Ross Brady. *Universal Logic*, CSLI Leture Notes, Number 109. CSLI Publications, Stanford, 2006. xii + 346 pp.

This solid volume with the ominous black cover and eerie glowing disc lettered with inscrutable strings of characters such as "$\mathrm{DN^dQ}$," "$\mathrm{LSDJ^d}$," and "$\mathrm{L_2LDJQ^{\pm}}$" is the fruit of over 30 years of Ross Brady's logical labours. And it is worth the wait.

Since the early 1970s, Brady has been interested in strong theories of sets, in which the comprehension axiom, to the effect that every open sentence $\phi(x)$ (in the language of first order logic with the binary predicate '$\in$' for membership) determines some set. That is, theories in which the comprehension axiom (CA)

$$(\exists y)(\forall x)(x \in y \leftrightarrow \phi(x))$$

holds completely unrestrictedly. This means that according to such theories, there is a universal set (take $\phi(x)$ to be $x = x$, or perhaps $(\exists z)(x \in z)$ if we have no identity in the language) and each set a has a complement (take $\phi(x)$ to be $x \notin a$). This is not a cumulative hierarchy of sets familiar to us in set theories in the vicinity of ZF. This is Frege's (and our) naïve notion of sets, where every "property" determines an extension. This causes *trouble* as is well known from the paradoxes of Burali-Forti, Russell and Curry. We can form troublesome extensions: (CA) assures us that there is is a set of all non-selfmembered sets (take $\phi(x)$ to be $x \notin x$). Is such a set a member of itself or not? (CA) tells us that it is a member of itself if and only if it is not. Now, for most of us, this is enough to reject (CA). Not so for some. For some, the problem lies not with this axiom, but with the underlying logic. This is a minority view, but it has always had its place in the literature [1, 2, 3, 5]. The motivation for this minority tradition is not hard to find. The paradoxes are genuinely puzzling and it seems that *every* option concerning them is worth exploring. We tend to treat proper classes as governed by the comprehension axiom in its unrestricted form, and in mathematical practice it is by no means clear that class talk is eliminable as a façon de parler. But once we treat classes as genuine members of the mathematical universe, they should be treated as members of classes in the same way as every other entity. So lies the way toward accepting (CA) *despite* the logical difficulties involved.

Those logical difficulties are not for the faint hearted. Fashioning one's logic in order to render a theory of comprehension non-trivial in the face of the paradoxes is not a straightforward matter. Or, rather, it is hard to do this and have a theory in which one can reconstruct mathematical reasoning. It is well-understood that moving to a three-valued logic such as Kleene's (in which we admit a 'gap' value, which is undesignated) or Priest's (with a 'glut' value, which is designated) renders the theory non-trivial. In the case of a Kleene-based theory, if r is the Russell class, $r \in r$ is neither true nor false (it takes the gap value) while with Priest, it is both true and false. However, I speak too swiftly, for this is not literally correct in the case of Kleene or Priest — for in these simple three-valued logics, the comprehension axiom is unstatable. These logics have no conditional or biconditional with which one can state (CA) and prove consequences form it. If we were to define $A \supset B$ as $\neg A \vee B$ in the usual manner, then in the case of Kleene's logic, $A \supset A$ is not valid (let A take the gap value), and in the case of Priest's, *modus ponens* for $\supset$ has counterexamples

($A \supset B$ and A may both be designated while B is not — let A be both true and false, and B be simply false). So, to have a recognisable conditional with which to represent the comprehension *axiom*, we must work harder. Curry's paradox (take for $\phi(x)$, $x \in x \to C$ where C is some statement we reject) shows us that any conditional present in a theory containing (CA) must reject inference principles such as *contraction* (from $p \to (p \to q)$ to infer $p \to q$) and an axiom form of *modus ponens* ($((p \to q) \wedge p) \to q$). Such logics are not easy to find. One such logic is Łukasiewicz's infinitely valued logic, but it is now known that although (CA) is consistent in this logic, adding extensionality renders the theory inconsistent [6].

Where do we find Brady's work in this tradition? He has set himself a simple task: to craft a logical system appropriate for (CA) with extensionality, to motivate the logic, to *prove* that (CA) with extensionality may live happily with in it, and then to go on to show that one can recover much of classical set theoretical reasoning *despite* the weakness of the underlying logic. The task may be simple, but the accomplishment of it is no mean feat—it is to single-handedly paint a comprehensive position diametrically opposed to the orthodoxy of the bulk of the last century's work on set theory. And somehow, in this book, Brady manages to just about pull it all off.

The book divides into ten chapters. The first four concern Brady's target logical system, DJ^dQ, a quantified *depth-relevant* logic. Brady is in the tradition in *relevant* logic stemming from the work of Ackermann, Anderson and Belnap. As an Australian, Brady's immediate ancestors are Robert K. Meyer, and especially, Richard Sylvan (formerly, Richard Routley). With Sylvan, Brady shows a deep and settled preference for very *weak* logical systems: in DJ^dQ, the conditional connective '$\to$' expresses a notion of *entailment*, and not mere contingent conditionality. Relevance considerations rule out claims such as $p \to (q \to q)$, since whether or not p is the case has nothing to do with the truth of $q \to q$, in general. The conditional $p \to (q \to q)$ is *irrelevant*, as there need be no content shared between the antecedent p and the consequent $q \to q$. Brady's concerns are not merely with *relevance*, but with a stricter condition, *depth* relevance. The inference from $p \to (p \to q)$ to $p \to q$ cannot be faulted on grounds of everyday irrelevance: the 'subject matter' is shared between the premise and the conclusion. However, Brady notices a difference: the premise contains q to a different *depth* than the conclusion. The q in $p \to (p \to q)$ appears at depth 2, since it is under *two* conditionals. In the consequent, it occurs at a different depth. In the axiom of *modus ponens*: $((p \to q) \wedge p) \to q$, notice that q occurs in the antecedent at depth 1, while in the consequent, it is at the depth 0—the surface. So, it fails to be *depth relevant*. In fact, depth relevance considerations rule out all the principles known to cause havoc in the face of the paradoxes. If we look out for depth relevance, when we find it, we have the bonus of paradox tolerance. What does the resulting logic look like? Not all implicational principles fail the criteria of depth relevance. Principles such as

$$(p \to q) \wedge (q \to r) \;\to\; p \to r$$

$$(p \to q) \wedge (p \to r) \;\to\; p \to (q \wedge r) \qquad (p \to r) \wedge (q \to r) \;\to\; (p \vee q) \to r$$

pass the test, and these principles form the basis of Brady's depth relevant logic

DJ^dQ: it is a logic in the relevant family, extending distributive lattice principles for $\wedge$ and $\vee$ with a de Morgan negation, with the aforementioned entailment connective '$\rightarrow$', and with quantifiers with standard properties. In the first four chapters, Brady motivates and presents the logic, not with the usual ternary relational semantics, familiar in the model theory of relevant logics, but with Brady's own favoured 'content' semantics, intended to cash out the way in which the *content* of the consequent of a true entailment is to be found in the content of the antecedent, and to motivate DJ^dQ as the one true logic of 'meaning containment.' This motivating account is, to my mind, the weakest aspect of the initial chapters of *Universal Logic*. The argument that considerations of 'meaning containment' pin down just *this* relevant logic is weak. It depends on unreflective intuitions to the effect this meaning is contained in that one, and that this other meaning is not contained in that. In the absence of at least the hint of a theory of what a meaning might *be*, it is hard to judge this kind of argument as anything more than a sketch of how one account of meaning containment might go. As a decisive motivation of *this* logic rather than any other, it is lacking. I found most unsatisfactory the short argument in footnote 12 on page 18, intended to show that the distribution principle $(p \wedge (q \vee r) \rightarrow (p \vee q) \wedge (p \vee r))$ is motivated on grounds of meaning containment to be quite unhelpful. It is one thing to simply prefer logics with distribution over those without it, and even to argue that it is necessary in developing a theory of sets. It is another to think that you have a good *argument* ruling out non-distributive logics as an account of meaning containment. Brady doesn't give us a good argument, but he does give us an explanation of why he prefers to take the path that he does. That is enough to motivate the reader to at least consider the journey down that road.

These quibbles are merely a minor matter — the highlight of this book is not the motivation of DJ^dQ, and neither is it the detailed exposition of the properties of this logic. The highlight is found in the middle of the book, and it is the demonstration that in DJ^dQ, (CA) with extensionality is consistent. Brady's argument is a tour de force. In fact, Brady proves more. He shows that a *generalised* comprehension axiom is consistent. (GCA) is the stronger axiom

$$(\exists y)(\forall x)(x \in y \leftrightarrow \phi(x, y))$$

where now ϕ can contain not only x free but also the variable y. This axiom has nowhere near the intuitive support as does (CA), but the fact that Brady's model construction allows us to interpret (GCA) as well as (CA) is striking. The construction is the usual kind of fixed point matter, introduced into this literature by Kripke [4]. We construct an interpretation in stages, defining the status of $a \in \{x : \phi(x)\}$ at stage $\alpha + 1$ in terms of the status $\phi(a)$ at stage α.[1] We must fiddle at limit stages, and we must fiddle even more to make sure that we interpret the conditional aright, since it has no straightforward truth-functional interpretation. With those finer points dealt with, we find a fixed point, at which the successor stage delivers no more an no less than what we already had, and this is our model — a model in which $a \in \{x : \phi(x)\}$

[1] For (GCA), we define the status of $a \in \{xy : \phi(x, y)\}$ at α in terms of $\phi(a, \{xy : \phi(x, y)\})$. This is no trouble than the case for (CA).

has the same semantic value as $\phi(x)$. Fixed points in the construction of a model allow us to model a theory in which every propositional function has a fixed point.[2]

Brady doesn't stop with the model construction. He defends a set-class distinction, in which (GCA) defines *classes*, and then we take *sets* to satisfy the traditional cumulative conception. The empty set is the empty class, and membership in this is unproblematically consistent and complete (everything is *not* a member). This means that classical reasoning applies here: we have no paradoxical funny business concerning $\emptyset$. It is plausible to think that classical reasoning applies also to those things constructed from $\emptyset$ — at the very least the hereditarily finite sets, but perhaps also to all sets of the cumulative hierarchy. In this way, Brady paints a picture of a well-behaved classical domain of the cumulative hierarchy living inside the wilds of the non-classically structured universe of classes. One can have ones cake ((GCA) and its consequences), and follow along with traditional ZFC set theory too.

What are we to make of this program, and the way that Brady has executed it? Without a doubt, it is the most well-worked out non-classical theory of general proper classes. This very difficult territory to navigate and Brady has the patience and eye for detail to carve out amazing theorems from unpromising material. The non-triviality proof is a marvel. It is a mindboggling marvel that is not easy to take in at a single glance, but it is a marvel nonetheless. The result is a theory with a unique take on foundational issues in set theory, with something to say both of the naïve conception which takes (CA) as the ground of the concept of collecting this together in a class, and the *classical* concept in the cumulative hierarchy. Nothing else in this field has quite the scope of Brady's project.

What my caveats about Brady's execution of this program? It is merely in the motivation for the position. The motivating ideas are not going to be to everyone's taste. They are not to mine. I agree that relevance is important and depth-relevance is intriguing. However, in the absence of a genuine theory of content, gestures concerning concept containment can only hint at why DJ^dQ is the right logic for content containment, and a lot more should be done to connect this concern to the concept of logical consequence if we are to say that this is the One True Logic, with Universal application in the way that Brady takes it to be.

Regardless. you don't need *those* ideas to motivate the book. DJ^dQ has defence enough when you see that with this contraption you can build the mighty and intriguing edifice of Brady's set/class theory. That is more than enough reason to be interested. All those interested in tackling the paradoxes at the limits of class theory would do well to invest some time and energy grappling with the results of this fine work.

References

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[2]The fixed point for negation is the statement $r \in r$, where r is $\{x : x \notin x\}$, since we have $r \in r \leftrightarrow \neg(r \in r)$. This generalises to any propositional function: we have $\{x : F(x \in x)\} \in \{x : F(x \in x)\} \leftrightarrow F(\{x : F(x \in x)\} \in \{x : F(x \in x)\})$ for any propositional function F.

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