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Subintuitionistic Logics

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Abstract Weakening the conditions on the Kripke semantics for propositional intuitionistic logic (**J**) unearths a family of logics below **J**. This paper provides a characterization of eleven such logics, using Kripke semantics, proof theory, and algebraic models. Questions about modelling quantification in these logics are also discussed.

1 Introduction Once the Kripke semantics for normal modal logics were introduced, a whole family of modal logics other than the Lewis systems **S1** to **S5** were discovered. These logics were obtained by changing the semantics in natural ways. The same can be said of the Kripke-style semantics for relevant logics: a whole range of logics other than the standard systems **R**, **E** and **T** were unearthed once a semantics was given (cf. Priest and Sylvan [6], Restall [7], and Routley et al. [8]). In a similar way, weakening the structural rules of the Gentzen formulation of classical logic gives rise to other 'substructural' logics such as linear logic (as in Girard [4]). This process of 'strategic weakening' is becoming popular today, with the discovery of applications of these logics to areas such as linguistics and the theory of computation (cf. van Benthem [1]).

Until now no-one has (to my knowledge) examined what the process of weakening does to the Kripke-style semantics of intuitionistic logic. This paper remedies the deficiency, introducing the family of *subintuitionistic logics*.

These systems have some appealing features. Unlike other substructural logics such as linear logic (which lack distribution of extensional disjunction over conjunction) they have a very natural Kripke-style worlds semantics. Also, the difficulties with regard to modelling quantification in these systems may be able to shed some light on the difficulties in naturally modelling quantification in relevant logics, as it must be admitted that the semantics currently available for quantified relevant logics are rather baroque (cf. Fine [3]). But most importantly, delving in the undergrowth of logics such as intuitionistic logic gives us a 'feel' for how such systems are put together, and what job is being done by each aspect of the modelling conditions in

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the semantics. There is no better way to see how something works than to take it apart and put it back together. So let's see what is lurking beneath Heyting's beautiful system.

2 The semantics Take a propositional language with a countable number of propositional variables, and the connectives

$$f \quad \wedge \quad \vee \quad \supset$$

of which the first is nullary (that is, a propositional constant) and the rest, binary. Formulas are defined inductively in the usual manner—we use parentheses to disambiguate bindings, and we leave them out whenever the context is clear. Conjunction and disjunction bind more strongly than implication, so $p \wedge q \supset r \vee s$ is short for $(p \wedge q) \supset (r \vee s)$. We use A, B as metavariables ranging over formulas, and $\sim A$ is defined as $A \supset f$.

The models of propositional intuitionistic logic that we will consider are quadruples $\langle g, W, R, \Vdash \rangle$, where g, called the *base world*, is a particular element of the set W of *worlds*. R is a reflexive and transitive binary relation on W representing *accessibility*. The base world can be taken to be *omniscient*, in that it accesses every world—so gRw for each $w \in W$. Each world *forces* the truth of formulas, and this relation is indicated by $\Vdash$ (so, $w \Vdash A$ is read as 'w forces A'). The forcing relation satisfies a few conditions: a world forces a conjunction whenever it forces its conjuncts, it forces a disjunction whenever it forces at least one disjunct, and it forces a conditional just in case at any world accessible from it, if the antecedent is forced there, so is the consequent. The constant f is never forced anywhere. Also, forcing is *hereditary*, in that if $w \Vdash A$ and wRv, then $v \Vdash A$ too. The theorems of intuitionistic logic are the formulas that are forced by g in *any* model. The valid rules are those that are preserved at g in any model.

It must be pointed out that this is not the standard way to present the Kripke semantics for intuitionistic logics. (For the original presentation, see Kripke [5]). Most often, no world is picked out as a base world, and validity may be characterized as truth-preservation at all worlds in all models. However, restricting our class of models to those that have base worlds, and defining validity in this way results in no loss of generality. (Given any model at all, we can cut it down to the submodel which contains only a particular world w, and all of the worlds it accesses, without any change in the truths of formulas at worlds.) The reason we consider models with an omniscient world is that it is important in the semantics of the systems below intuitionistic logic.

On a standard interpretation of this semantics, a world is a state of information, and state y is accessible from state x just in case y is an extension of x, so y will not reject any information that x accepts. This view motivates the conditions on the semantics—but it is clearly not the only possible view. Suppose that accessibility is related to time, and that we may *reject* some of our information as time progresses. Then if state y is accessible from state x just when y is later than x in our process of information gathering, there is no need for y to confirm all that x confirms. Heredity could be rejected on this interpretation. In this context, $A \supset (B \supset A)$ will not come out as a logical truth.

Another possibility is to reject the reflexivity of accessibility. Reflexivity ensures that states of information are closed under *modus ponens*. In contexts of vagueness we may want this to happen. A state of information might inform us that $10\,000$ grains of sand makes a heap, and that if n grains of sand make a heap, so do $n-1$. One way of dealing with vague predicates is to treat this state of information as a way of "stretching the truth" (Slaney [9]), and that the state of information that includes the fact that $9\,999$ grains of sand makes a heap, or the fact that if n grains of sand make a heap so do $n-2$, stretch the truth *further*. Treating worlds as ways of stretching the truth, we wouldn't want them to be closed under *modus ponens*—for each application of modus ponens might involve stretching the truth a little more, as our example illustrates. Under this interpretation, the clauses for the connectives still make sense—in particular, a degree of stretch will validate a conditional just in case we stretch the truth more to accommodate the antecedent, the consequent is also accommodated. So, on this interpretation of worlds, we have reason to reject reflexivity.

Rejecting the requirement that f never be forced at any world was introduced by Johannson in his minimal logic [2]. There is no reason to reiterate the reasons for this here.

It is also possible to reject the transitivity of accessibility. It is not easy to see any interpretation of accessibility that would motivate the rejection of this postulate, but as removing it does not make any of our proofs any more difficult, it is instructive to see what happens when it is absent. We leave the task of finding a use for this to a later occasion.

This leaves R bereft of conditions, except for the omniscience of g. This condition is vital for principles such as prefixing: $A \supset B \vdash (C \supset A) \supset (C \supset B)$. If we remove this requirement, we lose such principles. If g does not access itself, we even lose *modus ponens*. It would be interesting to see exactly what is left behind in these cases, but the methods used in this paper will not suffice to prove completeness of the resulting semantics. We will have to make do with the restriction we have on R, leaving it to another time and other methods to see what these *really* basic logics are like.

In the rest of this section we will introduce the system which is given by dropping each of the conditions on accessibility, and then in the next we see what happens when the conditions are added one at a time.

Definition 2.1 A *basic subintuitionistic model* is a quadruple $\langle g, W, R, \Vdash\rangle$, where g, called the *base world*, is a particular element of the set W of *worlds*. R is a binary relation on W, called *accessibility*. The base world is *omniscient*, so gRw for each $w \in W$. Each world *forces* the truth of formulas, and this relation is indicated by $\Vdash$. The relation satisfies the following conditions.

$w \Vdash A \wedge B$	if and only if	$x \Vdash A$ and $w \Vdash B$
$w \Vdash A \vee B$	if and only if	$x \Vdash A$ or $w \Vdash B$
$w \Vdash A \supset B$	if and only if	for each v where wRv, if $v \Vdash A$ then $v \Vdash B$

The constant f is left arbitrary. A model *forces* A just when its base world forces A. A set Σ of formulas is said to *give* A just when each model that forces every element of Σ also forces A. This is written $\Sigma \models A$. If Σ is empty, then this is written $\models A$.

To get a feel for the kinds of formulas forced at the base world, it is good to validate the following results.

Lemma 2.2 *In any model $\langle g, W, R, \Vdash\rangle$, for each $w \in W$ and each formula A, $w \Vdash A \supset A$.*

Theorem 2.3 *We have the following:*

$$\begin{array}{ll}
\models A \supset A & \models A \wedge (B \vee C) \supset (A \wedge B) \vee (A \wedge C) \\
\models A \wedge B \supset A & \models A \supset (B \supset B) \\
\models A \wedge B \supset B & \models (A \supset B) \wedge (A \supset C) \supset (A \supset B \wedge C) \\
\models A \supset A \vee B & \models (A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C) \\
\models B \supset A \vee B & \models (A \supset B) \wedge (B \supset C) \supset (A \supset C) \\
A,\ B \models A \wedge B & (A \supset B) \vee E,\ (C \supset D) \vee E \models ((B \supset C) \supset (A \supset D)) \vee E \\
A,\ A \supset B \models B & A \vee C,\ (A \supset B) \vee C \models B \vee C \\
 & A \supset B,\ C \supset D \models (B \supset C) \supset (A \supset D)
\end{array}$$

Proof: Proving these results is a trivial exercise of validating rules. Lemma 2.2 is useful for $\models A \supset (B \supset B)$. To show that $A \supset B,\ C \supset D \models (B \supset C) \supset (A \supset D)$ you need to use the omniscience of g.

The choice of theorems and valid rules in Theorem 2.3 is not completely arbitrary. They provide a Hilbert-style axiomatization for the logic of basic subintuitionistic models—which we will call **SJ**.

Definition 2.4 **SJ** is the logic given by the following axioms and rules

$$\begin{array}{ll}
\vdash A \supset A & \vdash A \wedge (B \vee C) \supset (A \wedge B) \vee (A \wedge C) \\
\vdash A \wedge B \supset A & \vdash A \supset (B \supset B) \\
\vdash A \wedge B \supset B & \vdash (A \supset B) \wedge (A \supset C) \supset (A \supset B \wedge C) \\
\vdash A \supset A \vee B & \vdash (A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C) \\
\vdash B \supset A \vee B & \vdash (A \supset B) \wedge (B \supset C) \supset (A \supset C) \\
A,\ B \vdash A \wedge B & (A \supset B) \vee E,\ (C \supset D) \vee E \vdash ((B \supset C) \supset (A \supset D)) \vee E \\
A,\ A \supset B \vdash B & A \vee C,\ (A \supset B) \vee C \vdash B \vee C \\
 & A \supset B,\ C \supset D \vdash (B \supset C) \supset (A \supset D)
\end{array}$$

Theorem 2.3 shows that the basic subintuitionistic models are sound with respect to **SJ**. That is, if $\Sigma \vdash A$, then $\Sigma \models A$. Completeness is not much more difficult—we will show that if $\Sigma \nvdash A$, then there is a particular model that forces each element of Σ, but doesn't force A. This model will be a *canonical* model, which has particular sets of formulas as worlds.

The completeness proof will then proceed as follows: We 'fill out' the set Σ to form a set Π that will serve as our base world, and we form a class of sets of sentences to serve as the other worlds. These sets must respect the modelling conditions for the connectives. If a world contains a conjunction, it must contain both conjuncts. If it contains a disjunction, it must contain one of the disjuncts. If $B \supset C$ is in Π and B is in some world, then C must be in that world too. This means that Π is closed under *modus ponens*, but the other worlds need not be so closed, because accessibility is no longer reflexive. Such sets of sentences are called *prime* Π-*theories*. The set of all nonempty and nonfull (that is, not containing *every* formula) prime Π-theories will count as our model. The accessibility relation is defined in the usual way—it relates Σ to Δ just when for each $B \supset C \in \Sigma$, if $B \in \Delta$, then $C \in \Delta$ too. The heart of the completeness proof involves showing that the resulting collection of worlds actually forms a model, in which the sentences forced by a world are just the elements of the world. The only difficult parts are showing that we can construct a Π that contains Σ yet excludes A, and that for each world Σ and $B \supset C \notin \Sigma$ there is a world Δ where $B \in \Delta$ but $C \notin \Delta$. These results require the well-ordering of our language, and the application of Zorn's lemma to provide prime theories that do the job. The rest of the proof is standard.

Definition 2.5

- If Π is a set of sentences, then $\Pi_{\supset}$ is the set of members of Π of the form $A \supset B$.
- If Π is a set of sentences, $\bigwedge \Pi$ and $\bigvee \Pi$ are the closure of Π under conjunction and disjunction respectively.
- $\Sigma \vdash_{\Pi} A$ if and only if $\Sigma \cup \Pi_{\supset} \vdash A$.
- Σ is a Π-theory if and only if:
  (i) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$, (i.e. $\bigwedge \Sigma = \Sigma$),
  (ii) if $\vdash_{\Pi} A \supset B$ then (if $A \in \Sigma$, $B \in \Sigma$).
- Σ is *prime* if and only if (if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$).
- W_{Π} is the set of all nonempty, nonfull prime Π-theories.
- R is defined on sets of formulas as follows:

$$\Sigma R \Delta \text{ if and only if (if } A \supset B \in \Sigma \text{ then (if } A \in \Delta \text{ then } B \in \Delta))$$

- $\Sigma \vdash_{\Pi} \Delta$ if and only if for some $B_1, \ldots, B_n \in \Delta$, $\Sigma \vdash_{\Pi} B_1 \vee \ldots \vee B_n$.
- $\vdash_{\Pi} \Sigma \supset \Delta$ if and only if for some $A_1, \ldots, A_n \in \Sigma$, and some $B_1, \ldots, B_n \in \Delta$, $\vdash_{\Pi} A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_n$.
- Σ is Π-deductively closed if and only if (if $\Sigma \vdash_{\Pi} A$ then $A \in \Sigma$).
- $\langle \Sigma, \Delta \rangle$ is a Π-partition if and only if $\Sigma \cup \Delta$ is the set of all formulas, and $\nvdash_{\Pi} \Sigma \supset \Delta$.

The worlds in our canonical models will be prime theories—we need results about the provability relation in the Hilbert system to show that particular prime theories exist.

Lemma 2.6 *If $A \vdash B$ then $C \vee A \vdash C \vee B$*

Proof: An easy induction on the length of the proof.

Corollary 2.7 *If $A \vdash C$ and $B \vdash C$, then $A \vee B \vdash C$.*

Proof: By the lemma, $A \vee B \vdash C \vee B$, and also, $C \vee B \vdash C \vee C$. But it is simple to show that $C \vee C \vdash C$, so $A \vee B \vdash C$ as required.

Now we need results concerning how to *construct* prime theories.

Lemma 2.8 *If $\langle \Sigma, \Delta \rangle$ is a Π-partition, then Σ is a prime Π theory.*

Proof: Take $A, B \in \Sigma$. Then $\nvdash_{\Pi} \Sigma \supset \Delta$ ensures that $A \wedge B \notin \Delta$, as $\vdash_{\Pi} A \wedge B \supset A \wedge B$. So, $A \wedge B \in \Sigma$. Take $A \in \Sigma$ and $\vdash_{\Pi} A \supset B$. Then $\nvdash_{\Pi} \Sigma \supset \Delta$ ensures that $B \notin \Delta$, so $B \in \Sigma$. So, Σ is a Π-theory.

Take $A \vee B \in \Sigma$. Then $\nvdash_{\Pi} \Sigma \supset \Delta$ ensures that either A or B is not in Δ, as $\vdash_{\Pi} A \vee B \supset A \vee B$. So, either A or B is in Σ. So, Σ is prime, giving us our result.

Lemma 2.9 *If $\Sigma \nvdash \Delta$, then there are $\Sigma' \supseteq \Sigma$, $\Delta' \supseteq \Delta$, where $\langle \Sigma', \Delta' \rangle$ is a partition, and Σ' is deductively closed.*

Proof: Consider the set $X = \{\langle \Sigma_*, \Delta_* \rangle : \Sigma \subseteq \Sigma_*,\ \Delta \subseteq \Delta_* \text{ and } \Sigma_* \nvdash \Delta_*\}$. It is nonempty, partially ordered by pairwise inclusion, and every chain has an upper bound (its union). Therefore, it has a maximal element by Zorn's lemma. Take one to be $\langle \Sigma', \Delta' \rangle$. It remains to be shown that it is a partition. Suppose it is not. Then there is a C where $C \notin \Sigma'$ and $C \notin \Delta'$, and as $\langle \Sigma', \Delta' \rangle$ is maximal, we must have $\Sigma' \cup \{C\} \vdash \Delta'$ and $\Sigma' \vdash \Delta' \cup \{C\}$. So there are $A_1, A_2 \in \bigwedge \Sigma'$ and $B_1, B_2 \in \bigvee \Delta'$ where

$$A_1 \wedge C \vdash B_1 \qquad A_2 \vdash B_2 \vee C$$

We will show that $A_1 \wedge A_2 \vdash B_1 \vee B_2$ contrary to the fact that $\Sigma' \nvdash \Delta'$, giving our result. It is easy to show that $A_1 \wedge A_2 \vdash (B_2 \vee C) \wedge A_1$. A distribution gives $A_1 \wedge A_2 \vdash B_2 \vee (A_1 \wedge C)$. But $A_1 \wedge C \vdash B_1$, so Lemma 2.6 gives $B_2 \vee (A_1 \wedge C) \vdash B_2 \vee B_1$, so transitivity gives $A_1 \wedge A_2 \vdash B_2 \vee B_1$, contradicting our assumption.

So $\langle \Sigma', \Delta' \rangle$ is a partition. To show that Σ' is deductively closed, suppose that $\Sigma' \vdash A$. If $A \notin \Sigma'$, we have $A \in \Delta'$. Thus $\Sigma' \vdash \Delta'$, contrary to our assumption. Thus $A \in \Sigma'$.

Corollary 2.10 *If $\Sigma \nvdash A$ then there is a $\Pi \supseteq \Sigma$ such that $A \notin \Pi$, $\Pi \in W_\Pi$, and Π is Π-deductively closed.*

Proof: $\Sigma \nvdash \{A\}$, so by the previous lemma, there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \{A\}$ where $\Sigma' \nvdash \Delta'$, $\langle \Sigma', \Delta' \rangle$ is a partition, and Σ' is deductively closed. Set $\Pi = \Sigma'$. As $\langle \Pi, \Delta' \rangle$ is a partition, Π is a prime theory, and as $A \in \Delta'$, $A \notin \Pi$. As $\Pi \vdash_\Pi A$ entails $\Pi \vdash A$, the deductive closure of Π ensures the Π-deductive closure of Π. To show that Π is a Π-theory, assume that $\vdash_\Pi A \supset B$ and $A \in \Pi$. Clearly, $\Pi \vdash B$, so $B \in \Pi$. Π is nonempty as it contains each theorem and nonfull as it doesn't include A.

Lemma 2.11 *If $\nvdash_\Pi \Sigma \supset \Delta$ there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a Π-partition.*

Proof: Take the proof of Lemma 2.9, and replace all things of the form $X \nvdash Y$ with $\nvdash_\Pi X \supset Y$. This is the proof.

Lemma 2.12 *If $\Sigma \in W_\Pi$ and $A \supset B \notin \Sigma$, then there is a $\Gamma \in W_\Pi$ where $\Sigma R \Gamma$, $A \in \Gamma$ and $B \notin \Gamma$.*

Proof: Consider the sets $\Gamma_0 = \{C : A \supset C \in \Sigma\}$ and $\Delta_0 = \{B\}$. It is easy to show that if $C \supset D \in \Sigma$ and $C \in \Gamma_0$ then $D \notin \Delta_0$. As in the proof of Lemma 2.9 we construct a set $X = \{\langle \Gamma_*, \Delta_* \rangle : \Gamma_0 \subseteq \Gamma_*, \Delta_0 \subseteq \Delta_* \text{ and if } C \supset D \in \Sigma \text{ and } C \in \Gamma_* \text{ then } D \notin \Delta_*\}$. X is nonempty, as it contains $\langle \Gamma_0, \Delta_0 \rangle$, it is partially ordered by inclusion, and chains are bounded by their pairwise union. Zorn's lemma gives us a maximal element. Call it $\langle \Gamma, \Delta \rangle$. It must be a partition, for if $C \notin \Gamma \vee \Delta$, then we must have $C' \in \Gamma$ and $D' \in \Delta$ such that $C \supset D' \in \Sigma$ (otherwise we could put C into Γ and get a bigger pair) and $C' \supset C \in \Sigma$ (otherwise we could put C into Δ). But this gives $C' \supset D' \in \Sigma$ by conjunctive syllogism, which would mean that $\langle \Gamma, \Delta \rangle \notin X$, contrary to our construction.

Now $\Sigma R \Gamma$ by the condition on being a member of X. $\langle \Gamma, \Delta \rangle$ is a Π-partition, for if $\vdash_\Pi \Gamma \supset \Delta$, by the conjunctive closure of Γ, and the disjunctive closure of Δ, we must have $C \in \Gamma$ and $D \in \Delta$ such that $\vdash_\Pi C \supset D$. This means that $C \supset D \in \Sigma$. So, $\Gamma \in W_\Pi$, $A \in \Gamma$, $B \notin \Gamma$ and $\Sigma R \Gamma$ as we desired.

Lemma 2.13 *If $\Pi \in W_\Pi$ then $\langle \Pi, W_\Pi, R, \Vdash_\Pi \rangle$ (where $\Delta \Vdash_\Pi A$ if and only if $A \in \Delta$) is a basic subintuitionistic model.*

Proof: Take $\Delta \in W_\Pi$. If $A \supset B \in \Pi$, then $\vdash_\Pi A \supset B$, so if $A \in \Delta$ then $B \in \Delta$. Thus $\Pi R \Delta$. It is sufficient then, to show that $\Vdash_\Pi$ satisfies the conditions for forcing. Clearly, $\Delta \Vdash_\Pi A \wedge B$ if and only if $\Delta \Vdash_\Pi A$ and $\Delta \Vdash_\Pi B$. And since Δ is prime, $\Delta \Vdash_\Pi A \vee B$ if and only if $\Delta \Vdash_\Pi A$ or $\Delta \Vdash_\Pi B$. If $\Delta \Vdash_\Pi A \supset B$ and $A \in \Gamma$, where $\Delta R \Gamma$, then $B \in \Gamma$ by the definition of R. Conversely, if $\Delta \nVdash_\Pi A \supset B$, then there is a $\Gamma \in W_\Pi$ where $\Delta R \Gamma$, $A \in \Gamma$ and $B \notin \Gamma$, by Lemma 2.12 (note that Γ is nonempty and nonfull, by construction). Thus $\Vdash_\Pi$ satisfies the conditions required, and our structure is a model.

Now we have enough to show that $\Sigma \not\models A$, given that $\Sigma \nvdash A$.

Theorem 2.14 *If $\Sigma \nvdash A$, then $\Sigma \not\models A$.*

Proof: By Corollary 2.10, there is a prime Π-theory $\Pi \supseteq \Sigma$ such that $A \notin \Pi$. The structure $\langle \Pi, W_\Pi, R, \Vdash_\Pi \rangle$ is a basic subintuitionistic model, and $\Pi \nVdash_\Pi A$, as $A \notin \Pi$. So, $\Sigma \not\models A$.

3 Extensions Now that we have the basics, we can add the other conditions to generate logics between our basic subintuitionistic logic and intuitionistic logic. These are *Transitivity*, *Reflexivity*, *Heredity* (also called *Weakening*) and *Absurdity*. It is a simple exercise to find candidate formulas that characterize the added conditions on R and $\Vdash$. My choice is:

Name	Code	Condition	Characteristic Formula
Transitivity	b	For all $u, v, w \in W$, if uRv then if vRw then uRw.	$(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
Reflexivity	w	For all $v \in W$, vRv.	$A \wedge (A \supset B) \supset B$
Heredity (Weakening)	k	If A is a propositional parameter or f, if $v, w \in W$, and vRw then if $v \Vdash A$ then $w \Vdash A$.	$A \supset (B \supset A)$
Absurdity	a	For all $v \in W$, $v \nVdash f$.	$f \supset A$

Table 1

Before we provide soundness and completeness for the additions, we need a result about the condition of heredity.

Lemma 3.1 *If heredity holds for propositional parameters and f, then heredity holds for all formulas, provided that R is transitive.*

Proof: The proof is an induction on the complexity of formulas. Heredity holds for propositional parameters and f. Suppose it holds for formulas less complex than A. If A is of the form $B \wedge C$, then if vRw and $v \Vdash B \wedge C$, we have $v \Vdash B$ and $v \Vdash C$. By heredity, $w \Vdash B$ and $w \Vdash C$. Thus $w \Vdash B \wedge C$, giving us the result. The case for disjunction is similar.

If A is of the form $B \supset C$, then if vRw and $v \Vdash B \supset C$, we have for each u where vRu, if $u \Vdash B$ then $u \Vdash C$. By the transitivity of R, if wRu' then vRu' so $u' \Vdash B$ gives $u' \Vdash C$. So, $y \Vdash B \supset C$ too.

Given this restriction on the application of heredity, we have 12 logics between **SJ** and **J**. For obvious reasons, we will call these the *subintuitionistic logics*. To name them we use the codes for each of the extensions. So the logic formed by adding transitivity, heredity and absurdity to **SJ** is called bka. In a diagram, the logics and their inclusions are

$$
\begin{array}{cccc}
 & & \bullet\ bwka = \mathbf{J} & \\
 & \bullet\ bwa & \bullet\ bka & \bullet\ bwk \\
\bullet\ wa & \bullet\ ba & \bullet\ bw & \bullet\ bk \\
\bullet\ a & \bullet\ w & \bullet\ b & \\
 & \bullet\ \mathbf{SJ} & &
\end{array}
$$

Theorem 3.2 *The class of subintuitionistic models satisfying any of the above conditions is sound and complete for the subintuitionistic logic with the addition of the corresponding characteristic formula (assuming for heredity that we also have transitivity).*

Proof: For soundness, we show that the addition of a condition to the semantics validates the corresponding characteristic formula. For completeness, we show that adding a characteristic formula to a logic ensures that the corresponding condition holds in the canonical models.

Transitivity: First show soundness. Assume that R is transitive. To show that $g \Vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$, assume that $w \Vdash A \supset B$, in order to show that $w \Vdash (B \supset C) \supset (A \supset C)$. If wRv, and $v \Vdash B \supset C$, we wish to show that $v \Vdash A \supset C$. Take z where vRz and $z \Vdash A$. Then as transitivity gives wRz, we have $z \Vdash B$. This gives $z \Vdash C$ as vRz and $v \Vdash B \supset C$. This gives us the result.

For completeness, assume that $\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$. We wish to show that in our canonical structure, R is transitive. Suppose that $\Sigma R\Delta$ and $\Delta R\Gamma$. To show that $\Sigma R\Delta$, take $A \supset B \in \Sigma$, and $A \in \Gamma$. As Δ is nonempty $B \supset B \in \Delta$, and as $\vdash (A \supset B) \supset ((B \supset B) \supset (A \supset B))$, we have $(B \supset B) \supset (A \supset B) \in \Sigma$. $\Sigma R\Delta$ gives $A \supset B \in \Delta$, and $\Delta R\Gamma$, with $A \in \Gamma$ gives $B \in \Gamma$ too.

Reflexivity: Assume that R is reflexive. To show that $g \Vdash A \wedge (A \supset B) \supset B$, assume that $w \Vdash A \wedge (A \supset B)$. As wRw, we see that $w \Vdash B$, as we desired. So we have soundness.

For completeness, assume that $\vdash A \wedge (A \supset B) \supset B$. To show that R is reflexive in the canonical structure, suppose that $A \supset B \in \Gamma$ and $A \in \Gamma$. We then have $A \wedge (A \supset B) \in \Gamma$, giving $B \in \Gamma$ as required.

Heredity: For soundness, assume that R is hereditary (and transitive). To show that $g \Vdash A \supset (B \supset A)$, assume that $w \Vdash A$. Take a v where wRv, and $v \Vdash B$. By the previous lemma $v \Vdash A$. So, $w \Vdash B \supset A$, and we have soundness.

For completeness, assume that $\vdash A \supset (B \supset A)$. To show that $\Vdash_\Pi$ is hereditary in the canonical structure, assume that $A \in \Gamma$ and $\Gamma R\Delta$. As Δ is nonempty, there is a $B \in \Delta$. The assumption gives $B \supset A \in \Gamma$, and so $A \in \Delta$, as desired.

Absurdity: For soundness, assume that f is forced nowhere. To show that $g \Vdash f \supset A$, assume that $w \Vdash f$. This doesn't happen, so (vacuously) $w \Vdash A$.

For completeness, assume that $\vdash f \supset A$. The canonical structure features nonempty, nontrivial prime Π-theories. Take one to be Σ. If $f \in \Sigma$ then $A \in \Sigma$ for any A. This doesn't happen, so $f \notin \Sigma$.

Theorem 3.3 *Each of the 12 subintuitionistic propositional logics are distinct.*

Proof: Four splitting models are required. We first split the pairs $(bwka, bwk)$, (bwa, bw), (bka, bk), (wa, w), (ba, b) and $(a, \mathbf{SJ})$. A model that does this has one world g, such that $g \Vdash f$, and $g \nVdash A$. R is the identity relation. In this model $g \nVdash f \supset A$, falsifying the axiom a. However, it is a model of bwk. So, the first logics of each of our pairs are shown to be strictly stronger than their counterparts that lack a.

Secondly, we split $(bwka, bwa)$, (bka, ba), (bwk, bw) and (bk, b) by giving a model of bwa that refutes k. For this we need two worlds g and v. Take R to be a universal relation on the two elements, and set $g \Vdash A$ and $v \Vdash B$, but no other atomic forcing relations. Clearly this models bwa. However, $g \nVdash B \supset A$, and so $g \nVdash A \supset (B \supset A)$. Each logic that has k is strictly stronger than its counterpart that lacks it.

Thirdly, to split $(bwka, bka)$, (bwk, bk), (bwa, ba), (bw, b), (wa, a) and $(w, \mathbf{SJ})$ consider the model of bka that has two worlds g and v, such that gRg and gRv, but v doesn't access anywhere. Set $g \Vdash A$, $v \Vdash A$, $g \nVdash B$ and $v \nVdash B$. We have $v \Vdash A \wedge (A \supset B)$, so $g \nVdash A \wedge (A \supset B) \supset B$. Logics with w are strictly stronger than their counterparts without it.

Finally, to split (bwa, wa), (bw, w), (ba, a) and $(b, \mathbf{SJ})$ we take a model of bwa with four worlds, g, v_1, v_2, v_3. The accessibility relation relates g to everything, v_1 to itself, v_2 and *not* v_3, v_3 to itself and v_4, and v_4 to itself. If we have $v_4 \Vdash A$, $v_4 \nVdash B$ and $v_4 \nVdash C$, then $v_3 \Vdash B \supset C$, $v_3 \nVdash A \supset C$ and $v_3 \nVdash A \supset B$. We can add $v_2 \nVdash A$, as we don't need to keep k. This gives $v_2 \Vdash A \supset B$, and $v_2 \nVdash (B \supset C) \supset (A \supset C)$. At last this leaves us with $g \nVdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$, which was what we needed to show that logics with b outstrip those without it.

4 Propositional structures The Kripke semantics provides one way of examining our logics. Another is given by the algebraic semantics—or *propositional structures*. These structures stand to our logics as boolean algebras stand to classical logic and Heyting algebras stand to intuitionistic logic. In other words, they model our logics, and provide a compact algebraic representation of propositions, which can be considered to be certain kinds of equivalence classes of sentences.

To assist us the formulation of the propositional structures, it is helpful to introduce a new connective into the language, inspired by its usefulness in the study of relevant logics. It is called 'fusion', written as '$\circ$', and it functions as the residual of the conditional. In other words, it satisfies

$$A \supset (B \supset C) \dashv\vdash A \circ B \supset C$$

In $\mathbf{J}$ fusion collapses into conjunction, but they are distinct in most weaker logics. For the proof-theoretic fusion to match the fusion we can model in our semantics, we will need an extra bookkeeping axiom:

$$(A \circ B) \wedge C \supset A \circ C$$

The reason for this is the modelling condition for fusion. For $A \circ B$ to satisfy the residuation condition, it *must* be modelled as follows:

$$w \Vdash A \circ B \text{ if and only if for some } v \text{ where } vRw \ v \Vdash A \text{ and } w \Vdash B$$

And it is easy to see that this verifies the bookkeeping axiom. To prove soundness and completeness, it is sufficient to note the following result.

Lemma 4.1 *If $\Sigma \in W_\Pi$ then $A \circ B \in \Sigma$ if and only if there is a $\Delta \in W_\Pi$ where $\Delta R \Sigma$, $A \in \Delta$ and $B \in \Sigma$.*

Proof: Firstly, suppose $A \circ B \in \Sigma$. Then $\vdash A \supset (B \supset B)$ gives $\vdash A \circ B \supset B$, and so $B \in \Sigma$. As a first approximation to Δ, take $\Delta_0 = \{C : \vdash_\Pi A \supset C\}$. To prime up Δ_0 define $\Theta_0 = \{C \supset D : C \in \Sigma,\ D \notin \Sigma\}$. Then it is easy to verify that Δ_0 is a Π-theory. To show that $\nvdash_\Pi \Delta_0 \supset \Theta_0$, assume without loss of generality that $(C_1 \supset D_1) \vee (C_2 \supset D_2) \in \bigvee \Theta_0$ where $C_1, C_2 \in \Sigma$ and $D_1, D_2 \notin \Sigma$, and that $\vdash_\Pi A \supset (C_1 \supset D_1) \vee (C_2 \supset D_2)$. It follows that $\vdash_\Pi A \supset (C_1 \wedge C_2 \supset D_1 \vee D_2)$, and so $\vdash_\Pi A \circ (C_1 \wedge C_2) \supset D_1 \vee D_2$. Now $A \circ B \in \Sigma$ and $C_1 \wedge C_2 \in \Sigma$, so our bookkeeping axiom gives $A \circ (C_1 \wedge C_2) \in \Sigma$, so $D_1 \vee D_2 \in \Sigma$ too. But Σ is prime, and $D_1, D_2 \notin \Sigma$, a contradiction.

Thus, by Lemma 2.11 there is a Π-partition $\langle \Delta, \Theta \rangle$ where $\Delta \supseteq \Delta_0$ and $\Theta \supseteq \Theta_0$. So, Δ is a prime Π-theory, containing A, and $\Delta R \Sigma$, for if $C \in \Sigma$, and $D \notin \Sigma$, then $C \supset D \in \Theta$, and so $C \supset D \notin \Delta$. This gives us the required prime Π-theory.

Conversely, if $\Delta R \Sigma$, $A \in \Delta$ and $B \in \Sigma$. Then $\vdash A \supset (B \supset A \circ B)$ gives $B \supset A \circ B \in \Delta$, and $\Delta R \Sigma$ ensures that $A \circ B \in \Sigma$, as we desired.

As well as the new connective, it is helpful to introduce a new propositional constant t that satisfies

$$t \supset A \dashv\vdash A$$

To model t it suffices to ensure that $v \Vdash t$ just in case v forces at least as much as g does. To do this in the presence of k is simple—$v \Vdash t$ for every v. This is sound and complete for the t-rule. Soundness is simple, and completeness follows from the fact that $\vdash t \supset (A \supset t)$ gives $\vdash A \supset t$, ($\vdash t$ is easy to show) and hence t is in each Π-theory.

For logics without k, it is a little more messy, but not significantly so. We need to model containment of worlds with a new binary relation $\sqsubseteq$. It must be reflexive and transitive relation, and forcing must be hereditary with respect to it. This is given by requiring the hereditary condition for atomic propositions and f, and making $\sqsubseteq$ satisfy the condition: $u \sqsubseteq v \Rightarrow (vRw \Rightarrow uRw)$, which gives heredity for conditionals across $\sqsubseteq$. The subset relation in a canonical model is a model for $\sqsubseteq$, as can be easily checked.

Given this, our condition for t is $v \models t$ iff $g \sqsubseteq v$. Soundness and completeness is simple. For soundness note that if $g \models t \supset A$ then $g \models t$ gives $g \models A$. Conversely, if $g \models A$ then for each $v \sqsupseteq g$, $v \models A$ too. So, $g \models t \supset A$. For completeness, note that $t \in \Sigma$ for each $\Sigma \supseteq \Pi$ in a canonical model. Conversely, if $t \in \Sigma$ we need

$\Pi \subseteq \Sigma$. The t rule shows that if $A \in \Pi$ then $t \supset A \in \Pi$. This gives $A \in \Sigma$ as desired.

Given a logic L without fusion or t, the logics L°, L^{t} and $L^{\circ t}$ are given by adding fusion or t or both. We then have the following result:

Theorem 4.2 *L°, L^{t} and $L^{\circ t}$ are conservative extensions of L, for every subintuitionistic logic L.*

Proof: Take any Σ and A where in L, $\Sigma \not\vdash A$. Then there is a Kripke-style countermodel. We have seen that this model can be equipped with $\circ$ and t, without disturbing the evaluation of A. This gives the result.

This gives us enough background to define the propositional structures and prove soundness and completeness for them.

Definition 4.3 A 6-tuple $\langle P, \leq, 0, F, \cdot, \Rightarrow \rangle$ is a *subintuitionistic propositional structure* if and only if it satisfies the following conditions:

- $\langle P, \leq \rangle$ is a *distributive lattice*. The meet and join of this lattice are written as '$\cup$' and '$\cap$' respectively.
- The binary operation $\cdot$ (called *fusion*) *respects the lattice*, in that $b \leq c$ only if $a \cdot b \leq a \cdot c$, and $b \cdot a \leq c \cdot a$. Equivalently, we have $a \cdot (b \cup c) = (a \cdot b) \cup (a \cdot c)$.
- The element 0 is a *left identity* for fusion: i.e. $0 \cdot a = a$ for each a.
- The element 0 is *prime*: i.e. $0 \leq a \vee b$ only if either $0 \leq a$ or $0 \leq b$.
- The binary operation $\Rightarrow$ is a *right residual* for fusion: i.e. $a \cdot b \leq c$ if and only if $a \leq b \Rightarrow c$.
- Fusion satisfies three *structural rules*: $b \cdot a \leq a$, $a \cdot b \leq a \cdot (a \cdot b)$ and $(a \cdot b) \cap c \leq a \cdot c$.
- The element F is an arbitrary member of P.

Examples of subintuitionistic propositional structures are easy to find—the two element boolean algebra is an example, where fusion is identified with lattice join, 0 is the top element, and $\supset$ is given in the usual fashion.

Definition 4.4 Each Kripke model with t, $\langle g, W, R, \sqsubseteq, \Vdash \rangle$ provides a *Kripke* propositional structure in the following way: Take the elements to be the subsets of W that are upwardly closed under $\sqsubseteq$. (Or if the model is hereditary, the R-closed subsets of W.) This forms a distributive lattice under subsethood. The identity 0 is the set $\{w : w \Vdash t\}$. This set is prime, as it is $\{w : g \sqsubseteq w\}$, (or under heredity, it is W), and if $\{w : g \sqsubseteq w\} \subseteq A \cup B$, then either $g \in A$ or $g \in B$, and as A and B are closed under $\sqsubseteq$, $\{w : g \sqsubseteq w\}$ is a subset of either A or B. F is taken to be $\{w : w \Vdash f\}$. The other operations are given as follows:

$$X \cdot Y = \{w : \text{for some } v \text{ where } vRw,\ v \in X \text{ and } w \in Y\}$$
$$X \Rightarrow Y = \{w : \text{for each } v \text{ where } wRv,\ v \in X \text{ only if } v \in Y\}$$

Rule chopping shows that this is a propositional structure.

Formulas can be interpreted in propositional structures in the usual way:

Definition 4.5 An *interpretation* is a map φ from sentences to elements of a propositional structure $\langle P, \leq, 0, F, \cdot, \Rightarrow \rangle$ satisfying the following inductive clauses:

$$\begin{array}{ll} \varphi(A \wedge B) = \varphi(A) \cap \varphi(B) & \varphi(f) = F \\ \varphi(A \vee B) = \varphi(A) \cup \varphi(B) & \varphi(t) = 0 \\ \varphi(A \supset B) = \varphi(A) \Rightarrow \varphi(B) & \varphi(A \circ B) = \varphi(A) \cdot \varphi(B) \end{array}$$

A sentence A is *true under* φ if and only if $0 \leq \varphi(A)$.

Lemma 4.6 *The propositional subintuitionistic logic,* $\vdash A$ *if and only if* A *is true under each interpretation in each propositional structure.* $\Sigma \vdash A$ *if and only if for each propositional structure and interpretation that makes each element of* Σ *true,* A *is also true.*

Proof: One half is the simple matter of showing that each axiom is true under each interpretation, and that the rules preserve truth. That is a tedious exercise left to the reader.

For the other half, take Σ and A such that $\Sigma \not\vdash A$. Then there is a Kripke model $\langle g, W, R, \Vdash\rangle$ in which $g \Vdash B$ for each $B \in \Sigma$, but $g \not\Vdash A$. The Kripke propositional structure given by the model can be equipped with an evaluation as follows:

$$\varphi(A) = \{w : w \Vdash A\}$$

As a result, we have $0 \leq \varphi(B)$ for each $B \in \Sigma$, but $0 \not\leq \varphi(A)$, as desired.

Expanding the result to deal with extensions of the basic subintuitionistic logic is no more difficult. Each extension of subintuitionistic logic is sound and complete with respect to algebraic semantics with the addition of the corresponding condition:

Name	**Condition**
Transitivity	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Reflexivity	$a \leq a \cdot a$
Heredity	$a \cdot b \leq a$
Absurdity	$F \leq a$

Table 2

The proof of this fact is left to the eager reader.

5 Quantification Providing an adequate modelling of quantification is an order of magnitude more difficult than the propositional case. In fact, the only results I have to report are of a negative nature. Be that as it may, we will add to our language the universal and existential quantifiers $\forall$ and $\exists$, with a denumerable set $\{x, y, z \ldots\}$ of variables, and a liberal supply of constants $\{a, b, c \ldots\}$. Exactly what quantificational axioms we ought to add is a moot point. To keep below **J**, we ought to add no more than the following:

$$\begin{array}{ll} \vdash \forall x A(x) \supset A(a) & \forall x(A \supset B(x)) \vdash A \supset \forall x B(x) \\ & \quad \text{where } A \text{ does not contain } x. \\ \vdash A(a) \supset \exists x A(x) & \forall x(A(x) \supset B) \vdash \exists x A(x) \supset B \\ & \quad \text{where } B \text{ does not contain } x. \end{array}$$

However, picking this logic out with the semantics proves difficult.

If we use the standard intuitionistic clause: that $w \Vdash \forall x A(x)$ just when for each v where wRv and each d in the domain of v, $v \models A(d)$, the models without reflexivity will no longer validate $\forall x A(x) \supset A(a)$. This is quite drastic. Of course, as gRg, we have universal instantiation in *rule* form, so it might be thought that we can live with that. But in fact, of our axioms and rules given above, only $\vdash A(a) \supset \exists x A(x)$ survives under this interpretation, and even plausible weakenings of the rules, such as $\forall x(A \supset B(x)), A \vdash \forall x B(x)$ (for x not free in A) fail under this interpretation. It is not clear that enough principles hold so that anything like the normal quantificational completeness proof can be made to work. So this is an open problem.

A standard *this worldly* interpretation of the quantifiers, in which $w \Vdash \forall x A(x)$ just when for each d in the domain of w, $w \models A(d)$ (and the dual clause for the existential quantifier) will validate $\forall x(F(x) \vee G(x)) \supset \forall x F(x) \vee \exists x G(x)$, which is not a theorem of **J**, and so, will not be a theorem of any properly *subintuitionistic* quantificational logic.

Worse than that, the $\supset$ clauses seem to wreak havoc with our rules for quantifiers, unless the domains are kept constant. Specifically, we have a countermodel to $\forall x(A \supset B(x)) \vdash A \supset \forall x B(x)$, by taking two worlds, g and v, such that $v \Vdash A$ and $v \nVdash B(a)$, so $v \nVdash \forall x B(x)$ and $g \nVdash A \supset \forall x B(x)$. This need not give $g \Vdash \forall x(A \supset B(x))$ unless a is in the domain of g.

Therefore, the constant domain, this-worldly approach to quantification seems to be the interesting one. However, proving completeness is not easy. Firstly, we need to give a formal characterization of the semantics.

Definition 5.1 A *quantified basic subintuitionistic model* for a language L is a structure $\langle g, W, R, D, E, \Vdash\rangle$, where g, W, R are as before, Q is a reflexive and transitive relation on W, and D is a *domain* of quantification. E defines the extensions of the predicates—if A is an n-ary predicate, then $E(A, a_1, \ldots, a_n)$ is the set of worlds in which $A(a_1, \ldots, a_n)$ comes out as true. The forcing relation is extended to operate on *D-sentences*. Forcing satisfies the requirements from the propositional semantics, in addition to the following conditions:

$$\begin{array}{rcl} w \Vdash A(a_1, \ldots, a_n) & \text{if and only if} & w \in E(A, a_1, \ldots, a_n) \\ w \Vdash \exists x A(x) & \text{if and only if} & w \Vdash A(d) \text{ for some } d \in D \\ w \Vdash \forall x A(x) & \text{if and only if} & w \Vdash A(d) \text{ for each } d \in D \end{array}$$

Soundness for this semantics (with respect to the axioms and rules given above, together with the confinement principle) is rather simple. Suppose $g \Vdash \forall x(A \supset B(x))$, and show that $g \Vdash A \supset \forall x B(x)$. Take a v where $v \Vdash A$ and $v \nVdash \forall x B(x)$. Then there is an a where $v \nVdash B(a)$, which means $g \nVdash A \supset B(a)$, which contradicts what we have seen. The other rule and the axioms are as easy to prove.

Completeness is not as easy. We need to show that there is a constant domain counterexample to any invalid inference. Specifically, the method to use would be this: Given an invalid inference $\Sigma \nvdash A$ to refute, we beef Σ up to Π, which is a prime Π-theory of D-sentences for some domain D, which doesn't contain A, and is $\forall$- and $\exists$-complete (this means, if $B(a) \in \Pi$ for each $a \in D$ then $\forall x B(x) \in \Pi$, and if $\exists x B(x) \in \Pi$, then $B(a) \in \Pi$ for some $a \in D$). This part is not difficult, and the standard methods will work. What is more difficult is showing that the collection of prime, $\forall$- and $\exists$-complete Π-theories *in this domain* gives us a quantified subintuitionistic model in the obvious way. The difficulty is showing that if $B \supset C \notin \Gamma$, there is a $\forall$- and $\exists$-complete Π-theories Δ such that $\Gamma R \Delta$, $B \in \Gamma$ and $C \notin \Delta$. The standard method involves adding *more* constants to ensure $\forall$- and $\exists$-completeness. No method for proving this in such weak logics has come to light (as far as I can tell), so completeness remains an open problem.*

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