B}$$

and abstracting away the choice of y, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{[y : A \to (A \to B)] \quad [x : A]}{(y\,x) : A \to B} \qquad [x : A]}{((y\,x)\,x) : B}}{\lambda x.((y\,x)\,x) : A \to B}}{\lambda y.\lambda x.((y\,x)\,x) : (A \to (A \to B)) \to (A \to B)}$$

so the diagonal function $\lambda y.\lambda x.((y\,x)\,x)$ has type $(A \to (A \to B)) \to (A \to B)$. It takes functions of type $A \to (A \to B)$ as input and returns an output of type $A \to B$.

Does that process look like something you have already seen?

We may use these λ-terms to represent proofs. Here are the definitions. We will first think of formulas as *types*.

$$\textsc{type} ::= \textsc{atom} \mid (\textsc{type} \to \textsc{type})$$

Then, given the class of types, we can construct terms for each type.

DEFINITION 1.29 [TYPED SIMPLE λ-TERMS] The class of typed simple λ-terms is defined as follows:

- For each type A, there is an infinite supply of variables x^A, y^A, z^A, w^A, x_1^A, x_2^A, etc.
- If M is a term of type $A \to B$ and N is a term of type A, then $(M\,N)$ is a term of type B.
- If M is a term of type B then $\lambda x^A.M$ is a term of type $A \to B$.

These formation rules for types may be represented in ways familiar to those of us who care for proofs. See Figure 1.3.

Sometimes we write variables without superscripts, and leave the typing of the variable understood from the context. It is simpler to write $\lambda y.\lambda x.((y\,x)\,x)$ instead of $\lambda y^{A\to(A\to B)}.\lambda x^A((y^{A\to(A\to B)}\,x^A)\,x^A)$.

$$\dfrac{M : A \to B \quad N : A}{(M\,N) : B}\,{\to}E \qquad \dfrac{\begin{matrix}[x : A]^{(i)} \\ \vdots \\ M : B\end{matrix}}{\lambda x.M : A \to B}\,{\to}I,i$$

Figure 1.3: RULES FOR λ-TERMS

Not everything that *looks* like a typed λ-term actually is. Consider the term

$$\lambda x.(x\,x)$$

There is no such simple typed λ-term. Were there such a term, then x would have to both have type $A \to B$ and type A. But as things stand now, a variable can have only one type. Not every λ-term is a *typed* λ-term.

Now, it is clear that typed λ-terms stand in some interesting relationship to proofs. From any typed λ-term we can reconstruct a unique proof.Take $\lambda x.\lambda y.(y\,x)$, where y has type $p \to q$ and x has type p. We can rewrite the unique formation pedigree of the term as a tree.

$$\dfrac{\dfrac{\dfrac{[y : p \to q] \quad [x : p]}{(y\,x) : q}}{\lambda y.(y\,x) : (p \to q) \to q}}{\lambda x.\lambda y.(y\,x) : p \to ((p \to q) \to q)}$$

and once we erase the terms, we have a proof of $p \to ((p \to q) \to q)$. The term is a compact, linear representation of the proof which is presented as a tree.

The mapping from terms to proofs is many-to-one. Each typed term constructs a single proof, but there are many different terms for the one proof. Consider the proofs

$$\dfrac{p \to q \quad p}{q} \qquad \dfrac{p \to (q \to r) \quad p}{(q \to r)}$$

we can label them as follows

$$\dfrac{x : p \to q \quad y : p}{(xy) : q} \qquad \dfrac{z : p \to (q \to r) \quad y : p}{(zy) : q \to r}$$

we could combine them into the proof

$$\dfrac{\dfrac{z : p \to (q \to r) \quad y : p}{(zy) : q \to r} \quad \dfrac{x : p \to q \quad y : p}{(xy) : q}}{(zy)(xy) : r}$$

but if we wished to discharge just *one* of the instances of p, we would have to have chosen a different term for one of the two subproofs. We could have chosen the variable w for the first p, and used the following term:

$$\dfrac{\dfrac{z : p \to (q \to r) \quad [w : p]}{(zw) : q \to r} \quad \dfrac{x : p \to q \quad y : p}{(xy) : q}}{\dfrac{(zw)(xy) : r}{\lambda w.(zw)(xy) : p \to r}}$$

So, the choice of variables allows us a great deal of choice in the construction of a term for a proof. The choice of variables both does *not* matter (who cares if we replace x^A by y^A) and *does* matter (when it comes to discharge an assumption, the formulas discharged are exactly those labelled by the particular free variable bound by λ at that stage).

DEFINITION 1.30 [FROM TERMS TO PROOFS AND BACK] For every typed term M (of type A), we find $\text{PROOF}(M)$ (of the formula A) as follows:

» $\text{PROOF}(x^A)$ is the identity proof A.

» If $\text{PROOF}(M^{A \to B})$ is the proof π_1 of $A \to B$ and $\text{PROOF}(N^A)$ is the proof π_2 of A, then extend them with one [$\to$*E*] step into the proof $\text{PROOF}(MN^B)$ of B.

» If $\text{PROOF}(M^B)$ is a proof π of B and x^A is a variable of type A, then construct the proof $\text{PROOF}((\lambda x.M)^{A \to B})$ of type $A \to B$ as follows: Extend the proof π by discharging each premise in π of type A labelled with the variable x^A.

Conversely, for any proof π, we find the set $\text{TERMS}(\pi)$ as follows:

» $\text{TERMS}(A)$ is the set of variables of type A. (Note that the term is an unbound variable, whose type is the only assumption in the proof.)

» If π_l is a proof of $A \to B$, and M (of type $A \to B$) is a member of $\text{TERMS}(\pi_l)$, and N (of type A) is a member of $\text{TERMS}(\pi_r)$, then (MN) (which is of type B) is a member of $\text{TERMS}(\pi)$, where π is the proof found by extending π_l and π_r by the [$\to$*E*] step. (Note that if the unbound variables in M have types corresponding to the assumptions in π_l and those in N have types corresponding to the assumptions in π_r, then the unbound variables in (MN) have types corresponding to the variables in π.)

» Suppose π is a proof of B, and we extend π into the proof π' by discharging some set (possibly empty) of instances of the formula A, to derive $A \to B$ using [$\to$*I*]. Then M is a member of $\text{TERMS}(\pi)$ for which a variable x labels *all* and *only* those assumptions A that are discharged in this [$\to$*I*] step, then $\lambda x.M$ is a member of $\text{TERMS}(\pi')$. (Notice that the free variables in $\lambda x.M$ correspond to the remaining active assumptions in π'.)

THEOREM 1.31 [TERMS ARE PROOFS ARE TERMS] *If* $M \in$ TERMS(π) *then* $\pi =$ PROOF(M). *Conversely,* $M' \in$ TERMS(PROOF(M)) *if and only if* M' *is a relabelling of* M.

Proof: For the first part, we proceed by induction on the proof π. If π is an atomic proof, then since TERMS(A) is the set of variables of type A, and since PROOF(x^A) is the identity proof A, we have the base case of the induction. If π is composed of two proofs, π_l of $A \rightarrow B$, and π_r of A, joined by an $[\rightarrow E]$ step, then M is in TERMS(π) if and only if $M = (N_1 N_2)$ where $N_1 \in$ TERMS(π_l) and $N_2 \in$ TERMS(π_r). But by the induction hypothesis, if $N_1 \in$ TERMS(π_l) and $N_2 \in$ TERMS(π_r), then $\pi_l =$ PROOF(N_1) and $\pi_r =$ PROOF(N_2), and as a result, $\pi =$ PROOF(M), as desired.

Finally, if π is a proof of B, extended to the proof π' of $A \rightarrow B$ by discharging some (possibly empty) set of instances of A, then if M is in TERMS(π) if and only if $M = \lambda x.N$, $N \in$ TERMS(π'), and x labels those (and only those) instances of A discharged in π. By the induction hypothesis, $\pi' =$ PROOF(N). It follows that $\pi =$ PROOF$(\lambda x.N)$, since x labels all and only the formulas discharged in the step from π' to π.

For the second part of the proof, if $M' \in$ TERMS(PROOF(M)), then if M is a variable, PROOF(M) is an identity proof of some formula A, and TERMS(PROOF(M)) is a variable with type A, so the base case of our hypothesis is proved. Suppose the hypothesis holds for terms simpler than our term M. If M is an application term $(N_1 N_2)$, then PROOF$(N_1 N_2)$ ends in $[\rightarrow E]$, and the two subproofs are PROOF(N_1) and (N_2) respectively. By hypothesis, TERM(PROOF(N_1)) is some relabelling of N_1 and TERM(PROOF(N_2)) is some relabelling of N_2, so TERM(PROOF$(N_1 N_2)$) may only be relabelling of $(N_1 N_2)$ as well. Similarly, if M is an abstraction term $\lambda x.N$, then PROOF$(\lambda x.N)$ ends in $[\rightarrow I]$ to prove some conditional $A \rightarrow B$, and PROOF(N) is a proof of B, in which some (possibly empty) collection of instances of A are about to be discharged. By hypothesis, TERM(PROOF(N)) is a relabelling of N, so TERM(PROOF$(\lambda x.N)$) can only be a relabelling of $\lambda x.N$. ∎

The following theorem shows that the λ-terms of different kinds of proofs have different features.

THEOREM 1.32 [DISCHARGE CONDITIONS AND TERMS] M *is a linear* λ*-term (a term of some linear proof) iff each* λ *expression in* M *binds exactly one variable.* M *is a relevant* λ*-term (a term of a relevant proof) iff each* λ *expression in* M *binds at least one variable.* M *is a an affine* λ*-term (a term of some affine proof) iff each* λ *expression binds at most one variable.*

Proof: Check the definition of PROOF(M). If M satisfies the conditions on variable binding, PROOF(M) satisfies the corresponding discharge conditions. Conversely, if π satisfies a discharge condition, the terms in TERM(π) are the corresponding kinds of λ-term. ∎

The most interesting connection between proofs and λ-terms is not simply this pair of mappings. It is the connection between *normalisation* and *evaluation*. We have seen how the application of a function, like $\lambda x.((y\,x)\,x)$ to an input like M is found by removing the lambda binder, and substituting the term M for each variable x that was bound by the binder. In this case, we get $((y\,M)\,M)$.

DEFINITION 1.33 [β REDUCTION] The term $\lambda x.M\,N$ is said to directly β-reduce to the term $M[x := N]$ found by substituting the term N for each free occurrence of x in M.

Furthermore, M β-reduces in one step to M' if and only if some subterm N inside M immediately β-reduces to N' and $M' = M[N := N']$. A term M is said to β-reduce to M^* if there is some chain $M = M_1, \cdots M_n = M^*$ where each M_i β-reduces in one step to M_{i+1}.

Consider what this means for *proofs*. The term $(\lambda x.M\,N)$ immediately β-reduces to $M[x := N]$. Representing this transformation as a proof, we have

$$\dfrac{\dfrac{\begin{array}{c}[x:A]\\ \vdots\,\pi_l\\ M:B\end{array}}{\lambda x.M : A \to B} \quad \begin{array}{c}\vdots\,\pi_r\\ N:A\end{array}}{(\lambda x.M\,N):B} \quad \Longrightarrow^{\beta} \quad \begin{array}{c}\vdots\,\pi_r\\ N:A\\ \vdots\,\pi_l\\ M[x:=N]:B\end{array}$$

and β-reduction corresponds to normalisation. This fact leads immediately to the following theorem.

THEOREM 1.34 [NORMALISATION AND β-REDUCTION] *A proof* PROOF(N) *is normal if and only if the term* N *does not* β*-reduce to another term. If* N β*-reduces to* N' *then a normalisation process sends* PROOF(N) *to* PROOF(N').

This natural reading of normalisation as function application, and the easy way that we think of $(\lambda x.M\,N)$ as *being identical to* $M[x := N]$ leads some to make the following claim:

If π and π' *normalise* to the same proof,
then π and π' are *really* the same proof.

We will discuss proposals for the identity of proofs in a later section.

1.5 | HISTORY

Gentzen's technique for natural deduction is not the only way to represent this kind of reasoning, with introduction and elimination rules for connectives. Independently of Gentzen, the Polish logician, Stanisław Jaśkowski constructed a closely related, but different system for presenting proofs in a natural deduction style. In Jaśkowski's system, a proof is a *structured list* of formulas. Each formula in the list is either a *supposition*, or it follows from earlier formulas in the list by means of the rule of *modus ponens* (conditional elimination), or it is proved by *conditionalisation*. To prove something by conditionalisation you first make a supposition of the antecedent: at this point you start a *box*. The contents of a box constitute a proof, so if you want to use a formula from outside the box, you may *repeat* a formula into the inside. A conditionalisation step allows you to exit the box, discharging the supposition you made upon entry. Boxes can be nested, as follows:

1.	$A \to (A \to B)$	Supposition
2.	A	Supposition
3.	$A \to (A \to B)$	1, Repeat
4.	$A \to B$	2, 3, Modus Ponens
5.	B	2, 4, Modus Ponens
6.	$A \to B$	2–5, Conditionalisation
7.	$(A \to (A \to B)) \to (A \to B)$	1–6, Conditionalisation

This nesting of boxes, and repeating or reiteration of formulas to enter boxes, is the distinctive feature of Jaśkowski's system. Notice that we could prove the formula $(A \to (A \to B)) \to (A \to B)$ without using a duplicate discharge. The formula A is used twice as a minor premise in a Modus Ponens inference (on line 4, and on line 5), and it is then discharged at line 6. In a Gentzen proof of the same formula, the assumption A would have to be made twice.

Jaśkowski proofs also straightforwardly incorporate the effects of a vacuous discharge in a Gentzen proof. We can prove $A \to (B \to A)$ using the rules as they stand, without making any special plea for a vacuous discharge:

1.	A	Supposition
2.	B	Supposition
3.	A	1, Repeat
4.	$B \to A$	2–3, Conditionalisation
5.	$A \to (B \to A)$	1–4, Conditionalisation

The formula B is supposed, and it is not *used* in the proof that follows. The formula A on line 4 occurs *after* the formula B on line 3, in the subproof, but it is harder to see that it is inferred *from* that$B$. Conditionalisation, in Jaśkowski's system, colludes with reiteration to allow the effect of vacuous discharge. It appears that the "fine control" over inferential connections between formulas in proofs in a Gentzen proof is somewhat obscured in the *linearisation* of a Jaśkowski proof. The fact that one formula occurs *after* another says nothing about how that formula is inferentially connected to its forbear.

Jaśkowski's account of proof was modified in presentation by Frederic Fitch (boxes become assumption *lines* to the left, and hence become somewhat simpler to draw and to typeset). Fitch's natural deduction ststem gained quite some popularity in undergraduate education in logic in the 1960s and following decades in the United States [41]. Edward Lemmon's text *Beginning Logic* [65] served a similar purpose in British logic education. Lemmon's account of natural deduction is similar to this, except that it does without the need to reiterate by *breaking the box*.

1	(1)	$A \to (A \to B)$	Assumption
2	(2)	A	Assumption
1,2	(3)	$A \to B$	1, 2, Modus Ponens
1,2	(4)	B	2,3, Modus Ponens
1	(5)	$A \to B$	2, 4, Conditionalisation
	(6)	B	1, 5, Conditionalisation

Now, line numbers are joined by *assumption numbers*: each formula is tagged with the line number of each assumption upon which that formula depends. The rules for the conditional are straightforward: If $A \to B$ depends on the assumptions X and A depends on the assumptions Y, then you can derive B, depending on the assumptions X, Y. (You should ask yourself if X, Y is the *set* union of the *sets* X and Y, or the *multiset* union of the *multisets* X and Y. For Lemmon, the assumption collections are *sets*.) For conditionalisation, if B depends on X, A, then you can derive $A \to B$ on the basis of X alone. As you can see, vacuous discharge is harder to motivate, as the rules stand now. If we attempt to use the strategy of the Jaśkowski proof, we are soon stuck:

1	(1)	A	Assumption
2	(2)	B	Assumption
$\vdots$	(3)	$\vdots$	

There is no way to attach the assumption number "2" to the formula A. The linear presentation is now explicitly *detached* from the inferential connections between formulas by way of the assumption numbers. Now the assumption numbers tell you all you need to know about the provenance of formulas. In Lemmon's own system, you *can* prove the formula $A \to (B \to A)$ but only, as it happens, by taking a detour through conjunction or some other connective.

1	(1)	A	Assumption
2	(2)	B	Assumption
1,2	(3)	$A \wedge B$	1,2, Conjunction intro
1,2	(4)	A	3, Conjunction elim
1	(5)	$B \to A$	2,4, Conditionalisation
	(6)	$A \to (B \to A)$	1,5, Conditionalisation

This seems quite unsatisfactory, as it breaks the normalisation property. (The formula $A \to (B \to A)$ is proved only by a non-normal proof—in this case, a proof in which a conjunction is introduced and then immediately eliminated.) Normalisation can be restored to Lemmon's system, but at the cost of the introduction of a new rule, the rule of *weakening*, which says that if A depends on assumptions X, then we can infer A depending on assumptions X together with another formula.

For more information on the history of natural deduction, consult Jeffrey Pelletier's article [83].

Notice that the lines in a Lemmon proof don't just contain *formulas* (or formulas tagged a line number and information about how the formula was deduced). They are *pairs*, consisting of a formula, and the formulas upon which the formula depends. In a Gentzen proof this information is implicit in the structure of the proof. (The formulas upon which a formula depends in a Gentzen proof are the leaves in the tree above that formula that are undischarged at the moment that this formula is derived.) This feature of Lemmon's system was not original to him. The idea of making completely explicit the assumptions upon which a formula depends had also occurred to Gentzen, and this insight is our topic for the next section.

» «

Linear, relevant and affine implication have a long history. Relevant implication bust on the scene through the work of Alan Anderson and Nuel Belnap in the 1960s and 1970s [1, 2], though it had precursors in the work of the Russian logician, I. E. Orlov in the 1920s [32, 79]. The idea of a proof in which conditionals could only be introduced if the assumption for discharge was genuinely *used* is indeed one of the motivations for relevant implication in the Anderson–Belnap tradition. However, *other* motivating concerns played a role in the development of relevant logics. For other work on relevant logic, the work of Dunn [34, 36], Routley and Meyer [98], Read [93] and Mares [68] are all useful. Linear logic arose much more centrally out of proof-theoretical concerns in the work of the proof-theorist Jean-Yves Girard in the 1980s [47, 48]. A helpful introduction to linear logic is the text of Troelstra [116]. Affine logic is introduced in the tradition of linear logic as a variant on linear implication. Affine implication is quite close, however to the implication in Łukasiewicz's infinitely valued logic—which is slightly stronger, but shares the property of rejecting all *contraction*-related principles [95]. These logics are all *substructural* logics [33, 81, 96]

The definition of normality is due to Prawitz [86], though glimpses of the idea are present in Gentzen's original work [43].

The λ-calculus is due to Alonzo Church [23], and the study of λ-calculi has found many different applications in logic, computer science, type theory and related fields [4, 54, 105]. The correspondence between formulas/proofs and types/terms is known as the Curry–Howard correspondence [26, 57].

1.6 | EXERCISES

Working through these exercises will help you understand the material. As with all logic exercises, if you want to deepen your understanding of these techniques, you should attempt the exercises until they are no longer difficult. So, attempt each of the different kinds of *basic* exercises, until you know you can do them. Then move on to the *intermediate* exercises, and so on. (The *project* exercises are not the kind of thing that can be completed in one sitting.)

I am not altogether confident about the division of the exercises into "basic," "intermediate," and "advanced." I'd appreciate your feedback on whether some exercises are too easy or too difficult for their categories.

BASIC EXERCISES

Q1 Which of the following formulas have proofs with no premises?

Formula 4 is *Peirce's Law*. It is a two-valued classical logic tautology.

1 : $p \to (p \to p)$
2 : $p \to (q \to q)$
3 : $((p \to p) \to p) \to p$
4 : $((p \to q) \to p) \to p$
5 : $((q \to q) \to p) \to p$
6 : $((p \to q) \to q) \to p$
7 : $p \to (q \to (q \to p))$
8 : $(p \to q) \to (p \to (p \to q))$
9 : $((q \to p) \to p) \to ((p \to q) \to q)$
10 : $(p \to q) \to ((q \to p) \to (p \to p))$
11 : $(p \to q) \to ((q \to p) \to (p \to q))$
12 : $(p \to q) \to ((p \to (q \to r)) \to (p \to r))$
13 : $(q \to p) \to ((p \to q) \to ((q \to p) \to (p \to q)))$
14 : $((p \to p) \to p) \to ((p \to p) \to ((p \to p) \to p))$
15 : $(p_1 \to p_2) \to ((q \to (p_2 \to r)) \to (q \to (p_1 \to r)))$

For each formula that can be proved, find a proof that complies with the strictest discharge policy possible.

Q2 Annotate your proofs from Exercise 1 with λ-terms. Find a most general λ-term for each provable formula.

Q3 Construct a proof from $q \to r$ to $(q \to (p \to p)) \to (q \to r)$ using vacuous discharge. Then construct a proof of $q \to (p \to p)$ (also using vacuous discharge). Combine the two proofs, using $[\to E]$ to deduce $q \to r$. Normalise the proof you find. Then annotate each proof with λ-terms, and explain the β reductions of the terms corresponding to the normalisation.

Then construct a proof from $(p \to r) \to ((p \to r) \to q))$ to $(p \to r) \to q$ using duplicate discharge. Then construct a proof from $p \to (q \to r)$ and $p \to q$ to $p \to r$ (also using duplicate discharge). Combine the two proofs, using $[\to E]$ to deduce q. Normalise the proof you find. Then annotate each proof with λ-terms, and explain the β reductions of the terms corresponding to the normalisation.

Q4 Find types and proofs for each of the following terms.

1 : $\lambda x.\lambda y.x$
2 : $\lambda x.\lambda y.\lambda z.((xz)(yz))$
3 : $\lambda x.\lambda y.\lambda z.(x(yz))$
4 : $\lambda x.\lambda y.(yx)$
5 : $\lambda x.\lambda y.((yx)x)$

Which of the proofs are linear, which are relevant and which are affine?

Q5 Show that there is no normal relevant proof of these formulas.

1 : $p \to (q \to p)$
2 : $(p \to q) \to (p \to (r \to q))$
3 : $p \to (p \to p)$

Q6 Show that there is no normal affine proof of these formulas.

1 : $(p \to q) \to ((p \to (q \to r)) \to (p \to r))$

2 : $(p \to (p \to q)) \to (p \to q)$

Q7 Show that there is no normal proof of these formulas.

1 : $((p \to q) \to p) \to p$
2 : $((p \to q) \to q) \to ((q \to p) \to p)$

Q8 Find a formula that can has both a relevant proof and an affine proof, but no linear proof.

INTERMEDIATE EXERCISES

Q9 Consider the following "truth tables."

GD3

$\to$	t	n	f
t	t	n	f
n	t	t	f
f	t	t	t

Ł3

$\to$	t	n	f
t	t	n	n
n	t	t	f
f	t	t	t

RM3

$\to$	t	n	f
t	t	f	f
n	t	n	f
f	t	t	t

A GD3 tautology is a formula that receives the value t in every GD3 valuation. An Ł3 tautology is a formula that receives the value t in every Ł3 valuation. Show that every formula with a standard proof is a GD3 tautology. Show that every formula with an affine proof is an Ł3 tautology.

Q10 Consider proofs that have paired steps of the form $[\to E/\to I]$. That is, a conditional is eliminated only to be introduced again. The proof has a sub-proof of the form of this proof fragment:

$$\dfrac{\dfrac{A \to B \quad [A]^{(i)}}{B}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to I, i}$$

These proofs contain redundancies too, but they may well be normal. Call a proof with a pair like this CIRCUITOUS. Show that all circuitous proofs may be transformed into non-circuitous proofs with the same premises and conclusion.

Q11 In Exercise 5 you showed that there is no normal relevant proof of $p \to (p \to p)$. By normalisation, it follows that there is no relevant proof (normal or not) of $p \to (p \to p)$. Use this fact to explain why it is more natural to consider relevant arguments with *multisets* of premises and not just *sets* of premises. (HINT: is the argument from p, p to p relevantly valid?)

Q12 You might think that "if … then …" is a slender foundation upon which to build an account of logical consequence. Remarkably, there is rather a lot that you can do with implication alone, as these next questions ask you to explore.

First, define $A \hat{\vee} B$ as follows: $A \hat{\vee} B ::= (A \to B) \to B$. In what way is "$\hat{\vee}$" like *disjunction*? What usual features of disjunction are not had by $\hat{\vee}$?

(Pay attention to the behaviour of $\hat{\vee}$ with respect to different discharge policies for implication.)

Q13 Think about what it would take to have introduction and elimination rules for $\hat{\vee}$ that do not involve the conditional connective $\rightarrow$. Can you do this?

Q14 Now consider *negation*. Given an ATOM p, define the p-negation $\neg_p A$ to be $A \rightarrow p$. In what way is "$\neg_p$" like negation? What usual features of negation are not had by $\neg_p$ defined in this way? (Pay attention to the behaviour of $\neg$ with respect to different discharge policies for implication.)

Q15 Provide introduction and elimination rules for $\neg_p$ that do not involve the conditional connective $\rightarrow$.

Q16 You have probably noticed that the inference from $\neg_p\neg_p A$ to A is not, in general, valid. Define a *new* language CFORMULA inside FORMULA as follows:

$$\text{CFORMULA} ::= \neg_p\neg_p\text{ATOM} \mid (\text{CFORMULA} \rightarrow \text{CFORMULA})$$

Show that $\neg_p\neg_p A \therefore A$ and $A \therefore \neg_p\neg_p A$ are valid when A is a CFORMULA.

Q17 Now define $A\dot{\wedge}B$ to be $\neg_p(A \rightarrow \neg_p B)$, and $A\dot{\vee}B$ to be $\neg_p A \rightarrow B$. In what way are $A\dot{\wedge}B$ and $A\dot{\vee}B$ like conjunction and disjunction of A and B respectively? (Consider the difference between when A and B are FORMULAS and when they are CFORMULAS.)

Q18 Show that if there is a normal relevant proof of $A \rightarrow B$ then there is an ATOM occurring in both A and B.

Q19 Show that if we have two conditional connectives $\rightarrow_1$ and $\rightarrow_2$ defined using different discharge policies, then the conditionals collapse, in the sense that we can construct proofs from $A \rightarrow_1 B$ to $A \rightarrow_2 B$ and *vice versa*.

Q20 Explain the significance of the result of Exercise 19.

Q21 Add rules the obvious introduction rules for a *conjunction* connective $\otimes$ as follows:

$$\frac{A \quad B}{A \otimes B}{\scriptstyle \otimes I}$$

Show that if we have the following two $\otimes E$ rules:

$$\frac{A \otimes B}{A}{\scriptstyle \otimes I_1} \qquad \frac{A \otimes B}{B}{\scriptstyle \otimes I_2}$$

we may simulate the behaviour of vacuous discharge. Show, then, that we may normalise proofs involving these rules (by showing how to eliminate all indirect pairs, including $\otimes I/\otimes E$ pairs).

ADVANCED EXERCISES

Q22 Another demonstration of the subformula property for normal proofs uses the notion of a *track* in a proof.

DEFINITION 1.35 [TRACK] A sequence $A_0, \ldots, A_n$ of formula instances in the proof π is a *track* of length $n + 1$ in the proof π if and only if

- A_0 is a *leaf* in the proof tree.
- Each A_{i+1} is immediately below A_i.
- For each $i < n$, A_i is not a minor premise of an application of $[{\to}E]$.

A track whose terminus A_n is the conclusion of the proof π is said to be a TRACK OF ORDER 0. If we have a track t whose terminus A_n is the minor premise of an application of $[{\to}E]$ whose conclusion is in a track of order n, we say that t is a TRACK OF ORDER $n + 1$.

The following annotated proof gives an example of tracks.

$$
\begin{prooftree}
\AxiomC{$\heartsuit [B \to C]^{(2)}$}
\AxiomC{$\spadesuit\ A \to ((D \to D) \to B)$}
\AxiomC{$\diamondsuit [A]^{(2)}$}
\RightLabel{$\scriptstyle \to E$}
\BinaryInfC{$\spadesuit\ (D \to D) \to B$}
\AxiomC{$\clubsuit [D]^{(1)}$}
\RightLabel{$\scriptstyle \to I,1$}
\UnaryInfC{$\clubsuit\ D \to D$}
\RightLabel{$\scriptstyle \to E$}
\BinaryInfC{$\spadesuit\ B$}
\RightLabel{$\scriptstyle \to E$}
\BinaryInfC{$\heartsuit\ C$}
\RightLabel{$\scriptstyle \to I,2$}
\UnaryInfC{$\heartsuit\ A \to C$}
\RightLabel{$\scriptstyle \to I,3$}
\UnaryInfC{$\heartsuit\ (B \to C) \to (A \to C)$}
\end{prooftree}
$$

(Don't let the fact that this proof has one track of each order 0, 1, 2 and 3 make you think that proofs can't have more than one track of the same order. Look at this example —

$$
\begin{prooftree}
\AxiomC{$A \to (B \to C)$}
\AxiomC{A}
\BinaryInfC{$B \to C$}
\AxiomC{B}
\BinaryInfC{C}
\end{prooftree}
$$

— it has two tracks of order 1.) The formulas labelled with $\heartsuit$ form one track, starting with $B \to C$ and ending at the conclusion of the proof. Since this track ends at the conclusion of the proof, it is a track of order 0. The track consisting of $\spadesuit$ formulas starts at $A \to ((D \to D) \to B)$ and ends at B. It is a track of order 1, since its final formula is the minor premise in the $[{\to}E]$ whose conclusion is C, in the $\heartsuit$ track of order 0. Similarly, the $\diamondsuit$ track is order 2 and the $\clubsuit$ track has order 3.

For this exercise, prove the following lemma by induction on the construction of a proof.

LEMMA 1.36 *In every proof, every formula is in one and only one track, and each track has one and only one order.*

Then prove this lemma.

LEMMA 1.37 *Let* $t : A_0, \ldots, A_n$ *be a track in a normal proof. Then*

a) *The rules applied within the track consist of a sequence* (*possibly empty*) *of* $[\rightarrow E]$ *steps and then a sequence* (*possibly empty*) *of* $[\rightarrow I]$ *steps.*

b) *Every formula* A_i *in* t *is a subformula of* A_0 *or of* A_n.

Now prove the subformula theorem, using these lemmas.

Q23 Consider the result of Exercise 19. Show how you might define a natural deduction system containing (say) both a linear and a standard conditional, in which there is *no* collapse. That is, construct a system of natural deduction proofs in which there are two conditional connectives: $\rightarrow_l$ for linear conditionals, and $\rightarrow_s$ for standard conditionals, such that whenever an argument is valid for a linear conditional, it is (in some appropraite sense) valid in the system you design (when $\rightarrow$ is translated as $\rightarrow_l$) and whenever an argument is valid for a standard conditional, it is (in some appropriate sense) valid in the system you design (when $\rightarrow$ is translated as $\rightarrow_s$). What mixed inferences (those using both $\rightarrow_l$ and $\rightarrow_s$) are valid in your system?

Q24 Suppose we have a new discharge policy that is "stricter than linear." The *ordered* discharge policy allows you to discharge only the *rightmost* assumption at any one time. It is best paired with a strict version of $[\rightarrow E]$ according to which the major premise $(A \rightarrow B)$ is on the left, and the minor premise (A) is on the right. What is the resulting logic *like*? Does it have the normalisation property?

Q25 Take the logic of Exercise 24, and extend it with *another* connective $\leftarrow$, with the rule $[\leftarrow E]$ in which the major premise $(B \leftarrow A)$ is on the *right*, and the minor premise (A) is on the *left*, and $[\leftarrow I]$, in which the *leftmost* assumption is discharged. Examine the connections between $\rightarrow$ and $\leftarrow$. Does normalisation work for *these* proofs? This is *Lambek's* logic for syntactic types [62, 63, 76, 77].

Q26 Show that there is a way to be even *stricter* than the discharge policy of Exercise 24. What is the *strictest* discharge policy for $\rightarrow I$, that will result in a system which normalises, provided that $\rightarrow E$ (in which the major premise is leftmost) is the only other rule for implication.

Q27 Consider the introduction rule for $\otimes$ given in Exercise 21. Construct an appropriate *elimination* rule for fusion which does not allow the simulation of vacuous (or duplicate) discharge, and for which proofs normalise.

Q28 Identify two proofs where one can be reduced to the other by way of the elimination of *circuitous* steps (see Exercise 10). Characterise the identities this provides among λ-terms. Can this kind if identification be maintained along with β-reduction?

PROJECT

Q29 Thoroughly and systematically explain and evaluate the considerations for choosing one discharge policy over another. This will involve looking at the different *uses* to which one might put a system of natural deduction, and then, relative to a use, what one might say in favour of a different policy.

SEQUENT CALCULUS 2

In this chapter we will look at a different way of thinking about proof and consequence: Gentzen's *sequent calculus*. The core idea is straightforward. We want to know what follows from what, so we will keep a track of facts of consequence: facts we will record in the following form:

$$A \vdash B$$

One can read "$A \vdash B$" in a number of ways. You can say that B follows from A, or that A entails B, that the argument from A to B is valid, or that asserting A clashes with denying B, or – and this is the understanding most appropriate for us – that there is a proof from A to B. The symbol used between A and B is sometimes called the TURNSTILE.

"Scorning a turnstile wheel at her reverend helm, she sported there a tiller; and that tiller was in one mass, curiously carved from the long narrow lower jaw of her hereditary foe. The helmsman who steered by that tiller in a tempest, felt like the Tartar, when he holds back his fiery steed by clutching its jaw. A noble craft, but somehow a most melancholy! All noble things are touched with that." — Herman Melville, *Moby Dick*.

Once we have this notion of consequence, we can ask ourselves what properties consequence has. There are many different ways you could answer this question. The focus of this section will be a particular technique, originally due to Gerhard Gentzen. We can think of consequence – relative to a particular *language* – like this: when we want to know about the relation of consequence, we first consider each different kind of formula in the language. To make the discussion concrete, let's consider a very simple language: the language of propositional logic with only two connectives, *conjunction* $\wedge$ and *disjunction* $\vee$. That is, we will now look at formulas expressed in a language with the following grammar:

$$\text{FORMULA} ::= \text{ATOM} \mid (\text{FORMULA} \wedge \text{FORMULA}) \mid (\text{FORMULA} \vee \text{FORMULA})$$

To characterise consequence relations, we need to characterise how consequence works on the *atoms* of the language, and then how the addition of $\wedge$ and $\vee$ expands the repertoire of facts about consequence. To do this, we need to know when $A \vdash B$ when A is a conjunction, or when A is a disjunction, and when B is a conjunction, or when B is a disjunction. In other words, for each connective, we need to know when we can prove something *from* a formula featuring that connective, and when we can prove a formula featuring that connective. Another way of putting it is that we wish to know how a connective behave on the left of the turnstile, and how it behaves on the right.

In a sequent system, we will have rules concerning statements about consequence—and these statements are the *sequents* at the heart of the system. Because we can make false claims as well as true ones, we will use the following *bent* turnstile for the general case of a sequent

I follow Lloyd Humberstone, who, as far as I am aware, introduced this convention for sequents [58]. Gentzen used the arrow, which we have already used for the object-language conditional.

$$A \succ B$$

and we reserve the straight turnstile $A \vdash B$ for when we wish to explicitly claim that the sequent $A \succ B$ is *derivable*. In what follows, $p \succ p \wedge q$

is a perfectly good sequent, though it will not be a derivable one (for $p \wedge q$ does not follow from p), so we will not have $p \vdash p \wedge q$.

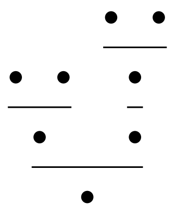
This is a *formal* account of consequence. We look only at the form of propositions and not their content. For atomic propositions (those with no internal form) there is nothing upon which we could pin a claim to consequence. Thus we never have $p \vdash q$ where p and q are different atoms, while $p \vdash p$ for all atoms p.

The answers for our language seem straightforward. For atomic formulas, p and q, the sequent $p \succ q$ is derivable only if p and q are the *same* atom: so we have $p \vdash p$ for each atom p. For conjunction, we can say that if $A \succ B$ and $A \succ C$ are derivable, then so is $A \succ B \wedge C$. That's how we can infer *to* a conjunction. Inferring *from* a conjunction is also straightforward. We can say that $A \wedge B \succ C$ when $A \succ C$, or when $B \succ C$. For disjunction, we can reason similarly. We can say $A \vee B \succ C$ when $A \succ C$ and $B \succ C$. We can say $A \succ B \vee C$ when $A \succ B$, or when $A \succ C$. This is *inclusive* disjunction, not exclusive disjunction.

You can think of these definitions as adding new material (in this case, conjunction and disjunction) to a pre-existing language. Think of the inferential repertoire of the basic language as settled (in our discussion this is *very* basic, just the atoms), and the connective rules are "definitional" extensions of the basic language. These thoughts are the raw materials for the development of an account of proof and logical consequence in general.

2.1 | DERIVATIONS

Like natural deduction proofs, derivations involving sequents are trees. The structure is as before:



Where each position on the tree follows from those above it. In a tree, the *order* of the branches does not matter. These are two different ways to present the *same* tree:

$$\frac{A \quad B}{C} \qquad \frac{B \quad A}{C}$$

In this case, the tree structure is at the one and the same time *simpler* and *more complicated* than the tree structure of natural deduction proofs. They are simpler, in that there is no discharge. They are more complicated, in that trees are not trees of formulas. They are trees consisting of *sequents*. As a result, we will call these structures DERIVATIONS instead of PROOFS. The distinction is simple. For us, a proof is a structure in which the *formulas* are connected by inferential relations in a tree-like structure. A proof will go *from* some formulas *to* other formulas, *via* yet other formulas. Our structures involving sequents are quite different. The last sequent in a tree (the *endsequent*) is itself a statement of consequence, with its own antecedent and consequent (or premise and conclusion, if you prefer.) The tree *derivation* shows you why (or perhaps how) you can

I say "tree-like" since we will see different structures in later chapters.

$$p \succ p \; Id \qquad \frac{L \succ C \quad C \succ R}{L \succ R} \; Cut$$

$$\frac{A \succ R}{A \land B \succ R} \land L_1 \qquad \frac{A \succ R}{B \land A \succ R} \land L_2 \qquad \frac{L \succ A \quad L \succ B}{L \succ A \land B} \land R$$

$$\frac{A \succ R \quad B \succ R}{A \lor B \succ R} \lor L \qquad \frac{L \succ A}{L \succ A \lor B} \lor R_1 \qquad \frac{L \succ A}{L \succ B \lor A} \lor R_2$$

Figure 2.1: A SIMPLE SEQUENT SYSTEM

infer from the antecedent to the consequent. The rules for constructing sequent derivations are found in Figure 2.1.

DEFINITION 2.1 [SIMPLE SEQUENT DERIVATION] If the leaves of a tree are instances of the *Id* rule, and if its transitions from node to node are instances of the other rules in Figure 2.1, then the tree is said to be a SIMPLE SEQUENT DERIVATION.

We must read these rules completely literally. Do not presume any properties of conjunction or disjunction other than those that can be demonstrated on the basis of the rules. We will take these rules as *constituting* the behaviour of the connectives $\land$ and $\lor$.

EXAMPLE 2.2 [EXAMPLE SEQUENT DERIVATIONS] In this section, we will look at a few sequent derivations, demonstrating some simple properties of conjunction, disjunction, and the consequence relation.

The first derivations show some commutative and associative properties of conjunction and disjunction. Here is the conjunction case, with derivations to the effect that $p \land q \vdash q \land p$, and that $p \land (q \land r) \vdash (p \land q) \land r$.

$$\frac{\dfrac{q \succ q}{p \land q \succ q} \land L_2 \quad \dfrac{p \succ p}{p \land q \succ p} \land L_1}{p \land q \succ q \land p} \land R \qquad \frac{\dfrac{\dfrac{p \succ p}{p \land (q \land r) \succ p} \land L_1 \quad \dfrac{\dfrac{q \succ q}{q \land r \succ q} \land L_1}{p \land (q \land r) \succ q} \land L_2}{p \land (q \land r) \succ p \land q} \land R \quad \dfrac{\dfrac{r \succ r}{q \land r \succ r} \land L_2}{p \land (q \land r) \succ r} \land L_2}{p \land (q \land r) \succ (p \land q) \land r} \land R$$

Here are the cases for disjunction. The first derivation is for the commutativity of disjunction, and the second is for associativity. (It is important to notice that these are not derivations of the commutativity or associativity of conjunction or disjunction in *general*. They only show the commutativity and associativity of conjunction and disjunction of *atomic* formulas. These are not derivations of $A \land B \succ B \land A$ (for example) since $A \succ A$ is not an axiom if A is a complex formula. We will see more on this in the next section.)

$$\dfrac{\dfrac{p \succ p}{p \succ q \lor p}{\scriptstyle \lor R_1} \qquad \dfrac{q \succ q}{q \succ p \lor q}{\scriptstyle \lor R_2}}{p \lor q \succ q \lor p}{\scriptstyle \lor L}$$

$$\dfrac{\dfrac{\dfrac{p \succ p}{p \succ p \lor (q \lor r)}{\scriptstyle \lor R_1} \qquad \dfrac{\dfrac{q \succ q}{q \succ q \lor r}{\scriptstyle \lor R_1}}{q \succ p \lor (q \lor r)}{\scriptstyle \lor R_2}}{p \lor q \succ p \lor (q \lor r)}{\scriptstyle \lor L} \qquad \dfrac{\dfrac{r \succ r}{r \succ q \lor r}{\scriptstyle \lor R_2}}{r \succ p \lor (q \lor r)}{\scriptstyle \lor R_2}}{(p \lor q) \lor r \succ p \lor (q \lor r)}{\scriptstyle \lor L}$$

Exercise 15 on page 113 asks you to make this duality precise.

You can see that the disjunction derivations have the same structure as those for conjunction. You can convert any derivation into another (its *dual*) by swapping conjunction and disjunction, and swapping the left-hand side of the sequent with the right-hand side. Here are some more examples of duality between derivations. The first is the dual of the second, and the third is the dual of the fourth.

$$\dfrac{p \succ p \quad p \succ p}{p \lor p \succ p}{\scriptstyle \lor L} \qquad \dfrac{p \succ p \quad p \succ p}{p \succ p \land p}{\scriptstyle \land R} \qquad \dfrac{p \succ p \quad \dfrac{p \succ p}{p \land q \succ p}{\scriptstyle \land L_1}}{p \lor (p \land q) \succ p}{\scriptstyle \lor L} \qquad \dfrac{p \succ p \quad \dfrac{p \succ p}{p \succ p \lor q}{\scriptstyle \lor R}}{p \succ p \land (p \lor q)}{\scriptstyle \land R}$$

You can use derivations you have at hand, like these, as components of other derivations. One way to do this is to use the *Cut* rule.

$$\dfrac{\dfrac{p \succ p \quad \dfrac{p \succ p}{p \land q \succ p}{\scriptstyle \land L_1}}{p \lor (p \land q) \succ p}{\scriptstyle \lor L} \qquad \dfrac{p \succ p \quad \dfrac{p \succ p}{p \succ p \lor q}{\scriptstyle \lor R}}{p \succ p \land (p \lor q)}{\scriptstyle \land R}}{p \lor (p \land q) \succ p \land (p \lor q)}{\scriptstyle Cut}$$

Notice, too, that each of these derivations we've seen so far move from less complex formulas at the top to more complex formulas, at the bottom. Reading from bottom to top, you can see the formulas decomposing into their constituent parts. This isn't the case for all sequent derivations. Derivations that use the *Cut* rule can include new (more complex) material in the process of deduction. Here is an example:

$$\dfrac{\dfrac{\dfrac{p \succ p}{p \succ q \lor p}{\scriptstyle \lor R_1} \qquad \dfrac{q \succ q}{q \succ q \lor p}{\scriptstyle \lor R_2}}{p \lor q \succ q \lor p}{\scriptstyle \lor L} \qquad \dfrac{\dfrac{q \succ q}{q \succ p \lor q}{\scriptstyle \lor R_1} \qquad \dfrac{p \succ p}{p \succ p \lor q}{\scriptstyle \lor R_2}}{q \lor p \succ p \lor q}{\scriptstyle \lor L}}{p \lor q \succ p \lor q}{\scriptstyle Cut}$$

We call the concluding sequent of a derivation the "ENDSEQUENT."

This derivation is a complicated way to deduce $p \lor q \succ p \lor q$, and it includes $q \lor p$, which is not a subformula of any formula in the final sequent of the derivation. Reading from bottom to top, the *Cut* step can introduce new formulas into the derivation.

2.2 | IDENTITY & CUT CAN BE ELIMINATED

The two distinctive rules in our proof system are *Id* and *Cut*. These rules are not about any particular kind of formula—they are *structural*, governing the behaviour of derivations, no matter *what* the nature of the formulas flanking the turnstiles. In this section we will look at the distinctive behaviour of *Id* and of *Cut*. We start with *Id*.

IDENTITY

This derivation of $p \vee q \succ p \vee q$ is *a* derivation of an identity (a sequent of the form $A \succ A$). There is a more systematic way to show that $p \vee q \succ p \vee q$, and any identity sequent. Here is a derivation of the sequent without *Cut*, and its dual, for conjunction.

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}{\scriptstyle \vee R_1} \quad \dfrac{q \succ q}{q \succ p \vee q}{\scriptstyle \vee R_2}}{p \vee q \succ p \vee q}{\scriptstyle \vee L} \qquad \dfrac{\dfrac{p \succ p}{p \wedge q \succ p}{\scriptstyle \wedge L_1} \quad \dfrac{q \succ q}{p \wedge q \succ q}{\scriptstyle \wedge L_2}}{p \wedge q \succ p \wedge q}{\scriptstyle \wedge R}$$

We can piece together these little derivations in order to derive any sequent of the form $A \succ A$. For example, here is the start of derivation of $p \wedge (q \vee (r_1 \wedge r_2)) \succ p \wedge (q \vee (r_1 \wedge r_2))$.

$$\dfrac{\dfrac{p \succ p}{p \wedge (q \vee (r_1 \wedge r_2)) \succ p}{\scriptstyle \wedge L_1} \quad \dfrac{q \vee (r_1 \wedge r_2) \succ q \vee (r_1 \wedge r_2)}{p \wedge (q \vee (r_1 \wedge r_2)) \succ q \vee (r_1 \wedge r_2)}{\scriptstyle \wedge L_2}}{p \wedge (q \vee (r_1 \wedge r_2)) \succ p \wedge (q \vee (r_1 \wedge r_2))}{\scriptstyle \wedge R}$$

It's not a complete derivation yet, as one leaf $q \vee (r_1 \wedge r_2) \succ q \vee (r_1 \wedge r_2)$ is not an axiom. However, we can add the derivation for it.

$$\dfrac{\dfrac{p \succ p}{p \wedge (q \vee (r_1 \wedge r_2)) \succ p} \quad \dfrac{\dfrac{\dfrac{q \succ q}{q \succ q \vee (r_1 \wedge r_2)}{\scriptstyle \vee R_1} \quad \dfrac{\dfrac{\dfrac{r_1 \succ r_1}{r_1 \wedge r_2 \succ r_1}{\scriptstyle \wedge L_1} \quad \dfrac{r_2 \succ r_2}{r_1 \wedge r_2 \succ r_2}{\scriptstyle \wedge L_2}}{r_1 \wedge r_2 \succ r_1 \wedge r_2}{\scriptstyle \wedge R}}{r_1 \wedge r_2 \succ q \vee (r_1 \wedge r_2)}{\scriptstyle \vee R_2}}{q \vee (r_1 \wedge r_2) \succ q \vee (r_1 \wedge r_2)}{\scriptstyle \vee L}}{p \wedge (q \vee (r_1 \wedge r_2)) \succ q \vee (r_1 \wedge r_2)}}{p \wedge (q \vee (r_1 \wedge r_2)) \succ p \wedge (q \vee (r_1 \wedge r_2))}$$

The derivation of $q \vee (r_1 \wedge r_2) \succ q \vee (r_1 \wedge r_2)$ itself contains a smaller identity derivation, for $r_1 \wedge r_2 \succ r_1 \wedge r_2$. The derivation displayed here uses shading to indicate the way the derivations are nested together. This result is general, and it is worth a theorem of its own.

THEOREM 2.3 [IDENTITY DERIVATIONS] *For each formula A, the sequent $A \succ A$ has a derivation. A derivation for $A \succ A$ may be systematically constructed from the identity derivations for the subformulas of A.*

Proof: We define Id_A, the IDENTITY DERIVATION FOR A by induction on the construction of A, as follows. Id_p is the axiom $p \succ p$. For complex formulas, we have

$$Id_{A\vee B}: \quad \dfrac{\dfrac{Id_A}{A \succ A \vee B}\vee R_1 \qquad \dfrac{Id_B}{B \succ A \vee B}\vee R_2}{A \vee B \succ A \vee B}\vee L \qquad Id_{A\wedge B}: \quad \dfrac{\dfrac{Id_A}{A \wedge B \succ A}\wedge L_1 \qquad \dfrac{Id_B}{A \wedge B \succ B}\wedge L_2}{A \wedge B \succ A \wedge B}\wedge R$$ ■

We say that $A \succ A$ is DERIVABLE in the sequent system. If we think of *Id* as a degenerate *rule* (a rule with no premise), then its generalisation, Id_A, is a *derivable rule*.

It might seem *crazy* to have a proof of identity, like $A \succ A$ where A is a complex formula. Why don't we take Id_A as an axiom? There are a few different reasons we might like to consider for taking Id_A as derivable instead of one of the primitive axioms of the system.

THE SYSTEM IS SIMPLE: In an axiomatic theory, it is always preferable to minimise the number of primitive assumptions. Here, it's clear that Id_A is derivable, so there is no need for it to be an axiom. A system with fewer axioms is preferable to one with more, for the reason that we have reduced derivations to a smaller set of primitive notions.

These are part of a general story, to be explored throughout this book, of what it is to be a logical constant. These sorts of considerations have a long history [51].

THE SYSTEM IS SYSTEMATIC: In the system without Id_A as an axiom, when we consider a sequent like $L \succ R$ in order to know whether it it is derived (in the absence of *Cut*, at least), we can ask two separate questions. We can consider L. If it is complex perhaps $L \succ R$ is derivable by means of a left rule like [$\wedge$L] or [$\vee$L]. On the other hand, if R is complex, then perhaps the sequent is derivable by means of a right rule, like [$\wedge$R] or [$\vee$R]. If both are primitive, then $L \succ R$ is derivable by identity only. And that is it! You check the left, check the right, and there's no other possibility. There is no other condition under which the sequent is derivable. In the presence of Id_A, one would have to check if $L = R$ as well as the other conditions.

THE SYSTEM PROVIDES A CONSTRAINT: In the absence of a general identity axiom, the burden on deriving identity is passed over to the connective rules. Allowing derivations of identity statements is a hurdle over which a connective rule might be able to jump, or over which it might *fail*. As we shall see later, this is provides a constraint we can use to sort out "good" definitions from "bad" ones. Given that the left and right rules for conjunction and disjunction tell you how the connectives are to be introduced, it would seem that the rules are defective (or at the very least, *incomplete*) if they don't allow the derivation of each instance of *Id*. We will make much more of this when we consider other connectives. However, before we make more of the philosophical motivations and implications of this constraint, we will add another possible constraint on connective rules, this time to do with the other rule in our system, *Cut*.

CUT

Some of the nice properties of a sequent system are as a matter of fact, the nice features of derivations that are constructed without the *Cut* rule. Derivations constructed without *Cut* satisfy the subformula property.

THEOREM 2.4 [SUBFORMULA PROPERTY] *If δ is a sequent derivation not containing Cut, then the formulas in δ are all subformulas of the formulas in the endsequent of δ.*

Proof: You can see this merely by looking at the rules. Each rule except for *Cut* has the subformula property. ∎

Notice how much simpler this proof is than the proof of Theorem 1.11.

A derivation is said to be CUT-FREE if it does not contain an instance of the *Cut* rule. Doing without *Cut* is good for some things, and bad for others. In the system of proof we're studying in this section, sequents have *very many* more proofs with *Cut* than without it.

EXAMPLE 2.5 [DERIVATIONS WITH OR WITHOUT CUT] $p \succ p \lor q$ has only one *Cut*-free derivation, it has infinitely many derivations using *Cut*.

You can see that there is only one *Cut*-free derivation with $p \succ p \lor q$ as the endsequent. The only possible last inference in such a derivation is $[\lor R]$, and the only possible premise for that inference is $p \succ p$. This completes that proof.

On the other hand, there are very many different last inferences in a derivation featuring *Cut*. The most trivial example is the derivation:

$$\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\RightLabel{Cut}
\BinaryInfC{$p \succ p \lor q$}
\end{prooftree}$$

which contains the *Cut*-free derivation of $p \succ p \lor q$ inside it. We can nest the cuts with the identity sequent $p \succ p$ as deeply as we like.

$$\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$p \succ p$}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\RightLabel{Cut}
\BinaryInfC{$p \succ p \lor q$}
\RightLabel{Cut}
\BinaryInfC{$p \succ p \lor q$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$p \succ p$}
\AxiomC{$p \succ p$}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\RightLabel{Cut}
\BinaryInfC{$p \succ p \lor q$}
\RightLabel{Cut}
\BinaryInfC{$p \succ p \lor q$}
\RightLabel{Cut}
\BinaryInfC{$p \succ p \lor q$}
\end{prooftree}
\qquad \cdots$$

However, we can construct quite different derivations of our sequent, and we involve different material in the derivation. For any formula A you wish to choose, we could implicate A (an "innocent bystander") in the derivation as follows:

$$\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor (q \land A)$}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\AxiomC{$q \succ q$}
\RightLabel{$\land L_1$}
\UnaryInfC{$q \land A \succ q$}
\RightLabel{$\lor R_2$}
\UnaryInfC{$q \land A \succ p \lor q$}
\RightLabel{$\lor L$}
\BinaryInfC{$p \lor (q \land A) \succ p \lor q$}
\RightLabel{Cut}
\BinaryInfC{$p \succ p \lor q$}
\end{prooftree}$$

In this derivation the *Cut* formula $p \lor (q \land A)$ is doing no genuine work. It is merely repeating the left formula p or the right formula q.

Well, it's doing *work*, in that $p \lor (q \land A)$ is, for many choices for A, genuinely intermediate between p and $p \lor q$. However, A is doing the kind of work that could be done by *any* formula. Choosing different values for A makes no difference to the shape of the derivation. A is doing the kind of work that doesn't require special qualifications.

So, using *Cut* makes the search for derivations rather difficult. There are very many more *possible* derivations of a sequent, and many more actual derivations. The search space is much more constrained if we are looking for *Cut*-free derivations instead. Constructing derivations, on the other hand, is easier if we are permitted to use *Cut*. We have very many more options for constructing a derivation, since we are able to pass through formulas "intermediate" between the desired antecedent and consequent.

Do we *need* to use *Cut*? Is there anything derivable with *Cut* that cannot be derived without? Take a derivation involving *Cut*, like this:

$$\dfrac{\dfrac{\dfrac{p \succ p}{p \land (q \land r) \succ p}\,{\scriptstyle \land L_1} \qquad \dfrac{\dfrac{q \succ q}{q \land r \succ q}\,{\scriptstyle \land L_1}}{p \land (q \land r) \succ q}\,{\scriptstyle \land L_2}}{p \land (q \land r) \succ p \land q}\,{\scriptstyle \land R} \qquad \dfrac{\dfrac{q \succ q}{p \land q \succ q}\,{\scriptstyle \land L_1}}{p \land q \succ q \lor r}\,{\scriptstyle \lor R_1}}{p \land (q \land r) \succ q \lor r}\,{\scriptstyle Cut}$$

This sequent $p \land (q \land r) \succ q \lor r$ did not have to be derived using *Cut*. We can *eliminate* the *Cut*-step from the derivation in a systematic way by showing that whenever we use a *Cut* in a derivation we could have either done without it, or used it *earlier*. For example in the last inference here, we did not need to leave the *Cut* until the last step. We could have *Cut* on the sequent $p \land q \succ q$ and left the inference to $q \lor r$ until later:

The systematic technique I am using will be revealed in detail very soon.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p \land (q \land r) \succ p}\,{\scriptstyle \land L_1} \qquad \dfrac{\dfrac{q \succ q}{q \land r \succ q}\,{\scriptstyle \land L_1}}{p \land (q \land r) \succ q}\,{\scriptstyle \land L_2}}{p \land (q \land r) \succ p \land q}\,{\scriptstyle \land R} \qquad \dfrac{q \succ q}{p \land q \succ q}\,{\scriptstyle \land L_1}}{p \land (q \land r) \succ q}\,{\scriptstyle Cut}}{p \land (q \land r) \succ q \lor r}\,{\scriptstyle \lor R_1}$$

Now the *Cut* takes place on the conjunction $p \land q$, which is introduced immediately before the application of the *Cut*. Notice that in this case we use the *Cut* to get us to $p \land (q \land r) \succ$, which is one of the sequents already seen in the derivation! This derivation repeats itself. (Do not be deceived, however. It is not a *general* phenomenon among proofs involving *Cut* that they repeat themselves. The original proof did not repeat any sequents except for the axiom $q \succ q$.)

The similarity with non-normal proofs as discussed in the previous section is *not* an accident.

No, the interesting feature of this new proof is that before the *Cut*, the *Cut* formula is introduced on the right in the derivation of left sequent $p \land (q \land r) \succ p \land q$, and it is introduced on the left in the derivation of the right sequent $p \land q \succ q$.

Notice that in general, if we have a *Cut* applied to a conjunction which is introduced on both sides of the step, we have a shorter route to $L \succ R$. We can sidestep the move through $A \wedge B$ to *Cut* on the formula A, since we have $L \succ A$ and $A \succ R$.

$$\dfrac{\dfrac{L \succ A \quad L \succ B}{L \succ A \wedge B}{\scriptstyle \wedge R} \qquad \dfrac{A \succ R}{A \wedge B \succ R}{\scriptstyle \wedge L_1}}{L \succ R}{\scriptstyle Cut}$$

In our example we do the same: We *Cut* with $p \wedge (q \wedge r) \succ q$ on the left and $q \succ q$ on the right, to get the first proof below in which the *Cut* moves *further* up the derivation. Clearly, however, this *Cut* is redundant, as cutting on an identity sequent does nothing. We could eliminate that step, without cost.

$$\dfrac{\dfrac{\dfrac{\dfrac{q \succ q}{q \wedge r \succ q}{\scriptstyle \wedge L_1}}{p \wedge (q \wedge r) \succ q}{\scriptstyle \wedge L_2} \qquad q \succ q}{p \wedge (q \wedge r) \succ q}{\scriptstyle Cut}}{p \wedge (q \wedge r) \succ q \vee r}{\scriptstyle \vee R_1} \qquad \dfrac{\dfrac{\dfrac{q \succ q}{q \wedge r \succ q}{\scriptstyle \wedge L_1}}{p \wedge (q \wedge r) \succ q}{\scriptstyle \wedge L_2}}{p \wedge (q \wedge r) \succ q \vee r}{\scriptstyle \vee R_1}$$

We have a *Cut*-free derivation of our concluding sequent.

As I hinted before, this technique is a general one. We may use exactly the same method to convert *any* derivation using *Cut* into a derivation without it. To do this, we will make explicit a number of the concepts we saw in this example.

definition 2.6 [active and passive formulas] The formulas L and R in each inference in Figure 2.1 are said to be *passive* in the inference (they "do nothing" in the step from top to bottom), while the other formulas are *active*.

A formula is active in a step in a derivation if that formula is either introduced or eliminated. The active formulas in the connective rules are the *principal* formula (the conjunction or disjunction introduced, below the line) or the *constituents* from which the principal formula is constructed. The active formulas in a *Cut* step are the two instances of the *Cut*-formula, present above the line, but absent below the line.

definition 2.7 [depth of an inference] The depth of an inference in a derivation δ is the number of nodes in the sub-derivation of δ in which that inference is the last step, minus one. In other words, it is the number of sequents above the conclusion of that inference.

Now we can proceed to present the technique for eliminating *Cut*s from a derivation. First we show that *Cut*s may be moved upward. Then we show that this process will terminate in a *Cut*-free derivation. This first lemma is the bulk of the procedure for eliminating *Cut*s from derivations.

LEMMA 2.8 [CUT-DEPTH REDUCTION] *Given a derivation δ of $A \succ C$, whose final inference is Cut, but which is otherwise Cut-free, and in which that inference has a depth of n, we can transform δ another derivation δ' of $A \succ C$ which is Cut-free, or in which each Cut step has a depth less than n.*

Proof: Our derivation δ contains two subderivations: δ_l ending in $A \succ B$ and δ_r ending in $B \succ C$. These subderivations are *Cut*-free.

$$\dfrac{\overset{\vdots\, \delta_l}{A \succ B} \quad \overset{\vdots\, \delta_r}{B \succ C}}{A \succ C}$$

To find our new derivation, we look at the two instances of the *Cut*-formula B and its roles in the final inference in δ_l and in δ_r. We have the following two cases: either B is passive in one or other of these inferences, or it is not.

CASE 1: THE CUT-FORMULA IS PASSIVE IN EITHER INFERENCE Suppose that the formula B is *passive* in the last inference in δ_l or passive in the last inference in δ_r. For example, if δ_l ends in $[\land L_1]$, then we may push the *Cut* above it like this:

The $\land L_2$ case is the same, except for the choice of A_2 instead of A_1.

$$\text{BEFORE:}\quad \dfrac{\dfrac{\overset{\vdots\, \delta_l'}{A_1 \succ B}}{A_1 \land A_2 \succ B}{\scriptstyle \land L_1} \qquad \overset{\vdots\, \delta_r}{B \succ C}}{A_1 \land A_2 \succ C}{\scriptstyle Cut} \qquad \text{AFTER:}\quad \dfrac{\dfrac{\overset{\vdots\, \delta_l'}{A_1 \succ B} \quad \overset{\vdots\, \delta_r}{B \succ C}}{A_1 \succ C}{\scriptstyle Cut}}{A_1 \land A_2 \succ C}{\scriptstyle \land L_1}$$

The resulting derivation has a *Cut*-depth lower by one. If, on the other hand, δ_l ends in $[\lor L]$, we may push the *Cut* above that $[\lor L]$ step. The result is a derivation in which we have duplicated the *Cut*, but we have reduced the *Cut*-depth more significantly, as the effect of δ_l is split between the two cuts.

$$\text{BEFORE:}\quad \dfrac{\dfrac{\overset{\vdots\, \delta_l^1}{A_1 \succ B} \quad \overset{\vdots\, \delta_l^2}{A_2 \succ B}}{A_1 \lor A_2 \succ B}{\scriptstyle \lor L} \qquad \overset{\vdots\, \delta_r}{B \succ C}}{A_1 \lor A_2 \succ C}{\scriptstyle Cut} \qquad \text{AFTER:}\quad \dfrac{\dfrac{\overset{\vdots\, \delta_l^1}{A_1 \succ B} \quad \overset{\vdots\, \delta_r}{B \succ C}}{A_1 \succ C}{\scriptstyle Cut} \qquad \dfrac{\overset{\vdots\, \delta_l^2}{A_2 \succ B} \quad \overset{\vdots\, \delta_r}{B \succ C}}{A_2 \succ C}{\scriptstyle Cut}}{A_1 \lor A_2 \succ C}{\scriptstyle \lor L}$$

The other two ways in which the *Cut* formula could be passive are when δ_2 ends in $[\lor R]$ or $[\land R]$. The technique for these is identical to the examples we have seen. The *Cut* passes over $[\lor R]$ trivially, and it passes over $[\land R]$ by splitting into two cuts. In every instance, the depth is reduced.

CASE 2: THE CUT-FORMULA IS ACTIVE In the remaining case, the *Cut*-formula formula B may be assumed to be active in the last inference in both δ_l and in δ_r, because we have dealt with the case in which it is passive in either inference. What we do now depends on the form of the formula B. In each case, the structure of the formula B determines the final rule in both δ_l and δ_r.

CASE 2A: THE CUT-FORMULA IS ATOMIC If the *Cut*-formula is an atom, then the only inference in which an atomic formula is active in the conclusion is *Id*. In this case, the *Cut* is redundant.

$$\text{BEFORE:}\quad \frac{p \succ p \qquad p \succ p}{p \succ p}\,{\scriptstyle Cut} \qquad\qquad \text{AFTER:}\quad p \succ p$$

CASE 2B: THE CUT-FORMULA IS A CONJUNCTION If the *Cut*-formula is a conjunction $B_1 \wedge B_2$, then the only inferences in which a conjunction is active in the conclusion are $[\wedge R]$ and $[\wedge L]$. Let us suppose that in the inference $[\wedge L]$, we have inferred the sequent $B_1 \wedge B_2 \succ C$ from the premise sequent $B_1 \succ C$. In this case, it is clear that we could have *Cut* on B_1 instead of the conjunction $B_1 \wedge B_2$, and the *Cut* is shallower.

The choice for $[\wedge L_2]$ instead of $[\wedge L_1]$ involves choosing B_2 instead of B_1.

$$\text{BEFORE:}\quad \frac{\dfrac{\begin{array}{c}\vdots\,\delta_l^1\\ A \succ B_1\end{array} \quad \begin{array}{c}\vdots\,\delta_l^2\\ A \succ B_2\end{array}}{A \succ B_1 \wedge B_2}\,{\scriptstyle \wedge R} \qquad \dfrac{\begin{array}{c}\vdots\,\delta_r'\\ B_1 \succ C\end{array}}{B_1 \wedge B_2 \succ C}\,{\scriptstyle \wedge L_1}}{A \succ C}\,{\scriptstyle Cut} \qquad \text{AFTER:}\quad \frac{\begin{array}{c}\vdots\,\delta_l^1\\ A \succ B_1\end{array} \quad \begin{array}{c}\vdots\,\delta_r'\\ B_1 \succ C\end{array}}{A \succ C}\,{\scriptstyle Cut}$$

CASE 2C: THE CUT-FORMULA IS A DISJUNCTION The case for disjunction is similar. If the *Cut*-formula is a disjunction $B_1 \vee B_2$, then the only inferences in which a conjunction is active in the conclusion are $\vee R$ and $\vee L$. Let's suppose that in $\vee R$ the disjunction $B_1 \vee B_2$ is introduced in an inference from B_1. In this case, it is clear that we could have *Cut* on B_1 instead of the disjunction $B_1 \vee B_2$, with a shallower *Cut*.

$$\text{BEFORE:}\quad \frac{\dfrac{\begin{array}{c}\vdots\,\delta_l'\\ A \succ B_1\end{array}}{A \succ B_1 \vee B_2}\,{\scriptstyle \vee R_1} \qquad \dfrac{\begin{array}{c}\vdots\,\delta_r^1\\ B_1 \succ C\end{array} \quad \begin{array}{c}\vdots\,\delta_r^2\\ B_2 \succ C\end{array}}{B_1 \vee B_2 \succ C}\,{\scriptstyle \vee L}}{A \succ C}\,{\scriptstyle Cut} \qquad \text{AFTER:}\quad \frac{\begin{array}{c}\vdots\,\delta_l'\\ A \succ B_1\end{array} \quad \begin{array}{c}\vdots\,\delta_r^1\\ B_1 \succ C\end{array}}{A \succ C}\,{\scriptstyle Cut}$$

In every case, then, we have traded in a derivation for a derivation either without *Cut* or with a shallower cut. ■

The process of reducing *Cut*-depth cannot continue indefinitely, since the starting *Cut*-depth of any derivation is finite. At some point we find a derivation of our sequent $A \succ C$ with a *Cut*-depth of *zero*: We find a derivation of $A \succ C$ without a *Cut*. That is,

THEOREM 2.9 [CUT ELIMINATION] *If a sequent is derivable with Cut, it is derivable without Cut.*

Proof: Given a derivation of a sequent $A \succ C$, take a *Cut* with no *Cut*s above it. This *Cut* has some depth, say n. Use the lemma to find a derivation with lower *Cut*-depth. Continue until there is no *Cut* remaining in this part of the derivation. (The depth of each *Cut* decreases, so this process cannot continue indefinitely.) Keep selecting cuts in the original derivation and eliminate them one-by-one. Since there are only finitely many cuts, this process terminates. The result is a *Cut*-free derivation. ■

This result is extremely powerful, and it has a number of fruitful consequences for our understanding of logical consequence, which we will consider in Section 2.4, but before that, we will extend our result to a richer language, putting together what natural deduction and the sequent calculus.

2.3 | COMPLEX SEQUENTS

Simple sequents are straightforward. They are simple – perhaps they are too simple to be a comprehensive analysis of the logical relationships between judgements involving conjunction and disjunction, let alone other connectives (like conditionals or negation). Staying with conjunction and disjunction for a moment: consider the sequent

$$p \land (q \lor r) \succ (p \land q) \lor (p \land r)$$

Is that sequent *valid*? It is not too hard to show that it has no *Cut*-free derivation.

EXAMPLE 2.10 [DISTRIBUTION IS NOT DERIVABLE] The sequent $p \land (q \lor r) \succ (p \land q) \lor r$ is not derivable.

Proof: Any *Cut*-free derivation of $p \land (q \lor r) \succ (p \land q) \lor r$ must end in either a $\land L$ step or a $\lor R$ step. Consider the two cases:

CASE 1: THE DERIVATION ENDS WITH [$\land$L]: Then we infer our sequent from either $p \succ (p \land q) \lor r$, or from $q \lor r \succ (p \land q) \lor r$. Neither of these are derivable. As you can see, $p \succ (p \land q) \lor r$ is derivable only, using $\lor R$ from either $p \succ p \land q$ or from $p \succ r$. The latter is not derivable (it is not an axiom, and it cannot be inferred from *anywhere*) and the former is derivable only when $p \succ q$ is — and it isn't. Similarly, $q \lor r \succ (p \land q) \lor r$ is derivable only when $q \succ (p \land q) \lor r$ is derivable, and this is only derivable when either $q \succ p \land q$ or when $q \succ r$ are derivable, and as before, neither of *these* are derivable either.

CASE 2: THE DERIVATION ENDS WITH [$\lor$R]: Then we infer our sequent from either $p \land (q \lor r) \succ p \land q$ or from $p \land (q \lor r) \succ r$. By dual reasoning, neither of these sequents are derivable. So, $p \land (q \lor r) \succ (p \land q) \lor r$ has no *Cut*-free derivation, and by Theorem 2.9 it has no derivation at all. ■

Reflecting on this sequent, though, it seems that on one account of proof involving disjunction and conjunction, it should be derivable. After all, can't we reason like this?

> Suppose $p \land (q \lor r)$. It follows that p, and that $q \lor r$. So we have two cases: Case (1) q holds, so by p we have $p \land q$, and therefore $(p \land q) \lor (p \land r)$. Case (2) r holds, so by p we have $p \land r$, and therefore $(p \land q) \lor (p \land r)$. So in either case we have $(p \land q) \lor (p \land r)$.

This looks like perfectly reasonable reasoning, and it's reasoning that cannot be reflected in our simple sequent system. The reason is that at the point we wish to split into two cases (on the basis of $q \vee r$), we want to *also* use the information that p holds. Our simple sequents have no space for that. If we expand them a little bit, to allow for more than one formula on the left, we could represent the reasoning like this:

Be careful: I have not yet defined the rules for conjunction or disjunction that this derivation is using.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad q \succ q}{p, q \succ p \wedge q}}{p, q \succ (p \wedge q) \vee (p \wedge r)} \quad \dfrac{\dfrac{p \succ p \quad r \succ r}{p, r \succ p \wedge r}}{p, r \succ (p \wedge q) \vee (p \wedge r)}}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}$$

Now, a sequent has the shape

$$X \succ B$$

where X is a collection of formulas, and B is a single formula. The flexibility of sequents like this allows for us to represent more reasoning. The sequent $X \succ B$ can be seen as making the claim that the conclusion B follows from the premises X. Once we allow sequents to have this structure, we should revisit our rules for conjunction and disjunction to see how they should behave in this new setting. Recall the simple sequent system rules in Figure 2.1 on page 51. These rules tell us the behaviour of identity sequents, the form of the *Cut* rules, and how to introduce a conjunction or a disjunction on the left hand side of a sequent, and on the right. We want to do the same in our new setting, where sequents can have multiple premises. Here is one option, where we take the old rules, and simply replace all parameters on the left (instances of L) with collections, and allow for extra formulas on the left where the rules allow. The result is in Figure 2.2. This is the simplest and most straightforward

$$p \succ p \; Id \qquad \dfrac{X \succ C \quad Y, C \succ R}{X, Y \succ R}\, Cut$$

$$\dfrac{X, A \succ R}{X, A \wedge B \succ R}\, \wedge L_1 \qquad \dfrac{X, A \succ R}{X, B \wedge A \succ R}\, \wedge L_2 \qquad \dfrac{X \succ A \quad X \succ B}{X \succ A \wedge B}\, \wedge R$$

$$\dfrac{X, A \succ R \quad X, B \succ R}{X, A \vee B \succ R}\, \vee L \qquad \dfrac{X \succ A}{X \succ A \vee B}\, \vee R_1 \qquad \dfrac{X \succ A}{X \succ B \vee A}\, \vee R_2$$

Figure 2.2: LATTICE RULES FOR MULTIPLE PREMISE SEQUENTS

extension of the simple sequent proof system to allow for sequents with multiple premises. Notice that each simple sequent derivation counts as a derivation in this new system, because we have just modified our rules to allow for more cases: now we allow more than one formula on the left of a sequent. As a result, in this new sequent system, identity sequents (of the form $A \succ A$) remain derivable, with the same derivations as before. Theorem 2.3 applies to this system of sequents. The subformula property holds, too, as each rule (other than *Cut*) still has the property that the formulas in the premise sequents of a rule remain (at least, as subformulas) in the concluding sequent of that rule.

The process for the elimination of the *Cut* rule in derivations is, however, more complicated. We cannot simply take the existing proof of the *Cut* elimination theorem and state that it applies here, too, for now, we have more flexibility in our derivations, and more ways that *Cut* can interact with a derivation. We will need to do more work to show how *Cut* can be eliminated from these derivations.

Before we do that, however, let's do a little more to explore the behaviour of these sequents. Notice the intuitive justification of the distribution sequent $p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)$ on page 61. In that derivation, we derive a sequent $p, q \succ p \wedge q$ on the way to proving our desired sequent. No doubt, $p, q \succ p \wedge q$ seems like a plausibly valid sequent (if p and q hold, so does $p \wedge q$). Can we derive it using the rules of our sequent system, as laid out in Figure 2.2?

It turns out that we can't, not as they stand. The sequents $p \succ p$ and $q \succ q$ are both instances of the identity axiom. We can derive $p \wedge q \succ p$ and $p \wedge q \succ q$ using the $[\wedge L]$ rules, but there is no way to derive $p, q \wedge p \wedge q$. To derive $X \succ p \wedge q$ using the $[\wedge R]$ rule we need to first derive $X \succ p$ and $X \succ q$, so to use $[\wedge R]$ to derive $p, q \succ p \wedge q$, we need to derive $p, q \succ p$ and $p, q \succ q$. But these are not axiomatic sequents. We can prove p from p, but not from p together with q.

This should remind you of the discussion of vacuous discharge in Chapter 1. It turns out that exactly the same phenomenon can be observed here with sequent derivations. We have some more options available to us, concerning how to use multiple premises in our sequents. These rules govern how the *structures* of our sequents behave, and as a result, they are called *structural rules*.

Contrast structural rules with the *connective* rules, which govern the behaviour of judgements featuring particular components, like conjunction, disjunction or the conditional.

» «

How should we derive $p, q \succ p$? The justification seems to not depend on any particular behaviour of q (the q could be *any* proposition at all), and there is nothing special in the behaviour of the sequent $p \succ p$ here, other than the fact that we've already derived *it*. There is a general principle at play. If we have derived a sequent $X \succ R$, then we could have the *weaker* sequent $X, A \succ R$, for any formula A. If X and A hold, so does R, because X holds. (We do not need to appeal to A in the justification of R. It stands unused.) In fact, the conclusion of this move of weakening is not general enough. The weakened-in item here need not be another formula. It could well be a collection of formulas all of its own. So, we have the general form of the structural rule of weakening:

$$\frac{X \succ A}{X, Y \succ A}\,{\scriptstyle K}$$

The label, *K*, comes from Schönfinkel's *Combinatory Logic* [26, 27, 102]. Now, we can go much further in our derivation of the distribution sequent:

You can remember it like this: **K** for wea**K**ening, but Schönfinkel took K to stand for *Konstanzfunktion*.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, q \succ p}{\scriptstyle K} \qquad \dfrac{q \succ q}{p, q \succ q}{\scriptstyle K}}{p, q \succ p \wedge q}{\scriptstyle \wedge R}}{p, q \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \vee R} \qquad \dfrac{\dfrac{\dfrac{p \succ p}{p, r \succ p}{\scriptstyle K} \qquad \dfrac{r \succ r}{p, r \succ p}{\scriptstyle K}}{p, r \succ p \wedge r}{\scriptstyle \wedge R}}{p, r \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \vee R}}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \vee L}$$

However, that is not enough to justify $p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)$, as our rules stand. We can continue like this

$$\dfrac{\dfrac{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}{p \wedge (q \vee r), q \vee r \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \wedge L}}{p \wedge (q \vee r), p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \wedge L}$$

and we are nearly at our desired conclusion. We just need some way to coalesce the two appeals to $p \wedge (q \vee r)$ into one. We want to appeal to the structural rule of *contraction*.

The '**W**' isn't due to Schönfinkel—this comes from Haskell Curry [26]. No, I will not attempt to justify the choice of letter by making some reference to cont**W**action. Curry's justification was the natural association of the letter with repetition.

$$\frac{X, Y, Y \succ R}{X, Y \succ R}\,{\scriptstyle W}$$

The full derivation of distribution, using weakening and contraction, goes as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, q \succ p}{\scriptstyle K} \qquad \dfrac{q \succ q}{p, q \succ q}{\scriptstyle K}}{p, q \succ p \wedge q}{\scriptstyle \wedge R}}{p, q \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \vee R} \qquad \dfrac{\dfrac{\dfrac{p \succ p}{p, r \succ p}{\scriptstyle K} \qquad \dfrac{r \succ r}{p, r \succ p}{\scriptstyle K}}{p, r \succ p \wedge r}{\scriptstyle \wedge R}}{p, r \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \vee R}}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \vee L}}{p \wedge (q \vee r), q \vee r \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \wedge L}}{p \wedge (q \vee r), p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle \wedge L}}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}{\scriptstyle W}$$

There are other possible structural rules you can explore [96], especially if you treat sequents as composed of lists or other structured collections of formulas. We will not explore those structural rules here. For us, the left hand side of a sequent will consist of a multiset of formulas: the sequent $A, B \succ C$ is literally the *same* sequent as $B, A \succ C$. Contraction and weakening will be enough structural rules for us to consider, just as vacuous discharge and duplicate discharge were enough for us to consider when it came to natural deduction.

If the left hand side of a sequent is a list of formulas, then $p, q \succ r$ is different from $q, p \succ r$.

» «

The parallels between structural rules and discharge policies are no coincidence. Given sequents that allow for more than one formula on the left, it is straightforward to give sequent rules for the conditional. Here is a natural [$\to R$] rule:

This can be motivated directly in terms of natural deduction proofs if you like. Given a proof from premises X together with A to B, if we discharge the A, we have constructed a proof from X to $A \to B$.

$$\frac{X, A \succ B}{X \succ A \to B}\,{\scriptstyle \to R}$$

What would a matching [$\to L$] rule look like? One constraint is that we would like to be able to derive $A \to B \succ A \to B$ from the sequents $A \succ A$ and $B \succ B$. So, we need to be able to derive $A \to B, A \succ B$ so we can go on like this:

$$\frac{A \to B, A \succ B}{A \to B \succ A \to B}\,{\scriptstyle \to R}$$

That much is sure. But think about this more generally: when should we be able to derive $A \to B, Z \succ R$ for arbitrary Z and R? Well, if we could use some of the formulas in Z to derive A, and we could use the other formulas in Z, together with B to derive R, then we'd have enough to conclude R on the basis of Z and $A \to B$. Splitting the Z up into two parts, we get this rule:

$$\frac{X \succ A \quad B, Y \succ R}{X, A \to B, Y \succ R}\,{\scriptstyle \to R}$$

This rule suffices to derive $A \to B, A \succ B$, since we have $A \succ A$ and $B \succ B$ (here X is A, Y is nothing at all, and R is B). Now we can see the connection between weakening and vacuous discharge. Here is a derivation of $p \succ q \to p$, paired with a natural deduction proof using vacuous discharge:

$$\frac{\dfrac{p \succ p}{p, q \succ p}\,{\scriptstyle K}}{p \succ q \to p}\,{\scriptstyle \to R} \qquad \frac{p}{q \to p}\,{\scriptstyle \to I, 1}$$

Similarly, duplicate discharge fits naturally with contraction.

$$\frac{\dfrac{\dfrac{p \succ p \quad \dfrac{p \succ p \quad q \succ q}{p \to q, p \succ q}\,{\scriptstyle \to L}}{p \to (p \to q), p, p \succ q}\,{\scriptstyle \to L}}{p \to (p \to q), p \succ q}\,{\scriptstyle W}}{p \to (p \to q) \succ p \to q}\,{\scriptstyle \to R} \qquad \frac{\dfrac{\dfrac{p \to (p \to q) \quad [p]^{(1)}}{p \to q}\,{\scriptstyle \to E} \quad [p]^{(1)}}{q}\,{\scriptstyle \to E}}{p \to q}\,{\scriptstyle \to I, 1}$$

The [$\to R$] rule has an interesting feature, not shared by any of the other rules in the system so far. The number of formulas on the left in the premise sequent $X, A \succ B$ is lowered by one in the conclusion sequent $X \succ A \to B$. This makes sense even if X is empty. Sequents, then, can have *empty* left hand sides. This corresponds to the move in a natural deduction proof when all premises have been discharged.

$$\dfrac{\dfrac{\dfrac{p \succ p \quad q \succ q}{p \to q, p \succ q}{\scriptstyle \to L}}{p \succ (p \to q) \to q}{\scriptstyle \to R}}{\succ p \to ((p \to q) \to q)}{\scriptstyle \to R} \qquad \dfrac{\dfrac{\dfrac{[p \to q]^{(1)} \quad [p]^{(2)}}{q}{\scriptstyle \to E}}{(p \to q) \to q}{\scriptstyle \to I,1}}{p \to ((p \to q) \to q)}{\scriptstyle \to I,2}$$

If the left hand side of a sequent can be empty, what about its right? If we generalise $A \succ B$ to $\succ B$, thinking of this sequent as signifying what can be derived with no active assumptions, then how should we think of $A \succ$, with an empty right? We know that if $\succ B$ is derivable, and $B \succ B'$ is derivable, then by *Cut*, we have $\succ B'$. Consequences of tautologies are tautologies. Applying this reasoning with *Cut* to empty conclusions, if $A \succ$ is derivable and $A' \succ A$ is also derivable, then so is $A' \succ$. The preservation goes the other way. The natural analogue is that if $A \succ$ is derivable, then A is a *contradiction*. Those things that entail contradictions are also contradictions. If the empty left hand side signifies truth on the basis of derivation alone, the empty right signifies falsity on the basis of derivation alone. This insight, and this slight expansion of the notion of a sequent gives us the scope to define *negation*. The natural rule for negation on the right is this:

$$\dfrac{X, A \succ}{X \succ \neg A}{\scriptstyle \neg R}$$

If X and A together are contradictory, then X suffices for $\neg A$, the negation of A. What is an appropriate rule for negation on the left of a sequent? The following rule:

$$\dfrac{X \succ A}{X, \neg A \succ}{\scriptstyle \neg L}$$

allows for the derivation of the identity sequent $\neg A \succ \neg A$, and seems well motivated.

Later in this section we'll see that it also allows for the elimination of *Cut*.

$$\dfrac{\dfrac{A \succ A}{\neg A, A \succ}{\scriptstyle \neg L}}{\neg A \succ \neg A}{\scriptstyle \neg R}$$

These rules, then, give us a logical system with a rich repertoire of propositional connectives: $\wedge, \vee, \to, \neg$. Here are some derivations showing some of the interactions between these connective rules. The first is a derivation of a principle connecting conditionals and conjunctions:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad q \succ q}{p \to q, p \succ q}{\scriptstyle \to L}}{(p \to q) \wedge (p \to r), p \succ q}{\scriptstyle \wedge L_1} \quad \dfrac{\dfrac{p \succ p \quad r \succ r}{p \to r, p \succ r}{\scriptstyle \to L}}{(p \to q) \wedge (p \to r), p \succ r}{\scriptstyle \wedge L_2}}{(p \to q) \wedge (p \to r), p \succ q \wedge r}{\scriptstyle \wedge R}}{(p \to q) \wedge (p \to r) \succ p \to (q \wedge r)}{\scriptstyle \to R}$$

Notice that this derivation does not use either contraction or weakening. Neither does this next derivation, connecting conditionals, conjunction and disjunction:

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$r \succ r$}
\RightLabel{$\scriptstyle \to L$}
\BinaryInfC{$p \to r, p \succ r$}
\RightLabel{$\scriptstyle \wedge L_1$}
\UnaryInfC{$(p \to r) \wedge (q \to r), p \succ r$}
\AxiomC{$q \succ q$}
\AxiomC{$r \succ r$}
\RightLabel{$\scriptstyle \to L$}
\BinaryInfC{$q \to r, q \succ r$}
\RightLabel{$\scriptstyle \wedge L_2$}
\UnaryInfC{$(p \to r) \wedge (q \to r), q \succ r$}
\RightLabel{$\scriptstyle \vee L$}
\BinaryInfC{$(p \to r) \wedge (q \to r), p \vee q \succ r$}
\RightLabel{$\scriptstyle \to R$}
\UnaryInfC{$(p \to r) \wedge (q \to r) \succ (p \vee q) \to r$}
\end{prooftree}
$$

This next pair of derivations shows that two de Morgan laws hold in this sequent system—again, without making appeal to any structural rules.

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\scriptstyle \vee R_1$}
\UnaryInfC{$p \succ p \vee q$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$\neg(p \vee q), p \succ$}
\RightLabel{$\scriptstyle \neg R$}
\UnaryInfC{$\neg(p \vee q) \succ \neg p$}
\AxiomC{$q \succ q$}
\RightLabel{$\scriptstyle \vee R_2$}
\UnaryInfC{$q \succ p \vee q$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$\neg(p \vee q), q \succ$}
\RightLabel{$\scriptstyle \neg R$}
\UnaryInfC{$\neg(p \vee q) \succ \neg q$}
\RightLabel{$\scriptstyle \wedge R$}
\BinaryInfC{$\neg(p \vee q) \succ \neg p \wedge \neg q$}
\RightLabel{$\scriptstyle \to R$}
\UnaryInfC{$\succ \neg(p \vee q) \to (\neg p \wedge \neg q)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$\neg p, p \succ$}
\RightLabel{$\scriptstyle \wedge L_1$}
\UnaryInfC{$\neg p \wedge \neg q, p \succ$}
\AxiomC{$q \succ q$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$\neg q, q \succ$}
\RightLabel{$\scriptstyle \wedge L_2$}
\UnaryInfC{$\neg p \wedge \neg q, q \succ$}
\RightLabel{$\scriptstyle \vee L$}
\BinaryInfC{$\neg p \wedge \neg q, p \vee q \succ$}
\RightLabel{$\scriptstyle \neg R$}
\UnaryInfC{$\neg p \wedge \neg q \succ \neg(p \vee q)$}
\RightLabel{$\scriptstyle \to R$}
\UnaryInfC{$\succ (\neg p \wedge \neg q) \to \neg(p \vee q)$}
\end{prooftree}
$$

For the other de Morgan laws, connecting the $\neg p \vee \neg q$ and $\neg(p \wedge q)$, one direction is derivable:

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$\neg p, p \succ$}
\RightLabel{$\scriptstyle \wedge L_1$}
\UnaryInfC{$\neg p, p \wedge q \succ$}
\AxiomC{$q \succ q$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$\neg q, q \succ$}
\RightLabel{$\scriptstyle \wedge L_2$}
\UnaryInfC{$\neg q, p \wedge q \succ$}
\RightLabel{$\scriptstyle \vee L$}
\BinaryInfC{$\neg p \vee \neg q, p \wedge q \succ$}
\RightLabel{$\scriptstyle \neg R$}
\UnaryInfC{$\neg p \vee \neg q \succ \neg(p \wedge q)$}
\RightLabel{$\scriptstyle \to R$}
\UnaryInfC{$\succ (\neg p \vee \neg q) \to \neg(p \wedge q)$}
\end{prooftree}
$$

The other direction, however, cannot be derived. There is no way to derive $\neg(p \wedge q) \succ \neg p \vee \neg q$. (The rule [$\neg L$] cannot be applied, since the right hand side of the sequent is not empty, and there is no way to apply [$\vee R$], since neither disjunct $\neg p$ or $\neg q$ can be derived from $\neg(p \wedge q)$.)

The logic of negation in this sequent system is *not* classical two valued – or Boolean – logic. While we can derive $p \succ \neg\neg p$:

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$p, \neg p \succ$}
\RightLabel{$\scriptstyle \neg R$}
\UnaryInfC{$p \succ \neg\neg p$}
\end{prooftree}
$$

there is no way to derive the converse, $\neg\neg p \succ p$. Again, we cannot apply [$\neg L$] to derive $\neg\neg p \succ p$ since the left hand side is not empty. For the same sort of reason, we cannot derive $\succ p \vee \neg p$.

If we had some way to move the $\neg p$ over to the right hand side while keeping the p on the right hand side as well, we could derive $\neg\neg p \succ p$, as we will see soon.

We have seen the behaviour of the conditional varies with respect to the presence or absence of structural rules, in a way that parallels the use of different discharge policies in natural deduction. The influence of these structural rules extends beyond the behaviour of the conditional, to the other connectives. We have seen this already with the distribution of $\land$ over $\lor$: $p \land (q \lor r) \succ (p \land q) \lor (p \land r)$ can be derived using contraction and weakening, as seen on page 63.

Here are two more negation principles derivable using contraction, are not derivable without it.

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ p}{\neg p, p \succ}\;{\scriptstyle \neg L}}{p \to \neg p, p, p \succ}\;{\scriptstyle \to L}}{p \to \neg p, p \succ}\;{\scriptstyle W}}{p \to \neg p \succ \neg p}\;{\scriptstyle \neg R}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\;{\scriptstyle \neg L}}{p, p \land \neg p \succ}\;{\scriptstyle \land L_2}}{p \land \neg p, p \land \neg p \succ}\;{\scriptstyle \land L_1}}{p \land \neg p \succ}\;{\scriptstyle W}}{\succ \neg(p \land \neg p)}\;{\scriptstyle \neg R}$$

The principle to the effect that the falsity of a conditional entails the falsity of its consequent can be derived using weakening, as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{q, p \succ p}\;{\scriptstyle K}}{q \succ p \to q}\;{\scriptstyle \to R}}{\neg(p \to q), q \succ}\;{\scriptstyle \neg L}}{\neg(p \to q) \succ \neg q}\;{\scriptstyle \neg R}$$

The shift to sequents with a collection of formulas on the left hand side has brought the question of structural rules to the fore. Do repeated formulas matter? — This is the question of contraction. Does the addition of an extra formula break validity? — This is the question of weakening.

With the introduction of the possibility of empty right hand sides in sequents, the question of weakening arises for the right hand side, too. If we have a derivation of $X \succ$ (that is, if the premises X are inconsistent), can move to the sequent $X \succ A$, adding the conclusion A where there was none before? This is the structural rule of weakening on the *right*. This means that we now have three different structural rules.

$$\dfrac{X, Y, Y \succ R}{X, Y \succ}\;{\scriptstyle W} \qquad \dfrac{X, Y, Y \succ R}{X, Y \succ R}\;{\scriptstyle KL} \qquad \dfrac{X \succ}{X \succ R}\;{\scriptstyle KR}$$

With weakening on the right, contradictions entail arbitrary conclusions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\neg L}{p, p \wedge \neg p \succ}\wedge L_2}{p \wedge \neg p, p \wedge \neg p \succ}\wedge L_1}{p \wedge \neg p \succ}W}{p \wedge \neg p \succ q}KR$$

We'll *prove* this in the next chapter, when we will see how sequents and derivations can be related to models. We'll show that the derivable sequents in this system match exactly those that hold in every Kripke model for intuitionistic logic.

The proof system with the full complement of structural rules is a sequent system for intuitionistic propositional logic [28, 56]. If we drop the rule of weakening on the right, we get *Minimal logic*, if we also drop the rule of weakening on the left, we get *Intuitionistic Relevant logic* (without distribution), and the system with out contraction or weakening is *Intuitionistic Linear logic*.

Or, *nearly*. We haven't quite modelled the whole of intuitionistic linear logic. We have missed out one connective, known in the linear logic and relevant logic traditions under various guises, such as *multiplicative conjunction* or *fusion*. Here is another way we could give left and right rules for a conjunction-like connective:

$$\dfrac{X, A, B \succ R}{X, A \otimes B \succ R}\otimes L \qquad \dfrac{X \succ A \quad Y \succ B}{X, Y \succ A \otimes B}\otimes R$$

Here, the connective $\otimes$ corresponds directly to the comma of premise combination in sequents. In the left rule, this is immediate. We trade in a comma on the left for a fusion. To derive the fusion of two formulas A and B, we derive A from X, and B from Y and combine the premises X and Y with another comma.

We can see immediately that the rules allow us to derive identity sequents for $\otimes$:

$$\dfrac{\dfrac{A \succ A \quad B \succ B}{A, B \succ A \otimes B}\otimes R}{A \otimes B \succ A \otimes B}\otimes L$$

and $\otimes$ is connected intimately with the conditional. The sequent $A \otimes B \succ C$ holds if and only if $A \succ B \to C$, as we can see using this pair of derivations:

$$\dfrac{\dfrac{\dfrac{\vdots}{A \succ B \to C} \quad \dfrac{B \succ B \quad C \succ C}{B \to C, B \succ C}\to L}{A, B \succ C}Cut}{A \otimes B \succ C}\otimes L \qquad \dfrac{\dfrac{\dfrac{A \succ A \quad B \succ B}{A, B \succ A \otimes B}\otimes R \quad \dfrac{\vdots}{A \otimes B \succ C}}{A, B \succ C}Cut}{A \succ B \to C}\to R$$

The connective $\otimes$ becomes salient in relevant and linear logic because in the context of intuitionistic or minimal logic – that is, in the presence of contraction and weakening (on the left) – $\otimes$ is another way of expressing the standard conjunction $\wedge$.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad q \succ q}{p, q \succ p \otimes q}\,{\scriptstyle \otimes R}}{p, p \wedge q \succ p \otimes q}\,{\scriptstyle \wedge L_2}}{p \wedge q, p \wedge q \succ p \otimes q}\,{\scriptstyle \wedge L_1}}{p \wedge q \succ p \otimes q}\,{\scriptstyle W} \qquad \dfrac{\dfrac{\dfrac{p \succ p}{p, q \succ p}\,{\scriptstyle KL} \quad \dfrac{q \succ q}{p, q \succ q}\,{\scriptstyle KL}}{p, q \succ p \wedge q}\,{\scriptstyle \wedge R}}{p \otimes q \succ p \wedge q}\,{\scriptstyle \otimes L}$$

In the presence of contraction, the lattice conjunction $\wedge$ is *at least as strong as* fusion $\otimes$. In the presence of weakening, fusion is *at least as strong as* lattice conjunction. So, in the presence of both contraction and weakening, the two coincide in strength, and it is straightforward to use the one to do the work of the other.

To round out the family of different possible connective rules, it will be worth our while to turn to another kind of concept that can be defined by way of left and right sequent rules: the propositional *constant*. One motivation for introducing propositional constants into our language is the connection between negation and the conditional. If you attend to the negation and the conditional rules, you will see some similarities between them. Consider the right rules, to start:

$$\dfrac{X, A \succ B}{X \succ A \to B}\,{\scriptstyle \to R} \qquad \dfrac{X, A \succ}{X \succ \neg A}\,{\scriptstyle \neg R}$$

Negation looks like the special case of a conditional where instead of a consequent, we have the empty right hand side of the sequent. The parallel is preserved when we consider the left rules.

$$\dfrac{X \succ A \quad B, Y \succ R}{X, A \to B, Y \succ R}\,{\scriptstyle \to L} \qquad \dfrac{X \succ A}{X, \neg A \succ}\,{\scriptstyle \neg L}$$

The negation rule is what one would get from the conditional rule where the consequent formula B somehow stands in for the empty right hand side. (Make Y and R empty, and let B be a formula that does the same job as the empty right hand side of the sequent, and the result is the [$\neg L$] rule. So, if we had a formula – call it f – that were governed by this pair of rules

$$f \succ \;{\scriptstyle fL} \qquad \dfrac{X \succ}{X \succ f}\,{\scriptstyle fR}$$

then it turns out that $\neg p$ is equivalent to $p \to f$.

$$\dfrac{\dfrac{p \succ p \quad f \succ}{p \to f, p \succ}\,{\scriptstyle \to L}}{p \to f \succ \neg p}\,{\scriptstyle \neg R} \qquad \dfrac{\dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,{\scriptstyle \neg L}}{\neg p, p \succ f}\,{\scriptstyle fR}}{\neg p \succ p \to f}\,{\scriptstyle \to R}$$

There is a sense in which the formula f is *false*—hence the choice of letter. It is a propositional constant, whose interpretation is fixed by its definition, unlike the other atoms, p, q, etc.

Just as we can have a formula corresponding to the behaviour of the empty right hand side of a sequent, we can have a formula which corresponds to the empty left hand side: t

$$\frac{X \succ R}{X, t \succ R}\,tL \qquad \succ t\ tR$$

Notice that for both f and for t, the identity sequent is derivable from the left and right rules, as for the other connectives, though in this case these do not 'connect' or combine other propositions.

$$\frac{f \succ}{f \succ f}\,fR \qquad \frac{\succ t}{t \succ t}\,tL$$

In the presence of weakening on the right [*KR*], we can derive $f \succ R$ from the axiom $f \succ$ – given [*KR*] f entails all propositions. In the presence of weakening on the left [*KL*], we can derive $X \succ t$ from the axiom $\succ t$ – given [*KL*] t is entailed by anything and everything. These facts does not hold in the absence of weakening. The role of the empty left hand side and the empty right hand side is separable from the role of the strongest and weakest propositions. We can introduce rules for *those* concepts, too, in a straightforward way:

To make sure you understand the difference between t and $\top$ and f and $\bot$, it's a useful exercise to show that t is an identity for $\otimes$, in that $t \otimes p$ is equivalent to p, while $\top$ is an identity for $\land$, in that $\top \land p$ is equivalent to p. Similarly, $\bot$ is an identity for $\lor$. What about f? Is there a connective for which f is an identity?

$$X, \bot \succ R\ \bot L \qquad X \succ \top\ \top R$$

Here there is no need to have a left rule for $\top$ or a right rule for $\bot$. The identity sequent $\top \succ \top$ is an instance of [$\top R$], and $\bot \succ \bot$ is an instance of [$\bot L$].

That is a great many connectives and rules! To help you keep stock, Figure 2.3 contains a summary of all of the sequent rules we have seen so far for each of these connectives. This family of rules (taken as a whole, altogether) defines a proof system for intuitionistic propositional logic. However, the rules give us more than that. They are a toolkit, which can be used to define a number of different proof systems. One axis of variation is the choice of structural rules: you can choose from any of the structural rules, independently of the others (giving eight different possibilities) and for each of the nine logical concepts which have been given rules ($\land, \lor, \otimes, \to, \neg$, t, f, $\top, \bot$) you have the choice of including that concept or not. That gives us $8 \times 2^9 = 4096$ different systems of rules, of varying degrees of expressive power and logical strength.

Actually, it defines intuitionistic propositional logic with two different conjunction connectives, $\land$ and $\otimes$, which are provably equivalent.

Not all of these 4096 systems differ in expressive power, of course. If we have negation at hand, and t, then $\neg t$ can be shown to be logically equivalent to f, so even if we do not include the f rules, we can get their effect using $\neg t$. But beware, you cannot, in general, define t as $\neg f$, and nor can you define $\bot$ as $\neg\top$, in the absence of weakening on the right.

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For all of the variety of these 4096 different systems, none of them give us a derivation of the following classically valid sequents:

$$\succ p \lor \neg p \qquad \neg\neg p \succ p \qquad \neg(p \land q) \succ \neg p \lor \neg q \qquad (p \to q) \to p \succ p$$

One way to strengthen this would be, for example, to allow for an inference rule that warrants the inference from $X \succ \neg\neg A$ to $X \succ A$ – the

$$p \succ p \; Id \qquad \frac{X \succ C \quad Y, C \succ R}{X, Y \succ R}\,Cut$$

$$\frac{X, Y, Y \succ R}{X, Y \succ R}\,W \qquad \frac{X \succ R}{X, Y \succ R}\,KL \qquad \frac{X \succ}{X \succ R}\,KR$$

$$\frac{X, A \succ R}{X, A \wedge B \succ R}\,\wedge L_1 \qquad \frac{X, A \succ R}{X, B \wedge A \succ R}\,\wedge L_2 \qquad \frac{X \succ A \qquad X \succ B}{X \succ A \wedge B}\,\wedge R$$

$$\frac{X, A \succ R \qquad X, B \succ R}{X, A \vee B \succ R}\,\vee L \qquad \frac{X \succ A}{X \succ A \vee B}\,\vee R_1 \qquad \frac{X \succ A}{X \succ B \vee A}\,\vee R_2$$

$$\frac{X, A, B \succ R}{X, A \otimes B \succ R}\,\otimes L \qquad \frac{X \succ A \quad Y \succ B}{X, Y \succ A \otimes B}\,\otimes R$$

$$\frac{X \succ A \quad B, Y \succ R}{X, A \to B, Y \succ R}\,\to L \qquad \frac{X, A \succ B}{X \succ A \to B}\,\to R$$

$$\frac{X \succ A}{X, \neg A \succ}\,\neg L \qquad \frac{X, A \succ}{X \succ \neg A}\,\neg R$$

$$\frac{X \succ R}{X, t \succ R}\,tL \quad \succ t \; tR \quad X \succ \top \; \top R \quad f \succ \; fL \quad \frac{X \succ}{X \succ f}\,fR \quad X, \bot \succ R \; \bot L$$

Figure 2.3: rules for multiple premise sequents

elimination of a double negation. We could, for example, derive Peirce's Law using this principle, together with contraction and weakening:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,\neg L}{\neg p, p \succ q}\,KR}{\neg p \succ p \to q}\,\to R \qquad p \succ p}{(p \to q) \to p, \neg p \succ p}\,\to L}{(p \to q) \to p, \neg p, \neg p \succ}\,\neg L}{(p \to q) \to p, \neg p \succ}\,W}{(p \to q) \to p \succ \neg\neg p}\,\neg R}{(p \to q) \to p \succ p}\,DNE$$

But there is something quite unsatisfying in this derivation. It passes through negation, when that concept isn't used in the end sequent. There should be some explanation for why $(p \to q) \to p$ classically entails p that doesn't appeal to negation, and which uses the rules for the conditional alone. Looking at the behaviour of $\neg p$ in the derivation, you can see that we use it purely to provide some way to apply contraction, duplicating that $\neg p$ (reading from bottom to top), giving us two copies, one for each premise of the [$\to L$] rule we need to apply. There is a sense in which a p on the right of a sequent (or a $\neg\neg p$) is like a $\neg p$ on the left. If we could keep the conclusion p on the right, and allow *it* to be duplicated there, we could avoid the whole charade of disguising the p-on-the-right as a $\neg p$-on-the-left. We could do this:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p \succ q, p}\,KR}{\succ p \to q, p}\,{\to}R \quad p \succ p}{(p \to q) \to p \succ p, p}\,{\to}L}{(p \to q) \to p \succ p}\,W$$

which exposes the central structure of the derivation, and trades in any $\neg p$-on-the-left for a p-on-the-right. Now the connective rules in this derivation involve the connective in the sequent itself: the derivation is much simpler.

But what does a sequent – like $\succ p \to q, p$ – with more than one formula on the right *mean*? The individual formulas on the right are the different *cases* that are being considered. In a general sequent $A, B \succ C, D$, we have that the two cases C, D follow from the premises A, B. In other words, if *all* of A and B are hold, then *some* of C and D do too. So, in our derivation, weakening on the right tells us that $p \succ q, p$ – if p holds, then at least one of q and p hold. Then we discharge the assumption p, and conclude that at least one of $p \to q$ and p hold. And indeed, in classical two-valued logic, this is correct: either $p \to q$ holds (if p is false) or p does (otherwise).

The move from multiple premise sequents $X \succ R$ representing proofs from premises X to a conclusion R to multiple premise and multiple conclusion sequents $X \succ Y$ representing proofs from a range of premises to a range of concluding *cases* was one of Gerhard Gentzen's great insights in proof theory in the 1930s [43, 44]. Sequents with this structure provide an elegant and natural proof system for classical logic.

Here is another derivation, showing how extra positions on the right of a sequent give us exactly the space we need to derive the formerly underivable sequents $\neg\neg p \succ p$ and $\neg(p \wedge q) \succ \neg p \vee \neg q$, this time without appealing to any structural rules.

$$\dfrac{\dfrac{p \succ p}{\succ \neg p, p}\,{\scriptstyle \neg R}}{\neg\neg p \succ p}\,{\scriptstyle \neg L} \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,{\scriptstyle \neg R}}{\succ p, \neg p \vee \neg q}\,{\scriptstyle \vee R_1} \quad \dfrac{\dfrac{q \succ q}{\succ q, \neg q}\,{\scriptstyle \neg R}}{\succ q, \neg p \vee \neg q}\,{\scriptstyle \vee R_2}}{\succ p \wedge q, \neg p \vee \neg q}\,{\scriptstyle \wedge R}}{\neg(p \wedge q) \succ \neg p \vee \neg q}\,{\scriptstyle \neg L}$$

These derivations use rules for the connectives which generalise the multiple premise rules to allow for multiple formulas in the conclusion of a sequent. Here are the negation rules, in their generality:

$$\dfrac{X \succ A, Y}{X, \neg A \succ Y}\,{\scriptstyle \neg L} \qquad \dfrac{X, A \succ Y}{X \succ \neg A, Y}\,{\scriptstyle \neg R}$$

Here X and Y are completely arbitrary collections of formulas. X may be empty, or many formulas. So may Y. The rules tell us that if I have can derive A (as one of the active cases), then I can ensure that it must be one of the other cases that hold if I add $\neg A$ to the stock of my premises. (Notice that this works equally well in the case of *no* other cases: then if A follows from X, then X and $\neg A$ are inconsistent). That is the [$\neg L$] rule. The [$\neg R$] rule tells us that if from X and A I can prove Y (a single formula, or a range of cases, or even no conclusion at all—if the premises are inconsistent), then from X alone, the cases that follow are Y or $\neg A$. This is eminently understandable classical reasoning.

The same can be said for the rules for the other connectives. Now that we have space for more than one formula on the right, we can add an intensional (multiplicative) *disjunction* $\oplus$ – called *fission* – to parallel our intensional conjunction $\otimes$, *fusion*. The rules for fission correspond to the rules for fusion, swapping left and right:

$$\dfrac{X, A \succ Y \quad X', B \succ Y'}{X, X', A \oplus B \succ Y, Y'}\,{\scriptstyle \oplus L} \qquad \dfrac{X \succ A, B, Y}{X \succ A \oplus B, Y}\,{\scriptstyle \oplus R}$$

We introduce $A \oplus B$ on the right by converting the two cases A and B into the one case $A \oplus B$. (A natural way to understand 'or'.) To derive some cases (Y and Y') *from* $A \oplus B$ (with other premises), you derive some cases (say Y) from A (with some of the other premises) and the other cases (say Y') from B (with the remaining premises).

Another variation in the rules for sequents with multiple premises and multiple conclusions is in the structural rules. In sequents with single or no conclusions, we had the option of weakening on the right as well as weakening on the left, and the left structural rule is independent from the right one. In multiple conclusion sequents, this changes. This change is most clear in the presence of negation. If we have weakening on the left, then weakening on the right follows, given negation. After all, if I want to weaken A into the right of the sequent $X \succ Y$, I can

For example, in minimal logic, we have weakening on the left without weakening on the right.

weaken $\neg A$ in on the left instead, and convert this to an A on the right, using a *Cut*.

$$\dfrac{\dfrac{A \succ A}{\succ A, \neg A}\,{\scriptstyle \neg R} \qquad \dfrac{\begin{array}{c}\vdots\\ X \succ Y\end{array}}{X, \neg A \succ Y}\,{\scriptstyle K}}{X \succ A, Y}\,{\scriptstyle Cut}$$

Even without the presence of negation, the need to have weakening on the right if we have weakening on the left is made clear in the process of eliminating *Cut*s from derivations. Consider a derivation in which a sequent $X, A \succ Y$ – where that A has been weakened in – is *Cut* with another sequent: $X' \succ A, Y'$.

$$\dfrac{\begin{array}{c}\vdots\\ X' \succ A, Y'\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ X \succ Y\end{array}}{X, A \succ Y}\,{\scriptstyle K}}{X, X' \succ Y, Y'}\,{\scriptstyle Cut}$$

We have moved from the sequent $X \succ Y$ to the weaker sequent $X, X' \succ Y, Y'$ where we have weakened on both sides. In sequent systems of this structure, it is hard separate weakening on one side of a sequent from weakening on the other. The same holds for contraction, for exactly the same reasons. For example, in the presence of negation and *Cut*, contraction on the left leads to contraction on the right.

$$\dfrac{\dfrac{A \succ A}{\succ A, \neg A}\,{\scriptstyle \neg R} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ X \succ A, A, Y\end{array}}{X, \neg A \succ A, Y}\,{\scriptstyle \neg R}}{X, \neg A, \neg A \succ Y}\,{\scriptstyle \neg R}}{X, \neg A \succ Y}\,{\scriptstyle W}}{X \succ A, Y}\,{\scriptstyle Cut}$$

For this reason, in these sequents we will not separate structural rules out into left and right versions. Rather, there is a single rule of weakening, that allows weakening in collections of formulas *simultaneously* on the right and left, and similarly for contraction.

This will make commuting a *Cut* over a structural rule much more straightforward. A *Cut* after a weakening converts into a weakening – not a series of weakenings – after a *Cut*, and a *Cut* after a contraction converts into a contraction – not a series of contractions – after a *Cut*.

The full complement of rules for multiple premise / multiple conclusion sequents is found in Figure 2.4.

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DEFINITION 2.11 [SEQUENT SYSTEMS] A sequent system is given by a selection of rules from Figures 2.3 or 2.4.

An *intuitionistic* or *multiple premise* sequent system has derivations involving sequents of the form $X \succ R$ where X is a multiset of formulas and R is either a single formula, or empty. The structural rules are *Id*, *Cut*, and any choice from among [*W*], [*KR*] and [*KL*], and the connective

$$p \succ p \; {\scriptstyle Id} \qquad \frac{X \succ C, Y \quad X', C \succ Y'}{X, X' \succ Y, Y'}\, {\scriptstyle Cut}$$

$$\frac{X, X', X' \succ Y, Y', Y'}{X, X' \succ Y, Y'}\, {\scriptstyle W} \qquad \frac{X \succ Y}{X, X' \succ Y, Y'}\, {\scriptstyle K}$$

$$\frac{X, A \succ Y}{X, A \land B \succ Y}\, {\scriptstyle \land L_1} \qquad \frac{X, A \succ Y}{X, B \land A \succ Y}\, {\scriptstyle \land L_2} \qquad \frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \land B, Y}\, {\scriptstyle \land R}$$

$$\frac{X, A \succ Y \quad X, B \succ Y}{X, A \lor B \succ Y}\, {\scriptstyle \lor L} \qquad \frac{X \succ A, Y}{X \succ A \lor B, Y}\, {\scriptstyle \lor R_1} \qquad \frac{X \succ A, Y}{X \succ B \lor A, Y}\, {\scriptstyle \lor R_2}$$

$$\frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\, {\scriptstyle \otimes L} \qquad \frac{X \succ A, Y \quad X' \succ B, Y'}{X, X' \succ A \otimes B, Y, Y'}\, {\scriptstyle \otimes R}$$

$$\frac{X, A \succ Y \quad X', B \succ Y'}{X, X', A \oplus B \succ Y, Y'}\, {\scriptstyle \oplus L} \qquad \frac{X \succ A, B, Y}{X \succ A \oplus B, Y}\, {\scriptstyle \oplus R}$$

$$\frac{X \succ A, Y \quad B, X' \succ Y'}{X, A \to B, X' \succ Y, Y'}\, {\scriptstyle \to L} \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}\, {\scriptstyle \to R}$$

$$\frac{X \succ A, Y}{X, \neg A \succ Y}\, {\scriptstyle \neg L} \qquad \frac{X, A \succ Y}{X \succ \neg A, Y}\, {\scriptstyle \neg R}$$

$$\frac{X \succ Y}{X, t \succ Y}\, {\scriptstyle tL} \qquad \succ t \; {\scriptstyle tR} \qquad X \succ \top, Y \; {\scriptstyle \top R}$$

$$f \succ \; {\scriptstyle fL} \qquad \frac{X \succ Y}{X \succ f, Y}\, {\scriptstyle fR} \qquad X, \bot \succ Y \; {\scriptstyle \bot L}$$

Figure 2.4: rules for multiple conclusion sequents

$$Id_{p}: \quad p \succ p$$

$$Id_{A\wedge B}: \quad \dfrac{\dfrac{Id_A}{A \wedge B \succ A}{\scriptstyle \wedge L_1} \qquad \dfrac{Id_B}{A \wedge B \succ B}{\scriptstyle \wedge L_2}}{A \wedge B \succ A \wedge B}{\scriptstyle \wedge R} \qquad Id_{A\vee B}: \quad \dfrac{\dfrac{Id_A}{A \succ A \vee B}{\scriptstyle \vee R_1} \qquad \dfrac{Id_B}{B \succ A \vee B}{\scriptstyle \vee R_2}}{A \vee B \succ A \vee B}{\scriptstyle \vee L}$$

$$Id_{A\otimes B}: \quad \dfrac{\dfrac{Id_A \qquad Id_B}{A, B \succ A \otimes B}{\scriptstyle \otimes R}}{A \otimes B \succ A \otimes B}{\scriptstyle \otimes L} \qquad Id_{A\oplus B}: \quad \dfrac{\dfrac{Id_A \qquad Id_B}{A \oplus B \succ A, B}{\scriptstyle \oplus L}}{A \oplus B \succ A \oplus B}{\scriptstyle \oplus R}$$

$$Id_{A\to B}: \quad \dfrac{\dfrac{Id_A \qquad Id_B}{A \to B, A \succ B}{\scriptstyle \to L}}{A \to B \succ A \to B}{\scriptstyle \to R} \qquad Id_{\neg A}: \quad \dfrac{\dfrac{Id_A}{A, \neg A \succ}{\scriptstyle \neg L}}{\neg A \succ \neg A}{\scriptstyle \neg R}$$

$$Id_{t}: \quad \dfrac{\succ t}{t \succ t}{\scriptstyle tL} \qquad Id_{\top}: \quad \top \succ \top \ {\scriptstyle \top R} \qquad Id_{f}: \quad \dfrac{f \succ}{f \succ f}{\scriptstyle fR} \qquad Id_{\bot}: \quad \bot \succ \bot \ {\scriptstyle \bot L}$$

Figure 2.5: IDENTITY DERIVATIONS FOR COMPLEX SEQUENTS

rules are the rules for any connective among ($\wedge$, $\vee$, $\otimes$, $\to$, $\neg$, t, $\top$, f, $\bot$). The sequent system for *intuitionistic logic* has all structural rules and all connectives.

For intuitionist and classical logic, it is typical to leave out some connectives which are definable in terms of the others. For example, in both intuitionist logic, f is equivalent to $\neg$t, and $\wedge$ is equivalent to $\otimes$. In classical logic, $\neg$ and $\wedge$ (for example), suffice to define *all* the connectives. However, for the sake of simplicity, and to show no favouritism between connectives, we will treat all of these concepts as independent, and we will allow the choice of any of them as primitive.

A *classical* or *multiple premise, multiple conclusion* sequent system has derivations involving sequents of the form $X \succ Y$ where X and Y are multisets of formulas. The structural rules are *Id*, *Cut*, and any choice from among [*W*], [*K*], and the connective rules are the rules for any connective among ($\wedge$, $\vee$, $\otimes$, $\oplus$, $\to$, $\neg$, t, $\top$, f, $\bot$). The sequent system for *classical logic* has all structural rules and all connectives.

THEOREM 2.12 [IDENTITY DERIVATIONS] *In any sequent system as defined in Definition 2.11, for any formula A, the identity sequent $A \succ A$ may be derived.*

Proof: For each formula A in the language, we define the identity derivation Id_A inductively, using the clauses in Figure 2.5. Notice that a derivation involving a concept (a connective or a propositional concept) uses only the sequent structure necessary for the rules involving that concept. So, for example, the identity derivation $Id_{p\wedge(q\vee r)}$ uses simple sequents only, and is a derivation in any sequent system with conjunction and disjunction in the vocabulary. The identity derivation for $\oplus$ requires multiple conclusions, so that is not a derivation in multiple premise sequent systems, but the connective $\oplus$ cannot be defined in such systems. The derivations for $\to$, $\neg$ and $\otimes$, require multiple premises, but none of the rest do. The derivations for f and $\neg$ require empty right hand sides, and the derivation for t requires an empty left hand side. ■

The last result for this section is generalising Theorem 2.9 for our sequent systems. We want to show that *Cut* is eliminable in each of oujr sequent systems. The strategy for the proof is exactly the same as the proof for simple sequents, but the details are more involved because there are more positions in our sequents on which rules can operate. The proof will proceed by showing how a *Cut* in a derivation, like this

$$\dfrac{\begin{array}{c}\vdots\,\delta_l\\ X \succ C, Y\end{array} \qquad \begin{array}{c}\vdots\,\delta_r\\ X', C \succ Y'\end{array}}{X, X' \succ Y, Y'}\,{\scriptstyle \mathit{Cut}}$$

can be made *simpler* in some way. Exactly which way depends on how the *Cut*-formula C behaves in the left and right derivations δ_l and δ_r. If C is a *passive* formula in the last inference of either δ_l or δ_r, the strategy is to pass the *Cut* upwards beyond that inference. If C is *active* in the last inference of δ_l and δ_r, we convert the *Cut* on C into *Cut*s on smaller formulas. The added complexity for our complex sequents is not first case, not the second. Showing how *Cut* formulas that are active in both premises of the *Cut* can be reduced is a matter of inspecting the left and right rules for each connective. The complexity arises out of the many more different ways that a *Cut* formula can be *passive* in an inference step.

How to treat the *Cut* depends, as before, on the behaviour of A in δ and δ', but now the structural rules and the more complex sequents give us many more options for ways in which a formula can be passive in a step in a derivation. Consider this example:

$$\dfrac{\begin{array}{c}\vdots\,\delta_l\\ X \succ C\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\delta_l\\ X', C, C \succ R\end{array}}{X, C \succ R}\,{\scriptstyle W}}{X, X' \succ R}\,{\scriptstyle \mathit{Cut}}$$

If we are to push this *Cut* over the contraction, the result is not a *Cut* on a simpler formula (a *Cut* with a smaller *rank*), nor is it a *Cut* closer to the leaves of the derivation tree (a *Cut* with a smaller *depth*). The result is *two Cuts*, with the same rank as before, and one of which has the same depth as the original *Cut*.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_l\\ X \succ C\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\delta_l\\ X \succ C\end{array} \qquad \begin{array}{c}\vdots\,\delta_r\\ X', C, C \succ R\end{array}}{X, X' \succ R}\,{\scriptstyle \mathit{Cut}}}{X, X', X' \succ R}\,{\scriptstyle \mathit{Cut}}}{X, X' \succ R}\,{\scriptstyle W}$$

It looks like we have made things worse by pushing the *Cut* upwards. Things are no better (and perhaps worse) in the case of sequents with multiple conclusions. Here, we could have contractions on both premises of a *Cut* step:

$$\dfrac{\dfrac{\dfrac{\vdots\ \delta_l}{X \succ C, C, Y}}{X \succ C, Y}\,{\scriptstyle W} \qquad \dfrac{\dfrac{\vdots\ \delta_r}{X', C, C \succ Y'}}{X', C \succ Y'}\,{\scriptstyle W}}{X, X' \succ Y, Y'}\,{\scriptstyle Cut}$$

Now, to push a *Cut* up the derivation, there would be wild proliferation, no matter what order we try to take. The process will not be quite as straightforward as in the case of simple sequents.

There are a number of options. Gentzen's original technique was that instead of eliminating *Cut*, we eliminate a more *general* inference rule, *Mix*.

$$\dfrac{X \succ C, \ldots, C, Y \quad X', C, \ldots, C \succ Y'}{X, X' \succ Y, Y'}\,{\scriptstyle Mix}$$

in which any number of instances of the *Mix*-formula may be removed. Then a *Mix* after a contraction (on either the left or the right) of a sequent can be traded in for a *Mix* processing more formulas before that contraction. That is an insightful idea, but eliminating *Mix* has other complications – such as the complication of processing the *Mix* when one *Mix* formula is active – and we will not follow Gentzen's lead.

Another approach is to rewrite the rules of the sequent system in order to get rid of contraction completely. We rearrange the rules to allow us to prove "contraction elimination" theorem, to the effect that if the premise sequent of a contraction step is derivable, then so is its conclusion, *without using contraction* [38, 78]. This involves rewriting each of the connective rules in such a way as to embed enough contraction into the rules that the separate rule is no longer required. For example, consider the conditional left rule:

$$\dfrac{X, A \succ Y \quad X', B \succ Y'}{X, A \to B, X' \succ Y, Y'}\,{\scriptstyle \to L}$$

Suppose some formula appears both in X and X', and we wish to contract it into one instance after we make this inference. Instead of doing this in an explicit step of contraction, we could eliminate the requirement by placing everything that occurs either in X or X' in *both* premises, and do the same for Y and Y', for we may need to contract formulas in the right, too. The result would be a variant of our inference rule:

$$\dfrac{X, A \succ Y \quad X, B \succ Y}{X, A \to B \succ Y}\,{\scriptstyle \to L'}$$

Now there is no need to contract formulas that end up duplicated in the concluding sequent by way of appearing in both premise sequents. But now, we have made both premise sequents *fatter*, by stuffing them with whatever appears in the endsequent of the inference, apart from the active formula. This is no problem if weakening is one of the structural rules, and it is in systems with contraction and weakening that this technique works best. Once the rules have been converted, the elimination of *Cut* is simpler, in that it never needs to be commuted over a separate contraction inference.

There are more details to take care of, especially in intuitionist logic. See exercise ?? for the details.

We won't follow the lead of this approach, either, because we our goal is an understanding of the process of *Cut* elimination that is relatively uniform between systems in which the structural rule is present and those in which it is absent – without modifying the connective rules. The target is an understanding of *Cut* elimination for any sequent system, as given in Definition 2.11. Tailoring the connective rules to incorporate the structural rules results a fine approach for particular sequent systems in which those structural rules are present, but it makes the general approach to *Cut* elimination less transparent. The most difficult case for us will be systems in which contraction is a structural rule but weakening is absent. The techniques of internalising contraction in the connective rules, as used in G3 systems, do not apply in this case. We must look elsewhere to understand how to take control of contraction and *Cut*.

Another, more recent approach to managing contraction and *Cut* is due to Katalin Bimbó [13, 14]. She has shown that if we keep track of the *contraction count* for an instance of *Cut* in a derivation (the number of applications of contraction above the *Cut*), then we can find a measure which always does decrease as the *Cut* makes its upward journey.

The approach we will take to eliminating *Cut* in these systems is originally due to Haskell Curry [25], and has been systematised and generalised by Nuel Belnap in his magisterial work on Display Logic [10], and further generalised by the current author [96]. The crucial idea is to understand the process of *Cut* elimination by keeping track of the *ancestors* of the *Cut* formula in a derivation – in either the left premise of the *Cut* step or the right – they form a *tree* above the *Cut* formula, and we perform the *Cut* instead at each of the *leaves* of that tree. At those steps the *Cut* formula is active in one premise. Then inspect the passive ancestors of the *Cut* formula in the other premise, and commute each *Cut* up to the leaves of *that* tree. The result is a derivation in which all of these *Cuts* now involve active formulas in both premises. And these can then be eliminated by reducing the *Cuts* to *Cuts* on simpler formulas. Let's illustrate this process with a concrete example. Here is a derivation with a single *Cut*, where the *Cut* formula, $\neg p$, is passive in both inferences leading up to the *Cut*.

...and in others, later in this book.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p \succ p \lor q}{\scriptstyle \lor R_1}}{\succ p \lor q, \neg p}{\scriptstyle \neg R}}{\neg(p \lor q) \succ \neg p}{\scriptstyle \neg L} \quad \dfrac{\dfrac{\dfrac{p \succ p}{\succ p, \neg p}{\scriptstyle \neg R}}{\neg p \succ \neg p}{\scriptstyle \neg L}}{\neg p \land r \succ \neg p}{\scriptstyle \land L_1}}{\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p, \neg p}{\scriptstyle \oplus L}}{\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p}{\scriptstyle W} \quad \dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{\succ p, \neg p}{\scriptstyle \neg R}}{\neg p \succ \neg p}{\scriptstyle \neg L} \quad \neg p \succ \top}{\neg p, \neg p \succ \neg p \otimes \top}{\scriptstyle \otimes R}}{\neg p \succ \neg p \otimes \top}{\scriptstyle W}}{\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top}{\scriptstyle Cut}$$

This derivation uses a large complement of concepts: $\land, \lor, \otimes, \oplus, \neg, \top$. The derivation doesn't use weakening, so fusion and fission differ in strength from lattice conjunction and disjunction.

In the derivation, the coloured boxes show the tree of ancestors of the *Cut* formula. The orange boxes trace the ancestry of the $\neg p$ as used in the left premise of the *Cut*, while the blue boxes, trace its ancestry as used in the right premise. Notice that the leaves of the orange tree both end in $[\neg R]$ steps, while one leaf of the blue tree introduces $\neg p$ in a $[\neg L]$ step, and the other introduces $\neg p$ in a $[\top R]$ axiom, where the $\neg p$ is passive. In the process of *Cut* reduction, we shift the *Cut* to the leaves of one of these trees. Let's first choose the orange tree. The result of commuting the *Cut* up the orange tree of passive ancestors is that we *Cut* the indicated sequents at the leaves of the orange tree of ancestors with the other *Cut* premise and we replace the $\neg p$ instances in that tree with the result of making the *Cut*, leaving the rest of the derivation undisturbed. Here is the result:

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p \lor q, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p$}
\AxiomC{$\neg p \succ \top$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\neg p, \neg p \succ \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\neg p \succ \neg p \otimes \top$}
\RightLabel{Cut}
\BinaryInfC{$\succ p \lor q, \neg p \otimes \top$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg(p \lor q) \succ \neg p \otimes \top$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p$}
\AxiomC{$\neg p \succ \top$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\neg p, \neg p \succ \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\neg p \succ \neg p \otimes \top$}
\RightLabel{Cut}
\BinaryInfC{$\succ p, \neg p \otimes \top$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p \otimes \top$}
\RightLabel{$\land L_1$}
\UnaryInfC{$\neg p \land r \succ \neg p \otimes \top$}
\RightLabel{$\oplus L$}
\BinaryInfC{$\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top$}
\end{prooftree}
$$

We have a tree deriving the required sequent, and the *Cut* has been driven up to the leaves of the orange tree. This means, in this case, that in the left premise of each *Cut* inference, the *Cut* formula is active. We can do the same, passing each *Cut* up to the leaves of the blue tree of passive instances of the *Cut* formula in consequent position.

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p \lor q, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p$}
\RightLabel{Cut}
\BinaryInfC{$\succ p \lor q, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p \lor q, \neg p$}
\AxiomC{$\neg p \succ \top$}
\RightLabel{Cut}
\BinaryInfC{$\succ p \lor q, \top$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\succ p \lor q, p \lor q, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\succ p \lor q, \neg p \otimes \top$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg(p \lor q) \succ \neg p \otimes \top$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p$}
\RightLabel{Cut}
\BinaryInfC{$\succ p, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\AxiomC{$\neg p \succ \top$}
\RightLabel{Cut}
\BinaryInfC{$\succ p, \top$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\succ p, p, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\succ p, \neg p \otimes \top$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p \otimes \top$}
\RightLabel{$\land L_1$}
\UnaryInfC{$\neg p \land r \succ \neg p \otimes \top$}
\RightLabel{$\oplus L$}
\BinaryInfC{$\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top$}
\end{prooftree}
$$

Here, the *Cuts* have been pushed up to the tops of both trees of ancestors of the *Cut* formula. Now each *Cut* formula is either either active in its premise, or passive in an axiom. The *Cut* steps featuring the axiomatic sequent $\neg p \succ \top$ may be immediately simplified. The $\neg p$ is passive in the axiom, and the conclusion of the *Cut* is another instance of the axiom, so we can simplify the derivation like this:

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p \lor q, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p$}
\RightLabel{Cut}
\BinaryInfC{$\succ p \lor q, \neg p$}
\AxiomC{$\succ p \lor q, \top$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\succ p \lor q, p \lor q, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\succ p \lor q, \neg p \otimes \top$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg(p \lor q) \succ \neg p \otimes \top$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p$}
\RightLabel{Cut}
\BinaryInfC{$\succ p, \neg p$}
\AxiomC{$\succ p, \top$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\succ p, p, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\succ p, \neg p \otimes \top$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p \otimes \top$}
\RightLabel{$\land L_1$}
\UnaryInfC{$\neg p \land r \succ \neg p \otimes \top$}
\RightLabel{$\oplus L$}
\BinaryInfC{$\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top$}
\end{prooftree}
$$

In the remaining *Cuts*, the *Cut* formula is active in both premises. We make this transformation:

$$
\begin{prooftree}
\AxiomC{$\vdots\ \delta_l$}
\noLine
\UnaryInfC{$X, C \succ Y$}
\RightLabel{$\neg R$}
\UnaryInfC{$X \succ \neg C, Y$}
\AxiomC{$\vdots\ \delta_r$}
\noLine
\UnaryInfC{$X' \succ C, Y'$}
\RightLabel{$\neg L$}
\UnaryInfC{$X', \neg C \succ Y'$}
\RightLabel{Cut}
\BinaryInfC{$X, X' \succ Y, Y'$}
\end{prooftree}
\quad \rightsquigarrow \quad
\begin{prooftree}
\AxiomC{$\vdots\ \delta_r$}
\noLine
\UnaryInfC{$X' \succ C, Y'$}
\AxiomC{$\vdots\ \delta_l$}
\noLine
\UnaryInfC{$X, C \succ Y$}
\RightLabel{Cut}
\BinaryInfC{$X, X' \succ Y, Y'$}
\end{prooftree}
$$

to reduce the degree of those *Cuts*, replacing each *Cut* on a negation, $\neg p$, by a *Cut* on p.

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{$\lor R_1$}
\UnaryInfC{$p \succ p \lor q$}
\RightLabel{Cut}
\BinaryInfC{$\succ p \lor q, \neg p$}
\AxiomC{$\succ p \lor q, \top$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\succ p \lor q, p \lor q, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\succ p \lor q, \neg p \otimes \top$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg(p \lor q) \succ \neg p \otimes \top$}
\AxiomC{$p \succ p$}
\RightLabel{$\neg R$}
\UnaryInfC{$\succ p, \neg p$}
\AxiomC{$p \succ p$}
\RightLabel{Cut}
\BinaryInfC{$\succ p, \neg p$}
\AxiomC{$\succ p, \top$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\succ p, p, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\succ p, \neg p \otimes \top$}
\RightLabel{$\neg L$}
\UnaryInfC{$\neg p \succ \neg p \otimes \top$}
\RightLabel{$\land L_1$}
\UnaryInfC{$\neg p \land r \succ \neg p \otimes \top$}
\RightLabel{$\oplus L$}
\BinaryInfC{$\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top, \neg p \otimes \top$}
\RightLabel{W}
\UnaryInfC{$\neg(p \lor q) \oplus (\neg p \land r) \succ \neg p \otimes \top$}
\end{prooftree}
$$

In the right *Cut* in the derivation, the right premise is an identity sequent, and a *Cut* on an identity is trivial (its other premise is its conclusion), so that can be deleted. In the *Cut* step on the left of the derivation, the *Cut* formula is passive in both sides. If we commute the *Cut* up in either direction, the result is a *Cut* free derivation of $\succ p \vee q, \neg q$ (the order of the $\vee R$ and $\neg R$ steps in that derivation depend on whether we push the *Cut* up the left branch or the right branch first). Pushing it up the left branch first, the result is this:

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\scriptstyle \vee R_1$}
\UnaryInfC{$p \succ p \vee q$}
\RightLabel{$\scriptstyle \neg R$}
\UnaryInfC{$\succ p \vee q, \neg p$}
\AxiomC{$\succ p \vee q, \top$}
\RightLabel{$\scriptstyle \otimes R$}
\BinaryInfC{$\succ p \vee q, p \vee q, \neg p \otimes \top$}
\RightLabel{$\scriptstyle W$}
\UnaryInfC{$\succ p \vee q, \neg p \otimes \top$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$\neg(p \vee q) \succ \neg p \otimes \top$}
\AxiomC{$p \succ p$}
\RightLabel{$\scriptstyle \neg R$}
\UnaryInfC{$\succ p, \neg p$}
\AxiomC{$\succ p, \top$}
\RightLabel{$\scriptstyle \otimes R$}
\BinaryInfC{$\succ p, p, \neg p \otimes \top$}
\RightLabel{$\scriptstyle W$}
\UnaryInfC{$\succ p, \neg p \otimes \top$}
\RightLabel{$\scriptstyle \neg L$}
\UnaryInfC{$\neg p \succ \neg p \otimes \top$}
\RightLabel{$\scriptstyle \wedge L_1$}
\UnaryInfC{$\neg p \wedge r \succ \neg p \otimes \top$}
\RightLabel{$\scriptstyle \oplus L$}
\BinaryInfC{$\neg(p \vee q) \oplus (\neg p \wedge r) \succ \neg p \otimes \top, \neg p \otimes \top$}
\RightLabel{$\scriptstyle W$}
\UnaryInfC{$\neg(p \vee q) \oplus (\neg p \wedge r) \succ \neg p \otimes \top$}
\end{prooftree}
$$

a derivation with no *Cut*s. Notice that the final derivation uses the same connective rules – introducing the same formulas – as were in the original derivation, but in a different order. The same structural rules were also used, but operating on *different* formulas. Had we pushed the original *Cut* up the right branch (to the leaves of the blue tree, the ancestors of the *Cut* formula in the right premise of the *Cut*), the resulting derivation would have been different, but we would also have found a *Cut*-free derivation.

This example illustrates four components of the general process of eliminating *Cut* from a derivation.

1. Defining the trees of ancestors of the *Cut* formulas in a derivation.
2. Transferring a *Cut* to the top of the tree of ancestors.
3. Eliminating a *Cut* when one instance of the *Cut* formula is introduced as a passive formula in a premise of the *Cut* (for example, in a weakening inference, or a [$\bot L$] or [$\top R$] axiom).
4. Replacing a *Cut* on a complex formula, active in both premises of that *Cut* with *Cut*s on subformulas of that formula, from the same premises.

To make these four components of the *Cut* elimination process precise, we need to extend our notion of *active* and *passive* formulas in inference rules in derivations from the case for simple sequents given Definition 2.6 on page 57.

DEFINITION 2.13 [ACTIVE AND PASSIVE FORMULAS] Any formula appearing as a component of the multisets X, Y, R in the rules for multiple premise sequents in Figure 2.3, or in the multisets X, X', Y, or Y' in multiple conclusion sequents in Figure 2.4 are said to be PASSIVE in that inference step. The other formulas are said to be ACTIVE in those inferences.

The presentation of the rules in Figures 2.3 and 2.4 not only allows us to define active and passive formulas in each instance of a rule. It also helps us define the ancestors of a formula in a derivation.

DEFINITION 2.14 [PARENTS, ORPHANS, ANCESTRY] When D is a formula occurring passively in a premise sequent of an inference falling under some rule, it is a PARENT of a single occurrence of the same formula D in the concluding sequent of that rule, where both occurrences fall under the same multiset (or formula) variable (X, Y, R, etc.) in the schematic statement of the rule. For example, in any instance of the rule $[\otimes R]$

$$\frac{X \succ A, Y \quad X' \succ B, Y'}{X, X' \succ A \otimes B, Y, Y'}\ {\otimes R}$$

A formula D occurring inside the multiset Y in the right hand side of the premise $X \succ A, Y$ of the rule is a parent of one occurrence of the formula D occurring in the right hand side of the conclusion $X, X' \succ A \otimes B, Y, Y'$. (There may be *other* instances of D occurring in that right hand side of the sequent, but they do not have our particular premise D as a parent.) If that D in the conclusion is also a member of Y, then its parent is another D occurring in the Y in the premise. If that D is in Y', its parent is in the other premise sequent. If that D is, on the other hand, the formula $A \otimes B$, then it is an active formula in this inference, and it has no parent in the premises of the inference. Passive formulas in the conclusions of most of our inference rules have single parents, but contraction brings us dual parentage. In an instance of contraction:

$$\frac{X, X', X' \succ Y, Y', Y'}{X, X' \succ Y, Y'}\ W$$

a formula in the X' or in Y' in the conclusion has two parents in the premise of the rule, while formulas in X and Y in the conclusion have a single parents in the premise.

Formulas occurring passively in inference rules (or axioms) need not have parents. The formulas in X' or Y' in a weakening inference

$$\frac{X \succ Y}{X, X' \succ Y, Y'}\ K$$

and the passive formulas in $[\bot L]$ and $[\top R]$ axioms are have no parents. They are said to be ORPHANS.

The ANCESTRY of a formula occurring passively in an inference rule of some derivation is the following collection of occurrences of the same formula in that derivation: Its ancestry is *empty* if that is an orphan. If it is not an orphan, its ancestry is its *parents*, together with the ancestries of those parents.

That completes the definitions of parents, of orphans, and of ancestry.

LEMMA 2.15 [ANCESTRY PRESERVES POSITION IN SEQUENTS] *The ancestors of a formula occurrence A in a derivation are on the same side of their sequents as the original occurrence A.*

This idea is, in fact, rather subtle, when sequents are composed of multisets of formulas. Take this $[\otimes R]$ inference:

$$\frac{p \succ p \quad p \succ \top}{p, p \succ p \otimes \top}\ {\otimes R}$$

Which occurrence of p in a premise of the inference is the ancestor of the *first* p in the sequent in the conclusion of the inference? This question, in fact, makes no sense, because the left hand side of the sequent $p, p \succ p \otimes p$ is a multiset and its members do not come in any order. The one formula p is a member of that multiset twice. We can depict the ancestry of formulas with colours like this:

$$\frac{p \succ p \quad p \succ \top}{p\,,\,p \succ p \otimes \top}\ {\otimes R}$$

but this does not pair the *first* member of the multiset $p\,,\,p$ with the p in the left hand side of the sequent $p \succ p$, for the multiset does not have a first member. This is exactly the same account of the ancestry of formulas as depicted here:

$$\frac{p \succ p \quad p \succ \top}{p\,,\,p \succ p \otimes \top}\ {\otimes R}$$

Proof: If you inspect the inference rules in Figures 2.3 and 2.4, you see that the multisets X, X', Y, Y', and the formula R never swaps sides from premise to conclusion of an inference. So a parent–child relationship is always between formulas on the same side of a sequent. ∎

The process of *Cut* elimination involves pushing a *Cut* up the tree of ancestors of the *Cut* formula, to the leaves. These orphans are either *active* in the inferences leading up to the *Cut* or *passive*. In either case, the *Cut* can be eliminated entirely, or converted into *Cut*s on subformulas of the *Cut* formula. Let's turn next to the results which allow us to transfer a *Cut* in a derivation up to the orphans in the ancestry of the *Cut* formula. To make the process precise, consider an arbitrary *Cut* step:

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ X \succ C, Y\end{array} \qquad \begin{array}{c}\vdots\,\delta_2\\ X', C \succ Y'\end{array}}{X, X' \succ Y, Y'}\;\textit{Cut}$$

If we are, for example, pushing the *Cut* up the tree of ancestors of C in δ_2, then the result of the process involve a *Cut* with δ_1 for each orphan at the leaves of the tree of ancestors, and the remaining Cs in the ancestry in δ_2 will be replaced by the result of the *Cut*. In general, this means that each non-orphan-C (in that ancestry in δ_2) will be replaced by each formula in X, and for good measure, for each such C we also add each formula in Y *on the other side of the sequent*. This process is *sequent-substitution*, as defined here:

DEFINITION 2.16 [SEQUENT SUBSTITUTION FOR MULTIPLE CONCLUSIONS] The result of substituting the sequent $X \succ Y$ for the given instance of C in $X', C \succ Y'$, or in $X' \succ C, Y$, is the sequent $X', X \succ Y', Y$. (To substitute a sequent for more than one instance of a formula in a sequent, perform that substitution once for each instance.)

In single conclusion sequents, sequent substitution needs more care, for there is less room for substitution in the right. In this case, the *Cut* has the following shape:

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ X \succ C\end{array} \qquad \begin{array}{c}\vdots\,\delta_2\\ X', C \succ R\end{array}}{X, X' \succ R}\;\textit{Cut}$$

Here, the ancestry of C in δ_1 is very simple (it is a single parent family until we reach an orphan), while its ancestry in δ_2 could be more complex. To substitute for C in $X', C \succ R$, we substitute a sequent of the form $X \succ$ for C, and the result is $X, X' \succ R$. If we substitute for C in $X \succ C$, we substitute $X' \succ R$ and the result is also $X, X' \succ R$.

DEFINITION 2.17 [SINGLE CONCLUSION SEQUENT SUBSTITUTION] The result of substituting $X \succ$ for C in $X', C \succ R$ is $X, X' \succ R$. The result of substituting $X \succ R$ for C in $X' \succ C$ is also $X, X' \succ R$. (To substitute a sequent for more than one instance of a formula in a sequent, perform the required substitution once for each instance of that formula.)

So, to pass the *Cut* up the ancestry of a formula in a derivation, we substitute the remainder of the *Cut* premise shifted up the derivation for the non-orphan formulas in the ancestry. For this to work, we need to show that substituting sequents for formulas in an ancestry does not invalidate any of the rules. That is, we will show that for any sequent rule

$$\frac{S_1 \ \cdots \ S_n}{S}$$

where some formulas C in S have parents in S_1 to S_n, then for any sequent $X \succ Y$, the result of substituting $X \succ Y$ for each instance of C in the ancestry is remains an instance of the same rule. That is, we need to show that our rules are *closed under substitution for parents and children*.

In our case, n is 1 or 2, but it rules could have more premises be more in general

LEMMA 2.18 [RULES ARE CLOSED UNDER SUBSTITUTION] *For each inference rule in Figures 2.3 and 2.4, and any formula* C *occurring passively in the conclusion of that rule, and any sequent* $U \succ V$ (*where this has the form* $U \succ R$ *in the case of single conclusion rules where* C *occurs in the right hand side of the conclusion, or* $U \succ$ *in the case of single conclusion rules where* C *occurs in the left hand side of the conclusion*), *the result of substituting* $U \succ V$ *for* C *and its parents in that inference remains an instance of the rule.*

Proof: This is verified by inspection of each of the rules. The multiple conclusion case is simplest. The passive formulas in the rules represented in Figure 2.4 are those occurring in X, X', Y or Y'. To substitute a sequent (say) $U \succ V$ for any such formula we need to find some place to slot in the extra formulas U and V. But the multisets X, Y, X', Y' are arbitrary and are as large as one likes. The crucial condition for the substitution is that no rules have a shape like

$$\frac{X \succ A}{X \succ A'}$$

for then, I could not always substitute $U \succ V$ for a formula in X, since there is nowhere for the formulas in V to go—the restriction on the right hand side to being a single formula blocks the substitution. Each of the rules in Figure 2.4 have places for arbitrary passive formulas on the left *and* the right, so replacing one formula in one such position with a family of others, both on the left and the right, results in another instance of the rule.

In the case of the contraction rule, if the formula being substituted for occurs once in the conclusion and twice in the premise, then the result of substituting $U \succ V$ for the contracted formula will have all of the formulas in U on the left and V on the right duplicated in the premise of the rule. This is why the rule of contraction has the form that it does

$$\frac{X, X', X' \succ Y, Y', Y'}{X, X' \succ Y, Y'}\,W$$

so, for example, the result of substituting $U \succ V$ for C in

$$\frac{X, A, A \succ Y}{X, A \succ Y}\,W$$

is the inference

$$\frac{X, U, U \succ Y, V, V}{X, U \succ Y, V}\, W$$

which is, indeed, another instance of the contraction rule.

For the case of rules in multiple premise single conclusion sequents, in Figure 2.3, we need to show that the more restrictive substitutions are, at least, possible. The restrictions on substitution are required, because many rules for single conclusion sequents do not have space for passive formulas on the right. The negation rules, in particular, have this shape:

$$\frac{X \succ A}{X, \neg A \succ}\, \neg L \qquad \frac{X, A \succ}{X \succ \neg A}\, \neg R$$

and here, there are no formulas on the right at all. Each of the connective right rules have no space for passive formulas on the right—the formula on the right is active in the rule. For these rules, the only formulas that could be substituted for are formulas on the left. And, in general, to substitute for C in $X, C \succ D$ we substitute a sequent $U \succ$, to get $X, U \succ D$. So, to substitute in inferences in which no formulas on the right are passive, we replace a single passive formula on the left by a multiset of formulas. But whenever a formula is passive on the left, the inference rule allows arbitrary multisets of passive formulas, so such a substitution is permissible. If a formula is passive on the right in a sequent (such as C in $X \succ C$) then we can substitute $U \succ R$ for *it*, to get $X, U \succ R$. For this to be acceptable, we need to check that any sequents in rules in which a formula on the right is passive (it is an R in the rule statements in Figure 2.3), then the rule also provides space for multiset of passive formulas on the left, available for substitution. This is the case for all such rules. (There is no rule with the shape $\succ R$ or $A \succ R$, where we are allowed arbitrary formulas on the right but not arbitrary multisets of passive formulas on the left.) So, each of our rules is general enough to allow for substitution. ■

Here is another fact about ancestry in derivations which will become important in understanding the behaviour of *Cut*s on orphan formulas.

LEMMA 2.19 [ACTIVE FORMULAS ARE SOLE ORPHANS] *If the formula C is an orphan in a sequent, and it is active in an inference leading up to that sequent, it is the only orphan in that sequent.*

Proof: You can see in all of the left and right connective rules, the only orphans in the rules are the active formula introduced. All the other formulas in the conclusion of the rule have parents in a premise sequent. (The exception to this is the additional *axioms* for $\top$ and $\bot$, which are not inference rules in the sense of this lemma.) ■

LEMMA 2.20 [CUTS CAN BE CONVERTED INTO ORPHAN ACTIVE CUTS] *In all sequent systems, a derivation ending in a Cut*

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ X \succ C, Y\end{array} \qquad \begin{array}{c}\vdots\,\delta_2\\ X', C \succ Y'\end{array}}{X, X' \succ Y, Y'}\,\textit{Cut}$$

(*and in which there are no other Cuts on* C) *can be systematically transformed into a derivation of the same conclusion in which the only Cuts on* C *are inferences in which the Cut formula* C *is an orphan and active in both premises. The resulting derivation contains no inference rules not present in the original derivation, and in particular, if* δ_1 *and* δ_2 *are Cut-free, then the only Cuts in the new derivation are those orphan Cuts on* C.

Proof: Consider the tree of ancestors of the cut formula C in δ_1. Substitute $X' \succ Y'$ for C for all non-orphan instances of C in the ancestry in δ_1. For each inference in which the substitution is made to premises and the conclusion, the inference is still an instance of that rule. (And, if C was not an orphan in the final inference of δ_1, the result is now the conclusion of the *Cut* step, $X, X' \succ Y, Y'$.) For sequents in δ_1 where C occurs as an orphan in this ancestry, either it is introduced in a connective rule or an *Id* axiom, and it is the sole orphan, or it is a family of passive instances, or it is $\top$ or $\bot$ in their axioms. In the former case (the sole orphan active formula), replace the component

$$\begin{array}{c}\vdots\\ U \succ C, V\end{array}$$

in which the indicated C is active, by the following instance of *Cut*.

$$\dfrac{\begin{array}{c}\vdots\\ U \succ C, V\end{array} \quad \begin{array}{c}\vdots\,\delta_2\\ X', C \succ Y'\end{array}}{U, X' \succ V, Y'}\,\textit{Cut}$$

The result of this step is the sequent $U \succ C, V$ where $X' \succ Y'$ has been substituted for C, so it leads appropriately into the rest of the derivation, where the substitution has occurred for non-orphan instances of C, including those which had *this* C as a parent.

For sequents in which there are a number of orphan instances of C in the ancestry, these are either *all* passive, or C is $\top$ and this is the [$\top R$] axiom, or C is $\bot$ and this is the [$\bot L$] axiom. In the case of all orphan instances of C being passive in the inference rule, this means that C is introduced by weakening (its parents are not in the premises, and it is not active), so the rule is also closed under substitution of $X' \succ Y'$ for C, so we may replace the conclusion of this rule by the desired substitution.

The remaining case is where C is either $\top$ and the sequent in which C is an orphan is a [$\top R$] axiom or it is $\bot$ and the sequent is a [$\bot L$] axiom. Consider the case for $\top$. Our sequent has the form

$$X \succ \top, \ldots, \top, Y$$

where one $\top$ is active and the others passive, and we wish to conclude

$$X, X', \ldots, X' \succ Y, Y', \ldots, Y'$$

substituting $X' \succ Y'$ for each instance of $\top$. We have a derivation δ_2:

$$\begin{array}{c} \vdots\ \delta_2 \\ X', \top \succ Y' \end{array}$$

In this derivation, $\top$ is passive *everywhere,* since there is no inference rule in which $\top$ is active on the left of a sequent. As a result, we can substitute $X, X', \ldots, X' \succ Y, Y', \ldots, Y'$ (with one fewer instances of X' and Y') for the ancestry of that $\top$ in δ_2. The result is a derivation of the required sequent.

So, we have pushed the *Cut* up the ancestry of C in δ_1. The same technique allows us to push the remaining *Cut*s up the ancestry of C in each instance of δ_2, and the result is a derivation in which the remaining *Cut*s on C feature C as active in both premises of the *Cut*. ■

The remaining component of the elimination of *Cut*s is reducing the complexity of *Cut* formulas.

LEMMA 2.21 [REDUCTION OF RANK FOR CUT FORMULAS] *In any derivation*

$$\dfrac{\begin{array}{c} \vdots\ \delta_1 \\ X \succ C, Y \end{array} \qquad \begin{array}{c} \vdots\ \delta_2 \\ X', C \succ Y' \end{array}}{X, X' \succ Y, Y'}\ \mathit{Cut}$$

where C *is active in the final inferences of* δ_1 *and* δ_2*, this Cut can be replaced by Cuts on subformulas of* C*, or by no Cuts at all.*

Proof: For *identity* sequents, the *Cut* reduction is immediate. A *Cut*

$$\dfrac{p \succ p \quad p \succ p}{p \succ p}\ \mathit{Cut}$$

can be replaced by the axiom $p \succ p$. Then for non-identity sequents, we replace *Cut*s on a case-by-case basis. For multiple premise and single conclusion sequents, use the reductions in Figure 2.6. For multiple conclusion sequents, use the reductions in Figure 2.7. ■

Putting these results together, we show how *Cut* can be eliminated from derivations in any of our sequent systems.

THEOREM 2.22 [CUT ELIMINATION IN SEQUENT SYSTEMS] *Any derivation* δ *in a sequent system may be systematically transformed into a derivation of the same sequent in which no Cuts are used.*

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l^1}{X \succ C_1} \quad \overset{\vdots\,\delta_l^2}{X \succ C_2}}{X \succ C_1 \wedge C_2}\,{\scriptstyle \wedge R} \qquad \dfrac{\overset{\vdots\,\delta_r}{X', C_i \succ R}}{X', C_1 \wedge C_2 \succ R}\,{\scriptstyle \wedge L_i}}{X, X' \succ R}\,{\scriptstyle Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots\,\delta_l^i}{X \succ C_i} \quad \overset{\vdots\,\delta_r}{X', C_i \succ R}}{X, X' \succ R}\,{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l}{X \succ C_i}}{X \succ C_1 \vee C_2}\,{\scriptstyle \vee R_i} \qquad \dfrac{\overset{\vdots\,\delta_r^1}{X', C_1 \succ R} \quad \overset{\vdots\,\delta_r^2}{X', C_2 \succ R}}{X', C_1 \vee C_2 \succ R}\,{\scriptstyle \vee L}}{X, X' \succ R}\,{\scriptstyle Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots\,\delta_l}{X \succ C_i} \quad \overset{\vdots\,\delta_r^i}{X', C_i \succ R}}{X, X' \succ R}\,{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l^1}{X \succ C_1} \quad \overset{\vdots\,\delta_l^2}{X' \succ C_2}}{X, X' \succ C_1 \otimes C_2}\,{\scriptstyle \otimes R} \qquad \dfrac{\overset{\vdots\,\delta_r}{X'', C_1, C_2 \succ R}}{X'', C_1 \otimes C_2 \succ R}\,{\scriptstyle \otimes L}}{X, X', X'' \succ R}\,{\scriptstyle Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots\,\delta_l^1}{X \succ C_1} \qquad \dfrac{\overset{\vdots\,\delta_l^2}{X' \succ C_2} \quad \overset{\vdots\,\delta_r}{X'', C_1, C_2 \succ R}}{X', X'', C_1 \succ R}\,{\scriptstyle Cut}}{X, X', X'' \succ R}\,{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l}{X, C_1 \succ C_2}}{X \succ C_1 \to C_2}\,{\scriptstyle \to R} \qquad \dfrac{\overset{\vdots\,\delta_r^1}{X' \succ C_1} \quad \overset{\vdots\,\delta_r^2}{X'', C_2 \succ R}}{X', X'', C_1 \to C_2 \succ R}\,{\scriptstyle \to L}}{X, X', X'' \succ R}\,{\scriptstyle Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\vdots\,\delta_r^1}{X' \succ C_1} \quad \overset{\vdots\,\delta_l}{X, C_1 \succ C_2}}{X, X' \succ C_2}\,{\scriptstyle Cut} \qquad \overset{\vdots\,\delta_r^2}{X', C_2 \succ R}}{X, X', X'' \succ R}\,{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l}{X, C \succ}}{X \succ \neg C}\,{\scriptstyle \neg R} \qquad \dfrac{\overset{\vdots\,\delta_r}{X' \succ C}}{X', \neg C \succ}\,{\scriptstyle \neg L}}{X, X' \succ}\,{\scriptstyle Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots\,\delta_r}{X' \succ C} \quad \overset{\vdots\,\delta_l}{X, C \succ}}{X, X' \succ}\,{\scriptstyle Cut}$$

$$\dfrac{\succ t \qquad \dfrac{\overset{\vdots\,\delta}{X \succ R}}{X, t \succ R}\,{\scriptstyle tL}}{X \succ R}\,{\scriptstyle Cut} \quad \rightsquigarrow \quad \overset{\vdots\,\delta}{X \succ R} \qquad\qquad \dfrac{\dfrac{\overset{\vdots\,\delta}{X \succ}}{X \succ f}\,{\scriptstyle fR} \qquad f \succ}{X \succ}\,{\scriptstyle Cut} \quad \rightsquigarrow \quad \overset{\vdots\,\delta}{X \succ}$$

Figure 2.6: ACTIVE FORMULA CUT REDUCTIONS: MULTIPLE PREMISE

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l^1}{X \succ C_1, Y} \quad \overset{\vdots\,\delta_l^2}{X \succ C_2, Y}}{X \succ C_1 \wedge C_2, Y}{\scriptstyle \wedge R} \qquad \dfrac{\overset{\vdots\,\delta_r}{X', C_i \succ Y'}}{X', C_1 \wedge C_2 \succ Y'}{\scriptstyle \wedge L_i}}{X, X' \succ Y, Y'}{\scriptstyle Cut} \;\rightsquigarrow\; \dfrac{\overset{\vdots\,\delta_l^i}{X \succ C_i, Y} \quad \overset{\vdots\,\delta_r}{X', C_i \succ Y'}}{X, X' \succ Y, Y'}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l}{X \succ C_i, Y}}{X \succ C_1 \vee C_2, Y}{\scriptstyle \vee R_i} \qquad \dfrac{\overset{\vdots\,\delta_r^1}{X', C_1 \succ Y'} \quad \overset{\vdots\,\delta_r^2}{X', C_2 \succ Y'}}{X', C_1 \vee C_2 \succ Y'}{\scriptstyle \vee L}}{X, X' \succ Y, Y'}{\scriptstyle Cut} \;\rightsquigarrow\; \dfrac{\overset{\vdots\,\delta_l}{X \succ C_i, Y} \quad \overset{\vdots\,\delta_r^i}{X', C_i \succ Y'}}{X, X' \succ Y, Y'}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l^1}{X \succ C_1, Y} \quad \overset{\vdots\,\delta_l^2}{X' \succ C_2, Y'}}{X, X' \succ C_1 \otimes C_2, Y, Y'}{\scriptstyle \otimes R} \qquad \dfrac{\overset{\vdots\,\delta_r}{X'', C_1, C_2 \succ Y''}}{X'', C_1 \otimes C_2 \succ Y''}{\scriptstyle \otimes L}}{X, X', X'' \succ Y, Y', Y''}{\scriptstyle Cut} \;\rightsquigarrow\; \dfrac{\overset{\vdots\,\delta_l^1}{X \succ C_1, Y} \quad \dfrac{\overset{\vdots\,\delta_l^2}{X' \succ C_2, Y'} \quad \overset{\vdots\,\delta_r}{X'', C_1, C_2 \succ Y''}}{X', X'', C_1 \succ Y', Y''}{\scriptstyle Cut}}{X, X', X'' \succ Y, Y', Y''}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l}{X \succ C_1, C_2, Y}}{X \succ C_1 \oplus C_2, Y}{\scriptstyle \oplus R} \qquad \dfrac{\overset{\vdots\,\delta_r^1}{X', C_1 \succ Y'} \quad \overset{\vdots\,\delta_r^2}{X'', C_2 \succ Y''}}{X', X'', C_1 \oplus C_2 \succ Y', Y''}{\scriptstyle \oplus L}}{X, X', X'' \succ Y, Y', Y''}{\scriptstyle Cut} \;\rightsquigarrow\; \dfrac{\dfrac{\overset{\vdots\,\delta_l}{X \succ C_1, C_2, Y} \quad \overset{\vdots\,\delta_r^1}{X', C_1 \succ Y'}}{X, X', C_2 \succ Y, Y'}{\scriptstyle Cut} \quad \overset{\vdots\,\delta_r^2}{X', C_2 \succ Y'}}{X, X', X'' \succ Y, Y', Y''}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l}{X, C_1 \succ C_2, Y}}{X \succ C_1 \to C_2, Y}{\scriptstyle \to R} \qquad \dfrac{\overset{\vdots\,\delta_r^1}{X' \succ C_1, Y'} \quad \overset{\vdots\,\delta_r^2}{X'', C_2 \succ Y''}}{X', X'', C_1 \to C_2 \succ Y', Y''}{\scriptstyle \to L}}{X, X', X'' \succ Y, Y', Y''}{\scriptstyle Cut} \;\rightsquigarrow\; \dfrac{\dfrac{\overset{\vdots\,\delta_r^1}{X' \succ C_1, Y'} \quad \overset{\vdots\,\delta_l}{X, C_1 \succ C_2, Y}}{X, X' \succ C_2, Y, Y'}{\scriptstyle Cut} \quad \overset{\vdots\,\delta_r^2}{X', C_2 \succ Y'}}{X, X', X'' \succ Y, Y', Y''}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{\vdots\,\delta_l}{X, C \succ Y}}{X \succ \neg C, Y}{\scriptstyle \neg R} \qquad \dfrac{\overset{\vdots\,\delta_r}{X' \succ C, Y'}}{X', \neg C \succ Y'}{\scriptstyle \neg L}}{X, X' \succ Y, Y'}{\scriptstyle Cut} \;\rightsquigarrow\; \dfrac{\overset{\vdots\,\delta_r}{X' \succ C, Y'} \quad \overset{\vdots\,\delta_l}{X, C \succ Y}}{X, X' \succ Y, Y'}{\scriptstyle Cut}$$

$$\dfrac{\succ t \qquad \dfrac{\overset{\vdots\,\delta}{X \succ Y}}{X, t \succ Y}{\scriptstyle tL}}{X \succ Y}{\scriptstyle Cut} \;\rightsquigarrow\; \overset{\vdots\,\delta}{X \succ Y} \qquad\qquad \dfrac{\dfrac{\overset{\vdots\,\delta}{X \succ Y}}{X \succ f, Y}{\scriptstyle fR} \qquad f \succ}{X \succ Y}{\scriptstyle Cut} \;\rightsquigarrow\; \overset{\vdots\,\delta}{X \succ Y}$$

Figure 2.7: ACTIVE FORMULA CUT REDUCTIONS: MULTIPLE CONCLUSION

Proof: We prove this by induction on the *Cut* complexity of the derivation, where this complexity is the sequence $\langle c_0, c_1, c_2, \ldots, c_m \rangle$ where the derivation has c_i *Cut*s of rank i, and no other *Cut*s. Cut complexity is ordered as usual, with higher ranks more significant than lower. Select a *Cut* in δ where there are no *Cut*s *above*. Use the process of the previous two lemmas to push that *Cut* up to orphans (temporarily blowing up the *Cut* measure by possibly duplicating it past contractions) and then replacing the *Cut* with *Cut*s on subformulas of C, *lowering* the *Cut* complexity. This process possibly duplicates material in the derivations, but since this derivation *up* to our *Cut* contains no *Cut*s, this does not increase the *Cut* measure by duplicating other *Cut*s. The result is a derivation with lower *Cut* complexity. Continue the process, and since there is no infinitely descending sequence of *Cut* complexity, the process terminates in a *Cut* free derivation. ∎

Recall the definition of the non-normality measure in Definition 1.16 on page 24.

We have shown that in each of our sequent systems, if a sequent $X \succ Y$ has a derivation, then it has a *Cut*-free derivation — and furthermore, the a *Cut*-free derivation can be found by *eliminating* the *Cut*s from the original derivation. This is a result which is rich in significance. In the next section, we will explore some of its consequences.

2.4 | CONSEQUENCES OF CUT ELIMINATION

A core consequence of *Cut* elimination is the subformula property.

THEOREM 2.23 [SUBFORMULA PROPERTY] *If δ is a Cut-free derivation of a sequent $X \succ Y$, then δ contains only subformulas of formulas in the endsequent $X \succ Y$.*

Proof: For any inference falling under a rule (other than *Cut*) in any of our sequent systems, the formulas in the premise sequents are subformulas of formulas in the concluding sequent. So, since any derivation is a tree of formulas structured in accordance with the rules, for any sequent in that tree, only subformulas of formulas in a given sequent can occur above that sequent in the tree. In particular, all formulas in a *Cut*-free derivation of $X \succ Y$ are subformulas of formulas in $X \succ Y$. ∎

This result holds for any of our sequent systems, so it holds for classical logic, intuitionistic logic, minimal logic, non-distributive relevant logic, linear logic, etc., and for fragments of these logics in which we have rules for only part of the traditional vocabulary. The phenomenon is robust. In particular, this result shows that our presentation of these logical systems is appropriately *modular*. For example, Peirce's Law

$$\succ ((p \to q) \to p) \to p$$

is derivable in classical logic. This means it has a *Cut* free derivation, and in particular, it has a derivation in which the only rules that apply act on subformulas of $((p \to q) \to p) \to p$. In particular, we do not need to use any rules except for [$\to L$] and [$\to R$], so we do not need to appeal to negation, conjunction, or any other logical concept. The rules for the conditional encapsulate the semantics of the conditional in a way that needs no supplementation by the rules for any other connective. This can be stated, formally, in the following theorem:

THEOREM 2.24 [CONSERVATIVE EXTENSION] *If we extend a sequent system $\mathfrak{S}$ for some subset of our family of connectives, by adding the left and rules for other connectives in our family to form sequent system $\mathfrak{S}'$, this addition is conservative, in the sense that system $\mathfrak{S}'$ can derive no new sequents $X \succ Y$ where X and Y are taken from the language of $\mathfrak{S}'$.*

Proof: Take a sequent $X \succ Y$ from the language of $\mathfrak{S}$, and which is derivable in $\mathfrak{S}'$. Take some *Cut*-free derivation of the sequent. The connective rules in this derivation apply only to subformulas of formulas in $X \succ Y$, and so, are rules from the system $\mathfrak{S}$. So this sequent could already have been derived in $\mathfrak{S}$. ■

This result means that the addition of new logical concepts gives us new concepts to express, and new ways to prove things – even in new ways to prove things from our old vocabulary – but it does not change the landscape of what can be derived in that old vocabulary. The significance of this result will be one of the central topics of the middle part of the book.

The subformula property is significant if you think of sequent rules for a connective as presenting the meaning of that connective. Consider the derivability of Peirce's Law. Not only does the separability of the system ensure that Peirce's Law holds in virtue of the rules [$\to L$] and [$\to R$] governing the conditional. The subformula property assures us that the sequent is derivable in virtue of the instances of those rules applying to subformulas of the formula itself. Peirce's Law $\succ ((p \to q) \to p) \to p$ holds in virtue of the semantic properties of $(p \to q) \to p$, $p \to q$, p and q. There is a profound sense in which the sequent is *analytic*. A *Cut*-free derivation of a sequent $X \succ Y$ shows that the sequent holds in virtue of an *analysis* of the sequent into its components. The sequent holds not in virtue of some relations that the components hold to *other* judgements, but in terms of the internal relationships between those components, and the *Cut*-free derivation of a sequent gives an analysis of the sequent into its components that suffices to establish that the sequent holds.

Excursus: This is not to say that all valid sequents have one and only one such analysis. The sequent calculus allows for sequents to hold for different reasons. The sequent $p \land q \succ p \lor q$, for example, has the following *Cut*-free derivations:

$$\dfrac{\dfrac{p \succ p}{p \land q \succ p}\,{\scriptstyle \land L_1}}{p \land q \succ p \lor q}\,{\scriptstyle \lor R_2} \qquad \dfrac{\dfrac{q \succ q}{p \land q \succ q}\,{\scriptstyle \land L_2}}{p \land q \succ p \lor q}\,{\scriptstyle \lor R_2}$$

where according to the first, the sequent holds in virtue of the p, shared between $p \land q$ and $p \lor q$, and according to the second, the sequent holds in virtue of the shared q. *End of Excursus*

Another consequence of the *Cut*-elimination theorem is the *decidability* of logical consequence in our languages. This is easiest to see in the case of simple sequents.

THEOREM 2.25 [SIMPLE SEQUENT DECIDABILITY] *There is an algorithm for determining whether or not a simple sequent* $A \succ B$ *is valid.*

To determine whether or not $A \succ B$ has a simple sequent derivation, we use the notion of a sequent's *possible ancestry*.

DEFINITION 2.26 [POSSIBLE ANCESTRY] Given any sequent, its POSSIBLE PARENTS are each of the sequents from which it could have been derived, using any rule other than *Cut*. That is, if the sequent $A \succ B$ is the concluding sequent in an instance of a rule, for which $C \succ D$ is a premise, then $C \succ D$ is one of the possible parents of $A \succ B$. The POSSIBLE ANCESTRY of the sequent $A \succ B$ is the tree with $A \succ B$ as its root, with links to each possible parent $C \succ D$, and then each of these sequents is further connected to *its* possible ancestry.

Figure 2.8 depicts the possible ancestry of $q \vee r \succ (p \wedge q) \vee r$.

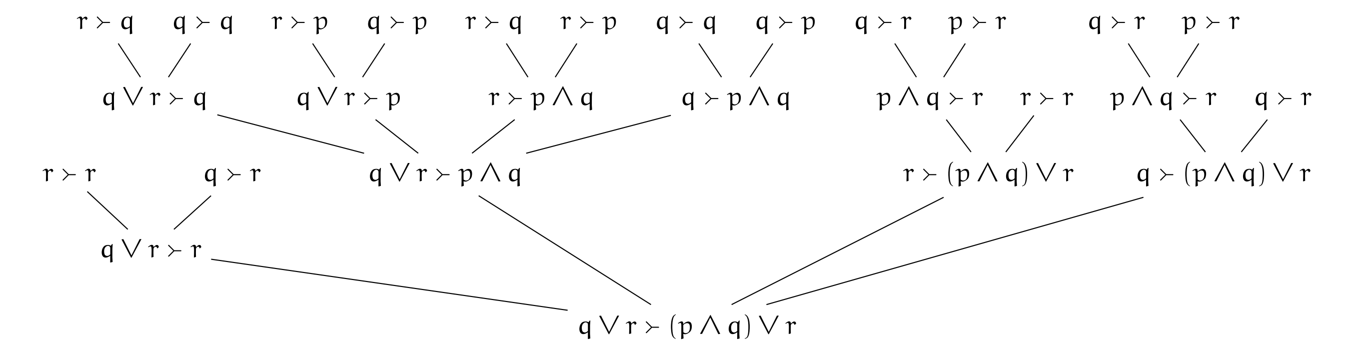


Figure 2.8: THE POSSIBLE ANCESTRY OF $q \vee r \succ (p \wedge q) \vee r$

LEMMA 2.27 [POSSIBLE ANCESTRY IS FINITE] *The possible ancestry of any sequent* $A \succ B$ *in a simple sequent system contains only finitely many nodes.*

Proof: We prove this by induction on the complexity of $A \succ B$. A sequent $p \succ q$ consisting of atoms has *no* possible parents, and so its ancestry is the trivial tree consisting of $p \succ q$ itself, and is finite.

Take any sequent $A \succ B$, and suppose that the hypothesis holds for simpler sequents. Inspecting the rules (see Figure 2.1, on page 51), we can see that each of its possible parents are simpler sequents, and, in addition, there are only finitely many possible parents. The hypothesis holds for each possible parents (their possible ancestries are finite) so the possible ancestry of $A \succ B$ is finite as well. ∎

Now we can use the possible ancestry of a sequent in order to find a derivation for that sequent—if it has one.

Proof: Given the possible ancestry of the sequent $A \succ B$, start at the leaves, and mark any leaf of the form $p \succ p$ as derivable, and delete (cross out) other leaf as underivable. We have marked the derivable sequents and deleted the underivable ones. Let's call this process *processing* the leaves. The marked sequents have derivations, and the deleted sequents do not. Let's suppose that all of the potential parents of the sequent $C \succ D$ have been processed, and we explain what it is to process $C \succ D$. Given a sequent $C \succ D$, examine its possible parents of $C \succ D$ which have not been deleted, and check, for each rule that can derive $C \succ D$, whether the premises of that rule have survived (been marked, not deleted). If so, mark $C \succ D$ as derivable, since it can be derived using any of the derivations of the parents, and the rule under which any of the surviving parents fall. If not enough parents have survived, the sequent $C \succ D$ has no derivation is deleted. This completes the process. ∎

We have already seen one example of this at the beginning of the previous section, when we saw that distribution $(p \land (q \lor r) \succ (p \land q) \lor (p \land r))$ is not derivable (see page 60), though now we can describe this process in terms of the possible ancestry of $p \land (q \lor r) \succ (p \land q) \lor (p \land r)$. This tree has $p \land (q \lor r) \succ (p \land q) \lor (p \land r)$ at its root, and this has four possible parents: $p \succ (p \land q) \lor (p \land r)$ and $q \lor r \succ (p \land q) \lor (p \land r)$ on the one hand, and $p \land (q \lor r) \succ p \land q$ and $p \land (q \lor r) \succ p \land r$ on the other. Each of these would suffice as sole parents (the relevant rules are [$\land L$] and [$\lor R$], which have single premises), but none of these are marked in *their* possible ancestries (see page 60 for the details), and as a result, $p \land (q \lor r) \succ (p \land q) \lor (p \land r)$ is not derivable.

Searching for derivations in the naïve manner described by this theorem is not as efficient as we can be: we don't need to search for *all* possible derivations of a sequent if we know about some of the special properties of the rules of the system. For example, consider the sequent $A \lor B \succ C \land D$ (where A, B, C and D are possibly complex statements). This is derivable in two ways (a) from $A \succ C \land D$ and $B \succ C \land D$ by [$\lor L$] or (b) from $A \lor B \succ C$ and $A \lor B \succ D$ by [$\land R$]. Instead of searching *both* of these possibilites, we may notice that *either* choice would be enough to search for a derivation, since the rules [$\lor L$] and [$\land R$] 'lose no information' in an important sense.

DEFINITION 2.28 [INVERTIBILITY] A sequent rule of the form

$$\frac{S_1 \ \cdots \ S_n}{S}$$

is *invertible* if and only if whenever the sequent S is derivable, so are the sequents $S_1, \ldots, S_n$.

THEOREM 2.29 [INVERTIBLE SEQUENT RULES] *The rules* [$\lor L$] *and* [$\land R$] *are invertible, but the rules* [$\lor R$] *and* [$\land L$] *are not.*

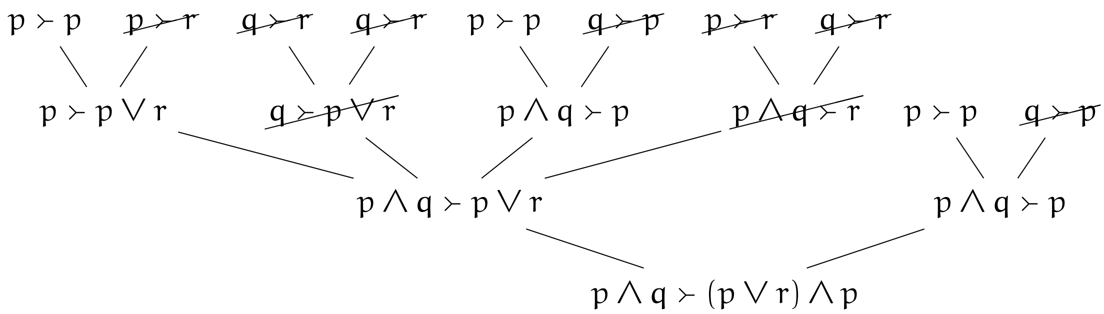
Proof: Consider [$\lor L$]. If $A \lor B \succ C$ is derivable, then since we have a derivation of $A \succ A \lor B$ (by [$\lor R$]), a use of *Cut* shows us that $A \succ C$

is derivable. Similarly, since we have a derivation of $B \succ A \vee B$, the sequent $B \succ C$ is derivable too. So, from the conclusion $A \vee B \succ C$ of a [$\vee L$] inference, we may derive the premises. The case for [$\wedge R$] is completely analogous.

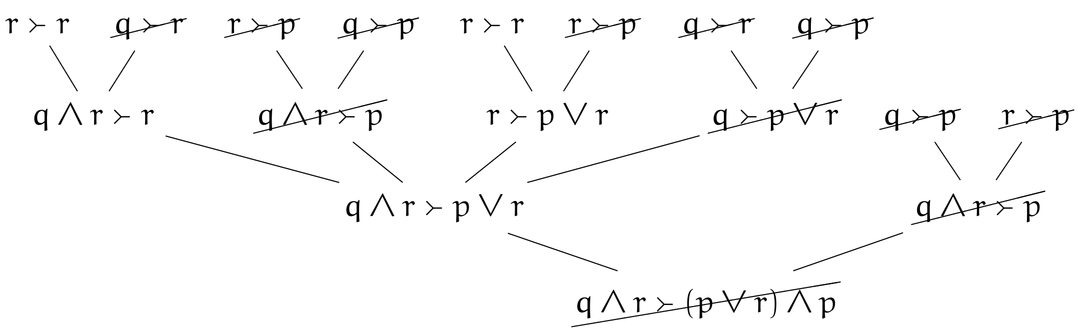
For [$\wedge L$], on the other hand, we have a derivation of $p \wedge q \succ p$, but no derivation of the premise $q \succ p$, so this rule is not invertible. Similarly, $p \succ q \vee p$ is derivable, but $p \succ q$ is not. ∎

It follows that when searching for a derivation of a sequent, instead of searching through its *entire* possible ancestry, if it may be derived from an invertible rule, we can look to the premises of *that* rule, and ignore the other branches of its ancestry.

EXAMPLE 2.30 [DERIVATION SEARCH] The sequent $(p \wedge q) \vee (q \wedge r) \succ (p \vee r) \wedge p$ is not derivable. By the invertibility of [$\vee L$], it *is* derivable only if (a) $p \wedge q \succ (p \vee r) \wedge p$ and (b) $q \wedge r \succ (p \vee r) \wedge p$ are both derivable. Here is the possible ancestry for $p \wedge q \succ (p \vee r) \wedge p$, where we, at the first step, appeal to the invertible rule [$\wedge R$], and we start from the top and strike out any underivable sequents.



The sequent at the root survives. It is derivable. The other required premise, for our target sequent, $q \wedge r \succ (p \vee r) \wedge p$, is less fortunate.



This sequent is not derivable, because $q \wedge r \succ p$ is underivable.

» «

This result can be generalised to apply to any of our complex sequent systems, but in the case of some of these systems, we need to do more work. Given a system $\mathfrak{S}$, and a sequent $X \succ Y$ we produce its possible ancestry, and use this to determine whether there the sequent has a derivation. In the case of systems without the contraction rule, this result is no more complex than for simple sequent systems.

THEOREM 2.31 [CONTRACTION FREE SYSTEMS ARE DECIDABLE] *Any of our sequent systems (multiple conclusion or single conclusion) without the contraction rule is decidable. In particular, given any sequent* $X \succ Y$, *its possible ancestry is finite.*

Proof: If you inspect every rule other than contraction and *Cut*, you can see that (a) the number of formulas in each premise sequent of a rule is less than or equal to the number of formulas in the conclusion sequent, and also, (b) each formula in a premise sequent must be a *subformula* of a formula in the concluding sequent, and (c) for each sequent, only a finite number of rules (other than *Cut*) could produce that sequent as a conclusion. So, the possible ancestry (defined as before) for a sequent $X \succ Y$ is a finitely branching tree, in which the sequents along a branch reduce in complexity, and each sequent contains only subformulas of sequents lower down in the branch. So, each branch has only a finite length. By König's Lemma, the possible ancestry is finite.

König's Lemma: A finitely branching tree in which every branch has finite length is itself finite.

We use the possible ancestry to check for the existence of a proof as before. Starting with the leaves of the tree (the sequents containing only atoms), we check for derivability immediately, marking those that survive, and crossing out those that are not axioms. Then, for nodes lower down in the tree, once all its parents have been processed, mark a sequent as derivable if any of the premises of a rule under which it falls have survived. If not enough possible parents have survived, and there are no premises upon which to derive the sequent, strike it out. Continue until the tree is complete. ∎

For systems *with* the contraction rule, things get more complicated, for with contraction, we can derive *smaller* sequents from *larger* sequents. This means that the possible ancestry for a sequent is no longer finite. In the case of systems with contraction and weakening, this is especially egregious. We can arbitrarily lengthen any derivation with moves like this: Replace a sequent $X, A \succ Y$ occurring in a derivation with this *weakening–contraction* two-step:

$$\dfrac{\dfrac{\vdots \atop X, A \succ Y}{X, A, A \succ Y}\,K}{\dfrac{X, A \succ Y}{\vdots}}\,W$$

to extend the derivation by two sequents. This can be repeated *ad libitum*. So, we need some way to limit the search for derivations. Clearly, when we search for derivations for a sequent, we want to limit the search space so we don't end up chasing our own tail. At least we should restrict our search to *concise* derivations.

DEFINITION 2.32 [CONCISE DERIVATIONS] A derivation δ is CONCISE iff each branch of the tree of sequents contains each sequent only once.

THEOREM 2.33 [DERIVATIONS CAN BE MADE CONCISE] *If a sequent* $X \succ Y$ *has a derivation (in some system), it has a concise derivation as a sub-tree of that derivation.*

Proof: Take a derivation δ of $X \succ Y$. If there is some brach where a sequent $U \succ V$ is repeated, delete all steps in the branch between the first occurrence of the sequent in this branch and the last (and one instance of $U \succ V$), including deleting any other branches of the tree which branch into this intermediate segment. The result is a smaller derivation of $X \succ Y$. If there are still repeated sequents in branches, continue the process. It cannot continue forever, as the derivation is smaller at each stage of the process. The end of the process is a concise derivation of $X \succ Y$ which found inside the original derivation δ. ∎

Searching for *concise* derivations cuts down on the search space. This step alone is not enough, though, for contraction is an insidious rule. When we ask ourselves whether the sequent $X, A \succ Y$ is derivable, perhaps it was derived from $X, A, A \succ Y$. And where was this derived? Perhaps from $X, A, A, A \succ Y$, and so on. If we search for concise derivations using contraction, the search space is still very large. Consider a derivation where contraction needs to be used a *lot*. The sequent $p \succ p \otimes (p \otimes (p \otimes p))$ is derivable using contraction.

$$\dfrac{\dfrac{\dfrac{p \succ p \quad \dfrac{p \succ p \quad \dfrac{p \succ p \quad p \succ p}{p, p \succ p \otimes p}\otimes R}{p, p, p \succ p \otimes (p \otimes p)}\otimes R}{p, p, p, p \succ p \otimes (p \otimes (p \otimes p))}\otimes R}{p, p \succ p \otimes (p \otimes (p \otimes p))}W}{p \succ p \otimes (p \otimes (p \otimes p))}W$$

In this derivation, we derive $p, p, p, p \succ p \otimes (p \otimes (p \otimes p))$, and then two steps of contraction reduce the four instances of p to one. When looking for a derivation of $p \succ p \otimes (p \otimes (p \otimes p))$, we can find one when we go through the more complex sequent $p, p, p, p \succ p \otimes (p \otimes (p \otimes p))$, having duplicated the p three times.

Why two steps? In the first, we contract two instances of p, p into one. In the second, two instances of p are contracted into one.

Why is so so much contraction *needed* in the derivation? They're required here because the repetitions of p are introduced by the [$\otimes R$] steps. To derive $p \succ p \otimes (p \otimes (p \otimes p))$, the only serious options are a contraction on the left or [$\otimes R$]. The [$\otimes R$] step cannot be immediately applied, since we have only one p to go around (the possible pairs of parents are $p \succ p$ and $\succ p \otimes (p \otimes p)$ or $\succ p$ and $p \succ p \otimes (p \otimes p)$, and in either case, at least one possible parent is underivable). So, a contraction it must be. But we don't *need* to apply contraction again to go to the four-way repetition of p, for the sequent $p, p \succ p \otimes (p \otimes (p \otimes p))$ is derivable by way of the two parents $p \succ p$ and $p \succ p \otimes (p \otimes p)$. The $p \succ p$ is an axiom, and again, we can derive $p \succ p \otimes (p \otimes p)$ by contraction from $p, p \succ p \otimes (p \otimes p)$, apply

There is nothing special with the numbers here. We could have used 3088 instances of p rather than 4, if we were deriving a large self-fusion of p.

An un-serious option is a contraction on the *right*, but why would you try that?

[$\otimes R$] to split again, and so on. The resulting derivation is different:

$$\dfrac{p \succ p \qquad \dfrac{\dfrac{p \succ p \qquad \dfrac{\dfrac{p \succ p \quad p \succ p}{p, p \succ p \otimes p}\,{\scriptstyle \otimes R}}{p \succ p \otimes p}\,{\scriptstyle W}}{p, p \succ p \otimes (p \otimes p)}\,{\scriptstyle \otimes R}}{p \succ p \otimes (p \otimes p)}\,{\scriptstyle W}}{\dfrac{p, p \succ p \otimes (p \otimes (p \otimes p))}{p \succ p \otimes (p \otimes (p \otimes p))}\,{\scriptstyle W}}\,{\scriptstyle \otimes R}$$

We apply contraction immediately after the duplication is incurred in the [$\otimes R$] step. And this generalises to the other rules. Contraction is required to the extent that different premise sequents in our rules pile up copies of formulas. If the rules are applied repeatedly, the piles of copies can be sizeable. However, if we “clean up” as we go, contracting formulas as the first occur together (when we indeed want to contract them), instead of delaying the process for later, the process is manageable. In fact, we can make the contraction step a part of the connective rule [$\otimes R$], like this: We could reformulate [$\otimes R$] to have the following shape:

There is an analogy somewhere in the vicinity about housework, or dealing with email, or other small but numerous recurring tasks, but I will not pause to make it.

$$\dfrac{X \succ A, Y \quad X' \succ B, Y'}{[X, X'] \succ [[A \otimes B], Y, Y']}\,{\scriptstyle \otimes R'}$$

where $[X, X']$ is some multiset formed from X, X', allowing for (but not requiring) any formula in X, X' to be contracted *once*, and $[[A \otimes B], Y, Y']$ is a multiset formed from $A \otimes B, Y, Y'$, allowing for (but not requiring) any formula in Y, Y' to be contracted *once*, and allowing (but not requiring) $A \otimes B$ to be contracted *once* or *twice*. This means that any new repetitions introduced in the output of this rule could be dealt with on the spot. Why would a contraction be required? Perhaps because a formula was supplied to the conclusion *both* from the left premise, and from the right premise, whereas I need only one in the resulting sequent. So contract the formula in X or X', or Y or Y', in this step. Or the formula $A \otimes B$ might already be in Y or in Y' – or in both. In that case, we can also contract also as needed. Clearly, if something is derivable using the old sequent rule [$\otimes R$], it is derivable using this new rule (nothing *forces* us to use contraction), and if something is derivable using the new rule [$\otimes R'$], and we have contraction available, we can derive it using the old rule too. Now the derivation of $p \succ p \otimes (p \otimes (p \otimes p))$ can be rewritten:

$$\dfrac{p \succ p \qquad \dfrac{p \succ p \qquad \dfrac{p \succ p \quad p \succ p}{p \succ p \otimes p}\,{\scriptstyle \otimes R'}}{p \succ p \otimes (p \otimes p)}\,{\scriptstyle \otimes R'}}{p \succ p \otimes (p \otimes (p \otimes p))}\,{\scriptstyle \otimes R'}$$

and we contract the duplicate ps in the left as we apply [$\otimes R'$]. No explicit contraction step is required. This is the genius of allowing contraction *inside* the rules.

We can do the same for *all* our rules. For any system $\mathfrak{S}$ including contraction, we call the system $\mathfrak{S}^W$ the system with contraction internalised into the rules.

DEFINITION 2.34 [CONTRACTED MULTISETS] Given multisets X, Y and Z, a multiset M is said to be an multiset of kind [[X], Y], Z if and only if its occurrences satisfy

$$\begin{array}{rcccl} \max(1, o_{X,Y,Z}(x) - 2) & \leq & o_M(x) & \leq & o_{X,Y,Z}(x) \quad \text{for } x \in X \\ \max(1, o_{X,Y,Z}(y) - 1) & \leq & o_M(y) & \leq & o_{X,Y,Z}(y) \quad \text{for } x \in Y \text{ and } x \notin X \\ & & o_M(z) & = & o_{X,Y,Z}(z) \quad \text{otherwise} \end{array}$$

That is, we allow for repeated members of X to be reduced by 2 (with a floor of 1 – you can only eliminiate copies when you have one copy left), and repeated members of Y (except for those in X) to be reduced by 1 (again, with a floor of 1), and members of Z (other than those occuring in X, Y) are be unchanged.

In this definition, any of X, Y, Z can be empty. For example, when X is empty, we have [Y], Z (allowing repeats to be reduced by one in Y), and if Z is empty, we have [[X], Y] (allowing for two repeats to be eliminated in X and one in Y), and in our notation we allow for the brackets to be in other orders: in other words, $[Y, [X]] = [[X], Y]$ and $[[X], Y], Z = [Y, [X]], Z = Z, [Y, [X]]$, etc.

EXAMPLE 2.35 The multiset p, q, r is of kind $[p, q], p, q, r$, as is p, p, q, r and p, q, q, r. The multisets of kind $[[p], p, q, r], p, r$ are

$$p, p, p, q, r, r \quad p, p, p, q, r \quad p, p, q, r, r \quad p, p, q, r \quad p, q, r, r \quad p, q, r$$

Given this notation for contracted multisets, we can specify the rules sequent systems with contraction folded into the rules. These rules are in Figure 2.9. Many of the rules are specified using contracted multisets in the conclusions. In these cases, the rules have more than one possible conclusion. You get a different instance of the rule for each different choice of a contracted multiset for the left hand side and right hand side of the sequent in the conclusion. For example, given the premise sequent $\neg p \succ p, \neg p$, these are both instances of $[\neg L']$, where the introduced formula is $\neg p$.

$$\frac{\neg p \succ p, \neg p}{\neg p, \neg p \succ \neg p}{\scriptstyle \neg L'} \qquad \frac{\neg p \succ p, \neg p}{\neg p \succ \neg p}{\scriptstyle \neg L'}$$

In the first case, there is no contraction applied, in the second, one instance of $\neg p$ is contracted on the left (as allowed in the specification of $[\neg L']$).

A crucial feature of these rules is that while contractions are applied, they are not applied *too much*.

LEMMA 2.36 [FINITELY MANY POSSIBLE PARENTS] *Each sequent* $X \succ Y$ *has only finitely many possible parents in the sequent rules with implicit contraction.*

$$p \succ p \; Id \qquad \frac{X \succ Y}{X, X' \succ Y, Y'} K$$

$$\frac{X, A \succ Y}{[A \land B], X \succ Y} \land L'_1 \qquad \frac{X, B \succ Y}{[A \land B], X \succ Y} \land L'_2 \qquad \frac{X \succ A, Y \qquad X \succ B, Y}{X \succ [A \land B], Y} \land R'$$

$$\frac{X, A \succ Y \qquad X, B \succ Y}{[A \lor B], X \succ Y} \lor L' \qquad \frac{X \succ A, Y}{X \succ [A \lor B], Y} \lor R'_1 \qquad \frac{X \succ B, Y}{X \succ [A \lor B], Y} \lor R'_2$$

$$\frac{X, A, B \succ Y}{X, [A \otimes B] \succ Y} \otimes L' \qquad \frac{X \succ A, Y \quad X' \succ B, Y'}{[X, X'] \succ [[A \otimes B], Y, Y']} \otimes R'$$

$$\frac{X, A \succ Y \quad X', B \succ Y'}{[X, X', [A \oplus B]] \succ [Y, Y']} \oplus L' \qquad \frac{X \succ A, B, Y}{X \succ [A \oplus B], Y} \oplus R'$$

$$\frac{X \succ A, Y \quad B, X' \succ Y'}{[[A \to B], X, X'] \succ [Y, Y']} \to L' \qquad \frac{X, A \succ B, Y}{X \succ [A \to B], Y} \to R'$$

$$\frac{X \succ A, Y}{X, [\neg A] \succ Y} \neg L' \qquad \frac{X, A \succ Y}{X \succ [\neg A], Y} \neg R'$$

$$\frac{X \succ Y}{X, t \succ Y} tL \qquad \succ t \; tR \qquad X \succ \top, Y \; \top R$$

$$f \succ \; fL \qquad \frac{X \succ Y}{X \succ f, Y} fR \qquad X, \bot \succ Y \; \bot L$$

Figure 2.9: SEQUENT RULES WITH IMPLICIT CONTRACTION

Proof: For each rule under which the conclusion might fall, there are only finitely many ways to split the conclusion up in order to find possible premises. Take, $[\otimes R']$ for example, and suppose the conclusion sequent has the shape $U \succ A \otimes B, V$. The possible premises have the form $X \succ A, Y$ and $X' \succ B, Y'$ for different choices of X, X', Y, Y', where $U \subseteq X \cup X'$, $X \subseteq U$ and $X' \subseteq U$, and $V \backslash (A \otimes B) \subseteq Y \cup Y'$, $Y \subseteq V$ and $Y' \subseteq V$. Given that U and V are finite multisets, there are finitely many choices for U and V in this case, and in the same way, in every other. ■

EXAMPLE 2.37 Given the sequent $p, q \succ r \otimes s$, the possible pairs of parents are:

You can see why I won't try drawing a tree for possible ancestry of *any* sequent in this system. It's just *too* large.

$p \succ r$ and $q \succ s$,	$p \succ s$ and $q \succ r$,
$p, q \succ r$ and $q \succ s$,	$p, q \succ s$ and $q \succ r$,
$q \succ r$ and $p, q \succ s$,	$q \succ s$ and $p, q \succ r$,
$p, q \succ r$ and $p, q \succ s$,	
$p \succ r, r \otimes s$ and $q \succ s$,	$p \succ s$ and $q \succ r, r \otimes s$,
$p, q \succ r, r \otimes s$ and $q \succ s$,	$p, q \succ s$ and $q \succ r, r \otimes s$.
$q \succ r, r \otimes s$ and $p, q \succ s$,	$q \succ s$ and $p, q \succ r, r \otimes s$,
$p, q \succ r, r \otimes s$ and $p, q \succ s$,	
$p \succ r$ and $q \succ s, r \otimes s$,	$p \succ s, r \otimes s$ and $q \succ r$,
$p, q \succ r$ and $q \succ s, r \otimes s$,	$p, q \succ s, r \otimes s$ and $q \succ r$,
$q \succ r$ and $p, q \succ s, r \otimes s$,	$q \succ s$ and $p, q \succ r$,
$p, q \succ r$ and $p, q \succ s, r \otimes s$,	
$p \succ r, r \otimes s$ and $q \succ s, r \otimes s$,	$p \succ s, r \otimes s$ and $q \succ r, r \otimes s$,
$p, q \succ r, r \otimes s$ and $q \succ s, r \otimes s$,	$p, q \succ s, r \otimes s$ and $q \succ r, r \otimes s$,
$q \succ r, r \otimes s$ and $p, q \succ s, r \otimes s$,	$q \succ s, r \otimes s$ and $p, q \succ r, r \otimes s$,
$p, q \succ r, r \otimes s$ and $p, q \succ s, r \otimes s$.	

There are 28 pairs of possible parents for this one sequent, following the $[\otimes R']$ rule. That is certainly much more than for rules where we *haven't* included contraction, but it is still finite.

It follows from this that the tree for possible ancestry is *finitely branching*. From any sequent you can trace only a finite number of possible parents. However, it is not the case that the possible ancestry of a sequent must be finite. Here is a fragment of the possible ancestry of $p \succ p \otimes p, p$, using our rules:

$$\dfrac{\dfrac{p \succ p \otimes p, p \qquad p \succ p \otimes p, p}{p \succ p \otimes p}\,{\scriptstyle \otimes R'}}{p \succ p \otimes p, p}\,{\scriptstyle K}$$

It follows that the *full* possible ancestry of $p \succ p \otimes p, p$ is infinite, for one of the ways we can derive $p \succ p \otimes p, p$ is from a pair of derivations of $p \succ p \otimes p, p$. We can stack derivations to an arbitrary depth – there is no limit on how long they might be. Clearly, when searching for a derivation for $X \succ Y$, if we find $X \succ Y$ in its own ancestry, we don't need to pursue *that* branch any further. There is no need to chase our own tail in the search for a derivation for $X \succ Y$. If all derivations of $X \succ Y$ went through earlier derivations of $X \succ Y$, this sequent wouldn't have *any* derivations.

But there are more ways to chase your tail than going around in a circle. We might also go around in an ever increasing *spiral*. Perhaps in my search for a derivation of $X, X' \succ Y, Y'$, I find that I could have done it by way of a derivation for $X, X', X' \succ Y, Y', Y'$. (We have already eliminated an *explicit* appeal to contraction in our rules, but we may still be able to mimic it through the contraction implicit in our connective rules.) We want to avoid having to derive $X, X' \succ Y, Y'$ through a derivation of a more complex sequent $X, X', X' \succ Y, Y', Y'$. To avoid this, we wish to use a stronger restriction on derivations than concision (Definition 2.32).

DEFINITION 2.38 [SUCCINCT DERIVATIONS] A derivation δ is SUCCINCT iff no branch of the tree that contains a sequent $X, X' \succ Y, Y'$ earlier contains a sequent $X, X', X' \succ Y, Y', Y'$ from which it could have been contracted.

To show that we can avoid searching for derivations that fail to be succinct, we prove the following lemma:

Anderson and Belnap call this Curry's Lemma, after Curry's 1950 proof of the result for classical and intuitionistic sequent systems [24].

LEMMA 2.39 [CURRY'S LEMMA] *In any system $\mathfrak{S}^W$, if a sequent $X, X', X' \succ Y, Y', Y'$ has a derivation with height n, then $X, X' \succ Y, Y'$ has a derivation with height $\leq n$.*

Proof: This is a straightforward induction on the length of the derivation of $X, X', X' \succ Y, Y', Y'$. If the sequent $X, X', X' \succ Y, Y', Y'$ is an axiom, its contraction $X, X' \succ Y, Y'$ is also an axiom. Suppose the result holds for derivations with height less than m and that $X, X', X' \succ Y, Y', Y'$ has a derivation of height m. If the formulas both occurrences of X' and Y' are all passive in the last step of the derivation, then the contraction could have occurred at the premise of the derivation, unless some components occurred in one premise and the others, in the other premise of the inference step. In that case, those premises may be contracted in this inference step. The only remaining case to consider is when one formula in X' or in Y' must is active in the final step of the conclusion. In that case, we are permitted to contract *that* instance as well in this inference step. This completes the proof. ∎

So, we can restrict our attention to succinct derivations.

THEOREM 2.40 [DERIVATIONS CAN BE MADE SUCCINCT] *In any system $\mathfrak{S}^W$, if a sequent $X \succ Y$ has a derivation, it also has a succinct derivation.*

Proof: Given any branch of the derivation including a sequent $U, U', U' \succ V, V', V'$ and later, its contraction $U, U' \succ V, V'$, by Curry's Lemma, the derivation of $U, U', U' \succ V, V', V'$ can be transformed into a derivation of $U, U' \succ V, V'$ of no greater height. Replace the larger derivation of $U, U' \succ V, V'$ with this new, smaller derivation. Continue the process in our derivation, until all failures of succinctness are dealt with. ∎

If I have a sequent $X \succ Y$ and I want to check it for derivability, I need only search the succinct possible ancestry. When I consider the sequents that could occur in a succinct derivation for $X \succ Y$, I need not worry about an unending sequence of larger and larger sequents, tracing contraction steps in reverse.

DEFINITION 2.41 [SUCCINCT POSSIBLE ANCESTRY] Given any sequent, its SUCCINCT POSSIBLE ANCESTRY is the tree constructed in the following way: start with the sequent itself (the root of the tree), and add a branch to the roots of the trees consisting of the succinct possible ancestry of the possible parents of that sequent. This tree is succinct (no branch contains a sequent and then, later a sequent from which is contracted) except concerning the root itself, which is new. Prune the tree by lopping off any branch at the point at which it contains a sequent from which the sequent at the root of the tree. The resulting tree is now the succinct possible ancestry of the starting sequent.

We have already shown that this tree is finitely branching. It remains to show that each branch is finite. This result was first proved by Saul Kripke in the late 1950s for the case of the sequent calculus for the implicational fragment of the relevant logic R [60], and so this result has his name:

LEMMA 2.42 [KRIPKE'S LEMMA] *Given a set $\mathcal{S}$ of sequents, each of which are comprised of formulas from some given finite set, and none of which is a contraction of any other sequent in the set, $\mathcal{S}$ is finite.*

Robert K. Meyer noticed, years later [73], that Kripke's Lemma follows from Dickson's Lemma, a result in number theory [31].

LEMMA 2.43 [DICKSON'S LEMMA] *For any infinite set S of n-tuples of natural numbers, there are at least two tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ where $a_i \leq b_i$ for each i.*

That Kripke's Lemma follows from Dickson's is immediate.

Proof: Given any set $\mathcal{S}$ of sequents, comprised from formulas from some given finite set, partition it into finitely many classes, where each two sequents $X \succ Y$ and $X' \succ Y'$ are in the same class if and only if X and X' contain the same formulas (with possibly different repetitions), and the same holds for Y and Y'. (There are only finitely many such classes since there are only finitely many formulas out of which each sequent can be constructed.) For each such class, apply Dickson's Lemma in the following way: if the class is contains sequents made up from formulas in $A_1, \ldots, A_n$ on the left and $B_1, \ldots, B_m$ on the right, assign the tuple $(j_1, \ldots, j_n, k_1, \ldots, k_m)$ to the sequent containing j_i repetitions of A_i and k_i repetitions of B_i. There are infinitely many tuples if and only if there are infinitely many sequents in this class. By Dickson's Lemma, if there are infinitely many tuples, then we have some pair of tuples $(j_1, \ldots, j_n, k_1, \ldots, k_m)$ and $(j'_1, \ldots, j'_n, k'_1, \ldots, k'_m)$ where $j_i \leq j'_i$ and $k_i \leq k'_i$ for each i, but this means that the sequent corresponding to the first tuple is a contraction of the sequent corresponding to the second. Given that there are only finitely many equivalence classes, the only way that the entire set could be infinite is if one class (at least), was infinite. This completes the proof of Kripke's Lemma. ∎

So, to prove Kripke's Lemma, we can simply refer to the general result of Dickson's Lemma and leave things at that. However, it is relatively straightforward to prove in its own right, so if you are interested, here is a proof:

Proof: A straightforward proof of Dickson's Lemma uses the fact that any infinite sequence $n_0, n_1, \ldots, n_i, \ldots$ of natural numbers has some *non-decreasing* infinite subsequence – that is, there is a selection of indices $i_0 < i_1 < i_2 < \cdots$ such that $n_{i_0} \leq n_{i_1} \leq n_{i_2} \leq \cdots$ – the subsequence $n_{i_0}, n_{i_1}, n_{i_2}, \ldots$ is never decreasing.

This reasoning is not at all constructive. It gives you no insight into what to do if you have not yet been able to verify that the original sequence is bounded above – and how *could* you verify this if I merely give you the sequence one item at a time? Consider the case where I feed you the sequence consisting of $1, 2, \ldots, n$ (for some *very* large number n) and only then continue with $0, 0, 0, \ldots$. The mathematics of providing an algorithm for *selecting* a subsequence is subtle [20, 82].

Why is there always such a subsequence? If the sequence is bounded above, then only finitely many numbers occur in the sequence, so at least one number occurs infinitely many times. Pick the constant subsequence consisting of one such number. On the other hand, if the sequence is unbounded, then define the sequence by setting $n_{i_0} = n_0$, and given n_{i_j}, for $n_{i_{j+1}}$, select the next item in the original sequence larger than n_{i_j}. Since the sequence is unbounded, there is always such a number.

Now, consider the our set S of n-tuples, and represent it as a sequence S_0, in some arbitrarily chosen order. We can define the sequence S_1 as the infinite subsequence of S_0 in which the *first* element of each tuple never decreases from one tuple to the next. There is always such an infinite subsequence, applying our lemma to the sequence consisting of the first element of each tuple. Continue for each position in the tuples in the sequences. That is, given that S_i defined so that the first 1 to i elements of each tuple are non-decreasing from one item to the next in the infinite sequence, define S_{i+1} as the infinite subsequence of the sequence S_i of tuples where the $(i+1)$st element of each tuple never decreases from one tuple to the next. The *final* sequence S_n is an infinite series of n-tuples where each tuple is dominated by each later tuple, so the first tuple $(a_1, \ldots, a_n)$ and second tuple $(b_1, \ldots, b_n)$ in the list are such that $a_i \leq b_i$ for each i, and Dickson's Lemma is proved. ∎

This completes all the components we need to show that all of our sequent systems are decidable.

THEOREM 2.44 [ALL SEQUENT SYSTEMS ARE DECIDABLE] *Any of our sequent systems (multiple conclusion or single conclusion). In particular, given any sequent $X \succ Y$, its possible ancestry (its succinct possible ancestry, in the case of systems with contraction) is finite.*

Proof: The proof takes the same shape as the proof for Theorem 2.31 on page 96, except in the presence of contraction, we use the *succinct* possible ancestry, terminating branches instead of adding nodes which are merely expansions of sequents we have already seen in the tree.

By Kripke's Lemma, the succinct possible ancestry is finite. Passing from the leaves to the root in the manner of the proof of Theorem 2.31, we have an algorithm for determining, for each sequent in the tree, whether it is derivable or not. ∎

This result shows that a our sequent systems have the wherewithal to give us an algorithm for determining derivability. Producing a succinct possible ancestry for a sequent is not the most *efficient* way to test for derivability. There are many techniques for making derivation search more tractable, and the discipline of automated theorem proving is thriving [12, 30, 42, 115].

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The elimination of *Cut* is useful for more than just limiting the search for derivations. The fact that any derivable sequent has a *Cut*-free derivation has other consequences. One consequence is the fact of *interpolation*.

THEOREM 2.45 [INTERPOLATION FOR SIMPLE SEQUENTS] *If* $A \succ B$ *is derivable in the simple sequent system, then there is a formula* C *containing only atoms present in both* A *and* B *such that* $A \succ C$ *and* $C \succ B$ *are derivable.*

This result tells us that if the sequent $A \succ B$ is derivable then that consequence "factors through" a statement in the vocabulary shared between A and B. This means that the consequence $A \succ B$ not only relies only upon the material in A and B and nothing *else* (that is due to the availability of a *Cut*-free derivation) but also in some sense the derivation 'factors through' the material in common between A and B. The result is a straightforward consequence of the *Cut*-elimination theorem. A *Cut*-free derivation of $A \succ B$ provides us with an interpolant.

Proof: We prove this by induction on the construction of the derivation of $A \succ B$. We keep track of the interpolant with these rules:

$$p \succ_p p \; \textit{Id}$$

$$\frac{A \succ_C R}{A \wedge B \succ_C R}\wedge L_1 \qquad \frac{A \succ_C R}{B \wedge A \succ_C R}\wedge L_2 \qquad \frac{L \succ_{C_1} A \qquad L \succ_{C_2} B}{L \succ_{C_1 \wedge C_2} A \wedge B}\wedge R$$

$$\frac{A \succ_{C_1} R \qquad B \succ_{C_2} R}{A \vee B \succ_{C_1 \vee C_2} R}\vee L \qquad \frac{L \succ_C A}{L \succ_C A \vee B}\vee R_1 \qquad \frac{L \succ_C A}{L \succ_C B \vee A}\vee R_2$$

We show by induction on the length of the derivation that if we have a derivation of $L \succ_C R$ then $L \succ C$ and $C \succ R$ and the atoms in C present in both L and in R. These properties are satisfied by the atomic sequent $p \succ_p p$, and it is straightforward to verify them for each of the rules. ∎

EXAMPLE 2.46 [A DERIVATION WITH AN INTERPOLANT] Take the sequent $p \wedge (q \vee (r_1 \wedge r_2)) \succ (q \vee r_1) \wedge (p \vee r_2))$. We may annotate a *Cut*-free derivation of it as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{q \succ_q q}{q \succ_q q \vee r}\vee R \qquad \dfrac{r_1 \succ_{r_1} r_1}{r_1 \wedge r_2 \succ_{r_1} r_1}\wedge L}{q \vee (r_1 \wedge r_2) \succ_{q \vee r_1} q \vee r_1}\vee L}{p \wedge (q \vee (r_1 \wedge r_2)) \succ_{q \vee r_1} q \vee r_1}\wedge L \qquad \dfrac{\dfrac{p \succ p}{p \succ_p p \vee r_2}\vee R}{p \wedge (q \vee (r_1 \wedge r_2)) \succ_p p \vee r_2}\wedge L}{p \wedge (q \vee (r_1 \wedge r_2)) \succ_{(q \vee r_1) \wedge p} (q \vee r_1) \wedge (p \vee r_2)}\wedge R$$

Notice that the interpolant $(q \vee r_1) \wedge p$ does not contain r_2, even though r_2 is present in both the antecedent and the consequent of the sequent. This tells us that r_2 is doing no 'work' in this derivation. Since we have

$$p \wedge (q \vee (r_1 \wedge r_2)) \succ (q \vee r_1) \wedge p, \quad (q \vee r_1) \wedge p \succ (q \vee r_1) \wedge (p \vee r_2)$$

We can replace the r_2 in either derivation with another statement – say r_3 – preserving the structure of each derivation. We get the more general fact:

$$p \wedge (q \vee (r_1 \wedge r_2)) \succ (q \vee r_1) \wedge (p \vee r_3)$$

We can extend interpolation to complex sequent systems, too, though this takes a little more work, since in these derivations, formulas can switch sides in sequents. To prove interpolation, we will prove a stronger hypothesis, according to which the *splitting* of the sequent may be independent of the division between left and right. Here is the target result:

THEOREM 2.47 [SPLITTING FOR COMPLEX SEQUENTS] *For any of our sequent systems, given any derivable sequent* $X, X' \succ Y, Y'$, *we can find a formula* I *where* (1) $X \succ I, Y$ *and* $X', I \succ Y'$ *are derivable, and* (2) I *is a formula whose atoms occur both in* $X \cup Y$ *and* $X' \cup Y'$.

To prove this, we will make use of a family of *split sequent* rules. These rules generalise the rules of the sequent calculus, to operate on *pairs* of multisets of formulas on the left and right, and each sequent will be subscripted with an interpolating formula. So, split sequents have this form:

$$X; X' \succ_I Y; Y'$$

and the intended interpretation is that $X \succ I, Y$ and $X', I \succ Y'$ are both derivable – and that I a formula whose atoms occur both in $X \cup Y$ and $X' \cup Y'$. Figure 2.10 gives the axioms and rules for the splitting sequent system.

A derivation in the split sequent system is a tree of sequents, starting with axioms, and developed according to the rules, in the usual fashion.

$$p;\ \succ_p\ ;p\ \textit{Id}\qquad p;\ \succ_\bot\ p;\ \textit{Id}\qquad ;p\ \succ_{\neg p}\ p;\ \textit{Id}\qquad ;p\ \succ_\top\ ;p\ \textit{Id}$$

$$X;X'\succ_\bot \top, Y;Y'\ {\top}\textit{R};\qquad X;X'\succ_\top Y;Y',\top\ ;{\top}\textit{R}\qquad \bot, X;X'\succ_\bot Y;Y'\ {\bot}\textit{L};\qquad X;X',\bot\succ_\top Y;Y'\ ;{\bot}\textit{L}$$

$$\frac{A,X;X'\succ_I Y;Y'}{A\land B,X;X'\succ_I Y;Y'}\,{\land}\textit{L}_1;\quad \frac{X;X',A\succ_I Y;Y'}{X;X',A\land B\succ_I Y;Y'}\,;{\land}\textit{L}_1\quad \frac{B,X;X'\succ_I Y;Y'}{A\land B,X;X'\succ_I Y;Y'}\,{\land}\textit{L}_2;\quad \frac{X;X',B\succ_I Y;Y'}{X;X',A\land B\succ_I Y;Y'}\,;{\land}\textit{L}_2$$

$$\frac{X;X'\succ_I A,Y;Y'\quad X;X'\succ_J B,Y;Y'}{X;X'\succ_{I\lor J} A\land B,Y;Y'}\,{\land}\textit{R};\qquad \frac{X;X'\succ_I Y;Y',A\quad X;X'\succ_J Y;Y',B}{X;X'\succ_{I\land J} Y;Y',A\land B}\,;{\land}\textit{R}$$

$$\frac{A,X;X'\succ_I Y;Y'\quad B,X;X'\succ_J Y;Y'}{A\lor B,X;X'\succ_{I\lor J} Y;Y'}\,{\lor}\textit{L};\qquad \frac{X;X',A\succ_I Y;Y'\quad X;X',A\succ_J Y;Y'}{X;X',A\lor B\succ_{I\land J} Y;Y'}\,;{\lor}\textit{L}$$

$$\frac{X;X'\succ_I A,Y;Y'}{X;X'\succ_I A\lor B,Y;Y'}\,{\lor}\textit{R}_1;\quad \frac{X;X'\succ_I Y;Y',A}{X;X'\succ_I Y;Y';A\lor B}\,;{\lor}\textit{R}_1\quad \frac{X;X'\succ_I B,Y;Y'}{X;X'\succ_I A\lor B,Y;Y'}\,{\lor}\textit{R}_2;\quad \frac{X\succ_I Y;Y',B}{X\succ_I Y;Y',A\lor B}\,;{\lor}\textit{R}_2$$

$$\frac{A,B,X;X'\succ_I Y;Y'}{A\otimes B,X;X'\succ_I Y;Y'}\,{\otimes}\textit{L};\qquad \frac{X;X',A,B\succ_I Y;Y'}{X;X',A\otimes B\succ_I Y;Y'}\,;{\otimes}\textit{L}$$

$$\frac{X_1;X_1'\succ_I A,Y_1;Y_1'\quad X_2;X_2'\succ_J B,Y_2;Y_2'}{X_{1,2};X_{1,2}'\succ_{I\oplus J} A\otimes B,Y_{1,2};Y_{1,2}'}\,{\otimes}\textit{R};\qquad \frac{X_1;X_1'\succ_I Y_1;Y_1',A\quad X_2;X_2'\succ_J Y_2;Y_2',B}{X_{1,2};X_{1,2}'\succ_{I\otimes J} Y_{1,2};Y_{1,2}',A\otimes B}\,;{\otimes}\textit{R}$$

$$\frac{A,X_1;X_1'\succ_I Y_1;Y_1'\quad B,X_2;X_2'\succ_J Y_2;Y_2'}{A\oplus B,X_{1,2};X_{1,2}'\succ_{I\oplus J} Y_{1,2};Y_{1,2}'}\,{\oplus}\textit{L};\qquad \frac{X_1;X_1',A\succ_I Y_1;Y_1'\quad X_2;X_2',B\succ_J Y_2;Y_2'}{X_{1,2};X_{1,2}',A\oplus B\succ_{I\otimes J} Y_{1,2};Y_{1,2}'}\,;{\oplus}\textit{L}$$

$$\frac{X;X'\succ_I A,B,Y;Y'}{X;X'\succ_I A\oplus B,Y;Y'}\,{\oplus}\textit{R};\qquad \frac{X;X'\succ_I Y;Y',A,B}{X;X'\succ_I Y;Y',A\oplus B}\,;{\oplus}\textit{R}$$

$$\frac{X_1;X_1'\succ_I A,Y_1;Y_1'\quad B,X_2;X_2'\succ_J Y_2;Y_2'}{A\to B,X_{1,2};X_{1,2}'\succ_{I\oplus J} Y_{1,2};Y_{1,2}'}\,{\to}\textit{L};\qquad \frac{X_1;X_1'\succ_I Y_1;Y_1',A\quad X_2;X_2',B\succ_J Y_2;Y_2'}{X_{1,2};X_{1,2}',A\to B\succ_{I\otimes J} Y_{1,2};Y_{1,2}'}\,;{\to}\textit{L}$$

$$\frac{A,X;X'\succ_I B,Y;Y'}{X;X'\succ_I A\to B,Y;Y'}\,{\to}\textit{R};\qquad \frac{X;X',A\succ_I Y;Y',A}{X;X'\succ_I Y;Y',A\to B}\,;{\to}\textit{R}$$

$$\frac{X;X'\succ_I A,Y;Y'}{\neg A,X;X'\succ_I Y;Y'}\,{\neg}\textit{L};\quad \frac{X;X'\succ_I Y;Y',A}{X;X',\neg A\succ_I Y;Y'}\,;{\neg}\textit{L}\quad \frac{A,X;X'\succ_I Y;Y'}{X;X'\succ_I \neg A,Y;Y'}\,{\neg}\textit{R};\quad \frac{X;X',A\succ_I Y;Y'}{X;X'\succ_I Y;Y',\neg A}\,;{\neg}\textit{R}$$

$$\frac{X;X'\succ_I Y,Y'}{t,X;X'\succ_I Y,Y'}\,t\textit{L};\qquad \frac{X;X'\succ_I Y,Y'}{X;X',t\succ_I Y,Y'}\,;t\textit{L}\qquad \succ_f t;\ t\textit{R};\qquad \succ_t ;t\ ;t\textit{R}$$

$$f;\ \succ_f\ f\textit{L};\qquad ;f\ \succ_t\ ;f\textit{L}\qquad \frac{X;X'\succ_I Y;Y'}{X;X'\succ_I f,Y;Y'}\,f\textit{R};\qquad \frac{X;X'\succ_I Y;Y'}{X;X'\succ_I Y;Y',f}\,;f\textit{R}$$

Figure 2.10: SPLITTING RULES FOR CONNECTIVES

$$\frac{X_1; X_1' \succ_I Y_1; Y_1'}{X_1, X_2; X_1', X_2' \succ_I Y_1, Y_2; Y_1', Y_2'}\,K \qquad \frac{X_1, X_2, X_2; X_1', X_2', X_2' \succ_I Y_1, Y_2, Y_2; Y_1', Y_2', Y_2'}{X_1, X_2; X_1', X_2' \succ_I Y_1, Y_2; Y_1', Y_2'}\,W$$

Figure 2.11: SPLIT STRUCTURAL RULES

LEMMA 2.48 [SPLITTING SEQUENTS INTERPOLATE] *If* $X; X' \succ_I Y; Y'$ *can be derived in the split sequent system with some choice of structural rules (from K and W), then* $X \succ I, Y$ *and* $X', I \succ Y'$ *are both derivable in the sequent system with those structural rules, and furthermore,* I *is formed from atoms which are shared between* $X \cup Y$ *and* $X' \cup Y'$.

Proof: This is a straightforward induction on the length of the split sequent derivation. Space does not permit checking each of the rules here (there are many), but here is an indicative sample to show how to perform the verifications.

IDENTITIES: We have $p; \succ_p ; p$ since $p \succ p$ and $p \succ p$ are derivable. (p is in the shared vocabulary of p and p), and we have $p; \succ \bot, p;$ since $p \succ \bot, p$, $\bot \succ$ are both derivable. ($\bot$ has no atoms, so it is in every vocabulary.) Similarly, we have $; p \succ_{\neg p} p;$ since $\succ \neg p, p$, and $p, \neg p \succ$ are both derivable. ($\neg p$ has atoms from the shared vocabulary of p and p.) And finally, we have $; p \succ_\top ; p$ since $\succ \top$ and $p, \top \succ p$ are both derivable. ($\top$ has no atoms, so it is in every vocabulary.)

WEAKENING AND CONTRACTION: Weakening and contraction do not modify the interpolating formula. The premise of the weakening rule is $X_1; X_1' \succ_I Y_1; Y_1'$, so we have derivations for $X_1 \succ I, Y_1$ and $X_1', I \succ Y_1'$ (and I is the vocabulary shared between $X_1 \cup Y_1$ and $X_1' \cup Y_1'$.) By weakening these underlying sequents, we have derivations for $X_1, X_2 \succ I, Y_1, Y_2$ and $X_1', X_2', I \succ Y_1', Y_2'$ (and I remains in the shared vocabulary), so we indeed have the conclusion of the splitting rule for weakening: $X_{12}; X_{12}' \succ_I Y_{12}; Y_{12}'$. The verification for contraction has exactly the asme form.

LATTICE CONNECTIVES: Let's check the [$\wedge L$] rules, and [$\wedge R$] rules. For [$\wedge L$;], if we have $A, X; X' \succ_I Y; Y'$ we have $A, X \succ I, Y$ and hence $A \wedge B, X \succ I, Y$, and since $X' \succ I, Y'$, we have $A \wedge B, X; X' \succ_I Y; Y'$. (Since the interpolant doesn't change, it remains in the shared vocabulary.) For [;$\wedge L$], if we have $X; X', A \succ_I Y; Y'$, then we have $X \succ I, Y$, and $X', A, I \succ Y'$, and hence $X', A \wedge B, I \succ Y'$, and so, $X; X', A \wedge B \succ_I Y; Y'$. (And again, the interpolant doesn't change, so it remains in the shared vocabulary.)

For [$\wedge R$;], if we have $X; X' \succ_I A, Y; Y'$ and $X; X' \succ_J B, Y; Y'$, then we can reason as follows:

$$\frac{\dfrac{X \succ I, A, Y}{X \succ I \vee J, A, Y}\,\vee R \qquad \dfrac{X \succ J, B, Y}{X \succ I \vee J, B, Y}\,\vee R}{X \succ I \vee J, A \wedge B, Y}\,\wedge R \qquad \frac{X', I \succ Y' \quad X', J \succ Y'}{X', I \vee J \succ Y'}\,\vee L$$

so we have $X \succ I \lor J, A \land B, Y$ and $X', I \lor J \succ Y'$ and hence $X; X' \succ_{I \lor J} A \land B, Y; Y'$ is derivable. And since I is in the vocabulary shared between A, X, Y and X', Y', and J is in the vocabulary shared between B, X, Y and X', Y', it follows that $I \lor J$ is in the vocabulary shared between $A \lor B, X, Y$ and X', Y', Z'.

For [$\land R$;], if we have $X; X' \succ_I Y; Y', A$ and $X; X' \succ_J Y; Y', B$, then we can reason as follows:

$$\dfrac{X \succ I, Y \quad X \succ J, Y}{X \succ I \land J, Y}\,{\scriptstyle \land R} \qquad \dfrac{\dfrac{X', I \succ Y', A}{X', I \land J \succ Y', A}\,{\scriptstyle \land L} \quad \dfrac{X', J \succ Y', B}{X', I \land J \succ Y', B}\,{\scriptstyle \land L}}{X', I \land J \succ Y', A \land B}\,{\scriptstyle \land R}$$

so we have $X; X' \succ_{I \lor J} Y; Y', A \land B$ as desired. And since I is in the vocabulary shared between A, X, Y and X', Y', and J is in the vocabulary shared between B, X, Y and X', Y', it follows that $I \land J$ is in the vocabulary shared between $A \lor B, X, Y$ and X', Y', Z'.

MULTIPLICATIVE CONNECTIVES: We'll check the conditional rules. (Fission and fusion are similar.) For [$\to L$;], if we have derivations of $X_1 \succ I, A, Y_1$ and $B, X_2 \succ J, Y_2$ and $X_1', I \succ Y_1'$ and $X_2', J \succ Y_2'$, then we can reason as follows:

$$\dfrac{\dfrac{X_1 \succ I, A, Y_1 \quad B, X_2 \succ J, Y_2}{A \to B, X_{12} \succ I, J, Y_{12}}\,{\scriptstyle \to L}}{A \to B, X_1, X_2 \succ I \oplus J, Y_{12}}\,{\scriptstyle \oplus R} \qquad \dfrac{X_1', I \succ Y_1' \quad X_2', J \succ Y_2'}{X_{12}', I \oplus J \succ Y_{12}'}\,{\scriptstyle \oplus L}$$

and similarly, for [;$\to L$], if we have derivations of $X_1 \succ I, Y_1$, $X_2 \succ J, Y_2$, $X_1', I \succ A, Y_1'$ and $B, X_2', J \succ Y_2'$, we have:

$$\dfrac{X_1 \succ I, Y_1 \quad X_2 \succ J, Y_2}{X_{12} \succ I \otimes Y_{12}}\,{\scriptstyle \otimes R} \qquad \dfrac{\dfrac{X_1', I \succ A, Y_1' \quad B, X_2', J \succ Y_2'}{A \to B, X_{12}', I, J \succ Y_{12}'}\,{\scriptstyle \to L}}{A \to B, X_{12}', I \otimes J \succ Y_{12}'}\,{\scriptstyle \otimes L}$$

and the right conditional rules are similarly verified, except the interpolating formula is constant, because we are not combining premise sequents. For [$\to R$;], if we have derivations for $A, X \succ I, B, Y$ and $X', I \succ Y'$ then to verify the split sequent $X; X' \succ_I A \to B, Y'; Y'$, we derive $X \succ I, A \to B, Y'$ and we are done. The same goes for the [;$\to R$] rule.

NEGATION: The negation rules work on the same principle as the [$\to R$] rules. The interpolating formula I remains constant as the formula A converted into $\neg A$ remains on the same side of the splitting as it shifts over the turnstile. Here is the case for [$\neg L$;], and the others are identical in form.

$$\dfrac{X \succ I, A, Y}{X, \neg A \succ I, Y}\,{\scriptstyle \neg L} \qquad X', I \succ Y'$$

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UNITS: For [$\mathsf{f}L$;], we have derivations of $\mathsf{f} \succ \mathsf{f}$ (the left splitting) and $\mathsf{f} \succ$ (the right). For [; $\mathsf{f}L$], we have $\succ \mathsf{t}$ (the left splitting) and $\mathsf{f}, \mathsf{t} \succ$ (the right). For the [$\mathsf{f}R$] rules, the interpolant is constant from premise to conclusion, and the rule merely inserts an extra f on the right hand side of a sequent (on one side of the splitting or the other), and that is an instance of the [$\mathsf{f}R$] rule of the underlying sequent calculus. (The verification for the t rules the the same form.)

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More consequences of *Cut*-elimination and the admissibility of the identity rules Id_A will be considered as the book goes on. Exercises 8–14 ask you to consider different possible connective rules, some of which will admit of *Cut*-elimination and *Id*-admissibility when added, and others of which which will not. In Chapter 4 we will look at reasons why this might help us demarcate definitions of a kind of *properly logical* concept from those which are not logical in that sense.

2.5 | HISTORY

The idea of taking the essence of conjunction and disjunction to be expressed in these sequent rules is to take conjunction and disjunction to form what is known as a *lattice*. A lattice is on *ordered* structure in which we have for every pair of objects a *greatest lower bound* and a *least upper bound*. A *greatest lower bound* of x and y is something below both x and y but which is greatest among such things. A *least upper bound* of x and y is something above both x and y but which is the least among such things. Among statements, taking $\succ$ to be the ordering, $A \wedge B$ is the greatest lower bound of A and of B (since $A \wedge B \succ A$ and $A \wedge B \succ B$, and if $C \succ A$ and $C \succ B$ then $C \succ A \wedge B$) and $A \vee B$ is their least upper bound (for dual reasons).

Here, $\succ$ is the ordering. If $A \succ B$, think of A as occuring 'below' B in the ordering from stronger to weaker.

Well, you need to *squint*, and take A and A' to be *the same* if $A \succ A'$ and $A' \succ A$ to make $A \wedge B$ *the unique* greatest lower bound. If it helps, don't think of the *sentence* A but the *proposition*, where two logically equivalent sentences express the same proposition.

Lattices are wonderful structures, which may be applied in many different ways, not only to logic, but in many other domains as well. Davey and Priestley's *Introduction to Lattices and Order* [29] is an excellent way into the literature on lattices. The concept of a lattice dates to the late 19th Century in the work of Charles S. Peirce and Ernst Schröder, who independently generalised Boole's algebra of propositional logic. Richard Dedekind's work on 'ideals' in algebraic number theory was an independent mathematical motivation for the concept. Work in the area found a focus in the groundbreaking series of papers by Garrett Birkhoff, culminating in a the book *Lattice Theory* [15]. For more of the history, and for a comprehensive state of play for lattice theory and its many applications, George Grätzer's 1978 *General Lattice Theory* [49], and especially its 2003 Second Edition [50] is a good port of call.

We will not study much algebra in this book. However, algebraic technqiues find a very natural home in the study of logical systems. Helena Rasiowa's 1974 *An Algebraic Approach to Non-Classical Logics* [92] was the first look at lattices and other structures as models of a wide range

Well, we won't study algebra *explicitly*. Algebraic considerations and sensibilities underly *much* of what will go on. But that will almost always stay under the surface.

of different systems. For a good guide to why this technique is important, and what it can do, you cannot go past J. Michael Dunn and Gary Hardegree's *Algebraic Methods in Philosophical Logic* [35].

The idea of studying derivations consisting of sequents, rather than proofs from premises to conclusions, is entirely due to Gentzen, in his groundbreaking work in proof theory. His motivation was to extend his results on normalisation from what we called the standard natural deduction system to classical logic as well as intuitionistic logic [43, 44]. To do this, it was fruitful to step back from proofs from premises X to a conclusion A to consider statements of the form '$X \succ A$,' making explicit at each step on which premises X the conclusion A depends. Then as we will see in the next chapter, normalisation 'corresponds' in some sense to the elimination of *Cut*s in a derivation. One of Gentzen's great insights was that sequents could be generalised to the form $X \succ Y$ to provide a uniform treatment of traditional Boolean classical logic. We will make much of this connection in the next chapter.

But for more connectives than just the conditional.

Gentzen didn't use the turnstile. His notation was '$\Gamma \rightarrow \mathfrak{A}$'. We use the arrow for a conditional, and the turnstile for a sequent separator.

Gentzen's own logic wasn't lattice logic, but traditional classical logic (in which the distribution of conjunction over disjunction—that is, $A \wedge (B \vee C) \succ (A \wedge B) \vee (A \wedge C)$—is valid) and intuitionistic logic. I have chosen to start with simple sequents for lattice logic for two reasons. First, it makes the procedure for the elimination of *Cut*s much more simple. There are fewer cases to consider and the essential shape of the argument is laid bare with fewer inessential details. Second, once we see the technique applied again and again, it will hopefully reinforce the thought that it is very general indeed. Sequents were introduced as a way of looking at an underlying proof structure. As a pluralist, I take it that there is more than one sort of underlying proof structure to examine, and so, sequents may take more than one sort of shape. Much work has been done recently on *why* Gentzen chose the rules he did for his sequent calculi. I have found papers by Jan von Plato [84, 85] most helpful. Gentzen's papers are available in his collected works [45], and a biography of Gentzen, whose life was cut short in the Second World War, has recently been written [70, 71].

2.6 | EXERCISES

BASIC EXERCISES

Q1 Find a derivation for $p \succ p \wedge (p \vee q)$ and a derivation for $p \vee (p \wedge q) \succ p$. Then find a *Cut*-free derivation for $p \vee (p \wedge q) \succ p \wedge (p \vee q)$ and compare it with the derivaion you get by joining the two original derivations with a *Cut*.

Q2 Show that there is no *Cut*-free derivation of the following sequents

1 : $p \vee (q \wedge r) \succ p \wedge (q \vee r)$
2 : $p \wedge (q \vee r) \succ (p \wedge q) \vee r$
3 : $p \wedge (q \vee (p \wedge r)) \succ (p \wedge q) \vee (p \wedge r)$

Q3 Suppose that there is a derivation of $A \succ B$. Let $C(A)$ be a formula containing A as a subformula, and let $C(B)$ be that formula with the subformula A replaced by B. Show that there is a derivation of $C(A) \succ C(B)$. Furthermore, show that a derivation of $C(A) \succ C(B)$ may be systematically constructed from the derivation of $A \succ B$ together with the context $C(-)$ (the shape of the formula $C(A)$ with a 'hole' in the place of the subformula A).

Q4 Find a derivation of $p \wedge (q \wedge r) \succ (p \wedge q) \wedge r$. Find a derivation of $(p \wedge q) \wedge r \succ p \wedge (q \wedge r)$. Put these two derivations together, with a *Cut*, to show that $p \wedge (q \wedge r) \succ p \wedge (q \wedge r)$. Then eliminate the cuts from this derivation. What do you get?

Q5 Do the same thing with derivations of $p \succ (p \wedge q) \vee p$ and $(p \wedge q) \vee p \succ p$. What is the result when you eliminate this cut?

Q6 Show that (1) $A \succ B \wedge C$ is derivable if and only if $A \succ B$ and $A \succ C$ is derivable, and that (2) $A \vee B \succ C$ is derivable if and only if $A \succ C$ and $B \succ C$ are derivable. Finally, (3) when is $A \vee B \succ C \wedge D$ derivable, in terms of the derivability relations between A, B, C and D.

Q7 Under what conditions do we have a derivation of $A \succ B$ when A contains only propositional atoms and *disjunctions* and B contains only propositional atoms and *conjunctions*.

Q8 Expand the system with the following rules for the propositional *constants* $\bot$ and $\top$.

$$A \succ \top \ [\top R] \qquad \bot \succ A \ [\bot L]$$

Show that *Cut* is eliminable from the new system. (You can think of $\bot$ and $\top$ as zero-place connectives. In fact, there is a sense in which $\top$ is a zero-place *conjunction* and $\bot$ is a zero-place *disjunction*. Can you see why?)

Q9 Show that simple sequents including $\top$ and $\bot$ are decidable, following Corollary 2.25 and the results of the previous question.

Q10 Show that every formula composed of just $\top$, $\bot$, $\wedge$ and $\vee$ is *equivalent* to either $\top$ or $\bot$. (What does this result remind you of?)

Q11 Prove the interpolation theorem (Corollary 2.45) for derivations involving $\wedge$, $\vee$, $\top$ and $\bot$.

Q12 Expand the system with rules for a propositional connective with the following rules:

$$\frac{A \succ R}{A \text{ tonk } B \succ R}\,\text{tonk } L \qquad \frac{L \succ B}{L \succ A \text{ tonk } B}\,\text{tonk } R$$

See Arthur Prior's "The Runabout Inference-Ticket" [89] for tonk's first appearance in print.

What new things can you derive using tonk? Can you derive $A \text{ tonk } B \succ A \text{ tonk } B$? Is *Cut* eliminable for formulas involving tonk?

Q13 Expand the system with rules for a propositional connective with the following rules:

$$\frac{A \succ R}{A \text{ honk } B \succ R}\,\text{honk } L \qquad \frac{L \succ A \quad L \succ B}{L \succ A \text{ honk } B}\,\text{honk } R$$

What new things can you derive using honk? Can you derive A honk $B \succ A$ honk B? Is *Cut* eliminable for formulas involving honk?

Q14 Expand the system with rules for a propositional connective with the following rules:

$$\frac{A \succ R \qquad B \succ R}{A \text{ plonk } B \succ R}\,\text{plonk}\,L \qquad \frac{L \succ B}{L \succ A \text{ plonk } B}\,\text{plonk}\,R$$

What new things can you derive using plonk? Can you derive A plonk $B \succ A$ plonk B? Is *Cut* eliminable for formulas involving plonk?

INTERMEDIATE EXERCISES

Q15 Give a formal, recursive definition of the *dual* of a sequent, and the *dual* of a derivation, in such a way that the dual of the sequent $p_1 \wedge (q_1 \vee r_1) \succ (p_2 \vee q_2) \wedge r_2$ is the sequent $(p_2 \wedge q_2) \vee r_2 \succ p_1 \vee (q_1 \wedge r_1)$. And then use this definition to prove the following theorem.

THEOREM 2.49 [DUALITY FOR DERIVATIONS] *A sequent* $A \succ B$ *is derivable if and only if its dual* $(A \succ B)^d$ *is derivable. Furthermore, the dual of the derivation of* $A \succ B$ *is a derivation of the dual of* $A \succ B$.

Q16 Even though the distribution sequent $p \wedge (q \vee r) \succ (p \wedge q) \vee r$ is not derivable (Example 2.10), some sequents of the form $A \wedge (B \vee C) \succ (A \wedge B) \vee C$ *are* derivable. Give an independent characterisation of the triples $\langle A, B, C\rangle$ such that $A \wedge (B \vee C) \succ (A \wedge B) \vee C$ is derivable.

Q17 Prove the invertibility result of Theorem 2.29 without appealing to the *Cut* rule or to *Cut*-elimination. (HINT: if a sequent $A \vee B \succ C$ has a derivation δ, consider the instances of $A \vee B$ 'leading to' the instance of $A \vee B$ in the conclusion. How does $A \vee B$ appear first in the derivation? Can you change the derivation in such a way as to make it derive $A \succ C$? Or to derive $B \succ C$ instead? Prove this, and a similar result for $\wedge L$.)

ADVANCED EXERCISES

Q18 Define a notion of reduction for simple sequent derivations parallel to the definition of reduction of natural deduction proofs in Chapter 1. Show that it is strongly normalising and that each derivation reduces to a unique *Cut*-free derivation.

Q19 Define *terms* corresponding to simple sequent derivations, in an analogy to the way that λ-terms correspond to natural deduction proofs for conditional formulas. For example, we may annotate each derivation with *terms* in the following way:

$$p \succ_x p \;\; \mathit{Id} \qquad \frac{L \succ_f A \quad A \succ_g R}{L \succ_{f \circ g} R}\,\mathit{Cut}$$

$$\frac{A \succ R}{A \land B \succ_{l[f]} R}{\scriptstyle \land L_1} \qquad \frac{B \succ R}{A \land B \succ_{r[f]} R}{\scriptstyle \land L_2} \qquad \frac{L \succ_f A \quad L \succ_g B}{L \succ_{f\|g} A \land B}{\scriptstyle \land R}$$

where x is an atomic term (of type p), f and g are terms, $l[\,]$ and $r[\,]$ are one-place term constructors and $\|$ is a two-place term constructor (of a kind of parallel composition), and $\circ$ is a two-place term constructor (of *serial composition*). Define similar term constructors for the disjunction rules.

Then reducing a *Cut* will correspond to simplifying terms by eliminating serial composition. A *Cut* in which $A \land B$ is active will take the following form of reduction:

$$(f\|g) \circ l[h] \text{ reduces to } f \circ h \qquad (f\|g) \circ r[h] \text{ reduces to } g \circ h$$

Fill out all the other reduction rules for every other kind of step in the *Cut*-elimination argument.

Do these terms correspond to anything like computation? Do they have any other interpretation?

PROJECTS

Q20 Provide sequent formulations for logics intermediate between simple sequent logic and the logic of *distributive lattices* (in which $p \land (q \lor r) \succ (p \land q) \lor r$). Characterise *which* logics intermediate between lattice logic (the logic of simple sequents) and distributive lattice logic *have* sequent presentations, and which do not. (This requires making explicit what counts as a *logic* and what counts as a sequent presentation of a logic.)

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