Just what is *Full-Blooded Platonism?*

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Abstract: Mark Balaguer has, in his *Platonism and Anti-Platonism in Mathematics* [Balaguer, 1998], given us an intriguing new brand of platonism, which he calls, plenitudinous platonism, or more colourfully, *full-blooded platonism*. In this paper, I argue that Balaguer's attempts to characterise full-blooded platonism fail. They are either too strong, with untoward consequences we all reject, or too weak, not providing a distinctive brand of platonism strong enough to do the work Balaguer requires of it.

Mark Balaguer's *Platonism and Anti-Platonism in Mathematics* [Balaguer, 1998] is an exciting and original addition to the recent literature in the metaphysics of mathematics. In it, Balaguer propounds an intriguing thesis: he argues that both platonism of a very particular sort, and fictionalism, are adequate metaphysical analyses of mathematics, and furthermore, that there is no telling between platonism and fictionalism about mathematical objects. There is literally *no fact of the matter* deciding between platonism and fictionalism.

I do not intend to discuss the major strand of Balaguer's project. Instead, I will focus on the first part of the project — Balaguer's exposition and defence of full-blooded platonism (henceforth 'FBP'). I will show that Balaguer has not defined a coherent variety of platonism, and that more work must be done to explain what FBP actually *is*.

At a first glance, FBP is relatively simple to state. It is the thesis that any mathematical object which *can* exist, *does* exist. Stating the thesis like this seems to commit one to two questionable things. First, it appears to appeal to some kind of quantification over the non-existent, for if the quantifier "any" merely quantifies over what *is* then we all, platonist and non-platonist together, can agree with the thesis. Any (existing) mathematical object which *can* exist, does. Second, the claim appears to commit one to *de re* modality concerning these possibilia. We ask of each x whether it possibly exists. We quantify into this modal context. Now, these are not significant difficulties if you wish to embrace Meinongian quantification and *de re* modality applying to possibilia. They *are* difficulties for Balaguer, for he wishes to embrace neither of these. For him, the modality in question is *logical* possibility, and for this, the bearer of possibility and necessity is the sentence, and not objects, or sentences with free first-order variables.

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So, if FBP is to be expressed clearly and formally, it must be expressed in some other way.

1 Not too Heavy?

Our first port of call in discussing FBP will be Balaguer's attempts in his introduction to formalise it. Balaguer is hesitant to formalise FBP, for two reasons. "First, I'm inclined to doubt that there is any really adequate way to formalise FBP, and second, I think that in any event, it is a mistake to think of FBP as a formal theory. FBP is, first and foremost an informal philosophy of mathematics ... " (page 6).[1] That caveat noted, we will consider Balaguer's formalisation, as he takes it to be useful in clarifying what FBP entails. After all, Balaguer takes his attempts at formalisations of FBP to be *true*.[2] On page 6, he formalises a *part* of FBP as follows:

$$(\forall Y)\big(\Diamond(\exists x)(Mx \wedge Yx) \supset (\exists x)(Mx \wedge Yx)\big) \tag{1}$$

This is a second-order modal sentence, where 'Mx' means 'x is a mathematical object.'[3] This formalisation clearly expresses the idea that any mathematical object (any mathematical bearer of Y) which could exist does exist. As so, it is a good candidate to express at least a *part* of FBP.[4] Note too that it does not objectually quantify into the scope of the modality, and it does not use an existence predicate. Instead, the second-order quantification provides the quantification under $\Diamond$, and the existential quantification inside $\Diamond$ does the work otherwise done by the existence predicate. We seem to be in good order for avoiding the problematic commitments of the initial attempt at understanding FBP.

Unfortunately, sentence (1) is almost certainly false. It entails that all mathematical objects are *modally fragile*. That is, any mathematical objects which do exist, exist in this world alone. They would not exist were anything different at all. Formally speaking, (1) entails

$$p \supset {\sim}\Diamond(\exists x)(Mx \wedge {\sim}p) \tag{2}$$

In English, if p is true then it is impossible that there both be a mathematical object and ${\sim}p$ be true.

The derivation of (2) from (1) is straightforward. First substitute ${\sim}p$ for Yx in (1). (This is an acceptable substitution instance of the second-order

[1] All quotations without a bibliography reference in this paper will be from *Platonism and Anti-Platonism in Mathematics* [Balaguer, 1998].

[2] But then again, Balaguer has stated in correspondence "as you noted in the paper, I don't think there is a really adequate formal statement of the view, and so I saw the whole passage as something of an aside." That said, if a philosophical position like FBP *cannot* be adequately formalised, we must ask where the problem lies. Perhaps it is a problem with our tools for formalisation, but perhaps the shortcoming lies elsewhere.

[3] Note too that it contains a modal operator within the scope of a second-order quantifier, and this raises the question of how such quantification is to be handled. Balaguer does not specify the behaviour of the quantifier. All that I require in what follows is that the standard substitutional rules apply, so my criticism will apply independently of any specification.

[4] Balaguer notes that alone, (1) is not sufficient to express FBP, since it does not entail the existence of any mathematical objects at all. He considers adding to it a conjunct of the form $\Diamond(\exists x)Mx$. We will not pursue this here, for we will see that (1) is too *strong*, not too weak.

quantifier. If substituting 'predicates' of order zero bothers you, substitute $(x = x) \wedge {\sim}p$ for Yx instead to get exactly the same effect.) You get the following instance of (1).

$$\Diamond(\exists x)(Mx \wedge {\sim}p) \supset (\exists x)(Mx \wedge {\sim}p)$$

A contraposition gives us

$${\sim}(\exists x)(Mx \wedge {\sim}p) \supset {\sim}\Diamond(\exists x)(Mx \wedge {\sim}p)$$

which entails (2), since p entails ${\sim}(\exists x)(Mx \wedge {\sim}p)$. So, (2) follows from (1).

But (2) is devastating. Take p to be the truth "Queensland won the Sheffield Shield in 1995" and detach p and we get the following consequence:

> It's not possible that there both be any mathematical objects and Queensland not win the Sheffield Shield in 1995.

and since Balaguer takes $\Diamond$ to be *logical* possibility, we get the rather strong consequence:

> It is *logically impossible* that Queensland lose the Sheffield Shield in 1995 and mathematical objects exist.

This cannot be correct. If mathematical objects *could* exist, they do not depend for their existence on each and every contingency. It follows that FBP cannot come down to (1). Balaguer's attempt at formalising FBP fails.[5]

However, not all is lost. There was an obvious problem in formalising FBP with such a second-order quantification as found in (1). $(\exists x)(Mx \wedge Yx)$ only talks of a different sort mathematical object if the property picked out by Y is a *mathematical* property. To be sure, we can agree that there might be a mathematical object such that Queensland lost the Sheffield Shield in 1995. It is not incumbent on any kind of platonism, no matter how full-blooded, to tell us that it follows that Queensland did lose the Sheffield Shield in 1995. Perhaps we can fix the problem with (1) as follows. It is not that any possible mathematical object satisfying any Y exists, but that any possible mathematical object satisfying any *mathematical property* Y must exist. We get, then the following modification,

$$(\forall Y)\big(\mathrm{Math}(Y) \wedge \Diamond(\exists x)(Mx \wedge Yx) \supset (\exists x)(Mx \wedge Yx)\big) \tag{3}$$

where 'Math' is the *third*-order predicate meaning 'is a mathematical property.' Perhaps this will salvage FBP from ruinous consequences. At least, it blocks the inference involving arbitrary propositions such as those involving Queensland's cricket victories. Unfortunately, any victory is short lived. I take it as unproblematic that $x = 2 \wedge x^{\aleph_0} = \aleph_1$ and $x = 2 \wedge x^{\aleph_0} \neq$

[5] You must be very careful in interpreting this result. (1) and (2) leave unscathed the existence of mathematical objects like 3088, π and $\aleph_1$ in other worlds — provided that in those worlds these objects are not *mathematical*. For all (1) and (2) say, the same objects might exist in every world — it is just that the extension of M is empty in any world other than the actual one. So, had Queensland lost the Sheffield Shield in 1995, the number 3088 may still have existed, but had it existed, it wouldn't have been a *number*, or any other kind of mathematical object. This, too, is an upalatable consequence of (1).

$\aleph_1$ are both mathematical predicates (taken as ascribing a property to x). Furthermore, I take it that for the platonist

$$\Diamond(\exists x)(Mx \ \wedge \ x = 2 \ \wedge \ x^{\aleph_0} = \aleph_1)$$

is true, as it asserts that it is *logically* possible that 2 is a mathematical object and that $2^{\aleph_0} = \aleph_1$, and the continuum hypothesis seems to be a *logical* possibility. Similarly,

$$\Diamond(\exists x)(Mx \ \wedge \ x = 2 \ \wedge \ x^{\aleph_0} \neq \aleph_1)$$

is true, provided that it is logically possible that the continuum hypothesis is *false*. But given these two truths, and given (3) we get

$$2^{\aleph_0} = \aleph_1 \ \wedge \ 2^{\aleph_0} \neq \aleph_1$$

a contradiction. So it appears that (3) will not stand either. If two conflicting mathematical propositions are both possible we have a contradiction.

Perhaps I have been too hasty. The choice of the continuum hypothesis as a logically 'open' issue is perhaps unfortunate, for some have taken CH to be decided by logic alone — by unrestricted second-order logic [Shapiro, 1991]. However, there is no such exit from our problem. We can choose *other* mathematical sentence undecided by logic alone. Large cardinal axioms prove to be a good candidate. Both of the following propositions seem to be true

$$\Diamond(\exists x)(Mx \ \wedge \ x = \omega \ \wedge \ (\exists y > x)(y \text{ is an inaccessible}))$$

$$\Diamond(\exists x)(Mx \ \wedge \ x = \omega \ \wedge \ {\sim}(\exists y > x)(y \text{ is an inaccessible}))$$

since the 'height' of the sequence of cardinals is left open by logic alone. But given this, (3) allows us to deduce yet another contradiction:

$$(\exists y > \omega)(y \text{ is an inaccessible}) \ \wedge \ {\sim}(\exists y > \omega)(y \text{ is an inaccessible})$$

So, for (3) to prove consistent, all false mathematical existence claims must be logically impossible, in the following sense. We may contrapose (3) to deduce:

$$(\forall Y)\big(\text{Math}(Y) \wedge {\sim}(\exists x)(Mx \wedge Yx) \supset \Box{\sim}(\exists x)(Mx \wedge Yx)\big) \quad (4)$$

Any true claim of the form ${\sim}(\exists x)(Mx \wedge Yx)$, where Y is a mathematical property, is necessary. It follows then that (3) is consistent only at the price of logicism about negative existence claims.[6] As we saw in the cases of CH and large cardinal axioms, it is simple to construct negative existence claims equivalent to other, more interesting mathematical claims. The further this goes, the further logicism spreads. It follows, then, that (3) does not succeed as a plausible theory, at least for the defender of FBP, who (like Balaguer, see page 74) rejects logicism. Instead of pursuing further modifications of formal accounts such as (1) and (3), we will consider the *informal* expositions Balaguer gives of FBP to see whether these are any more promising.

[6] As a referee noted at this point, most varieties of logicism make FBP trivially true anyway: Frege, for example, took it to be necessary that any existing mathematical objects exist necessarily. A kind of FBP follows immediately, but this is, in itself, no guide to what mathematical objects exist.

2 Not too Light?

One way to clarify a philosophical thesis is to examine how it is *used*. One important way that Balaguer uses FBP is to defend a new epistemology for mathematics. As Beall writes,

> ... FBP is supposed to solve the [epistemological] problem by expanding platonic heaven to such a degree that one's cognitive faculties can't miss it (as it were). If you're having trouble hitting the target, then just make your target bigger! [Beall, 1999]

Let's see how FBP actually helps in this project of expanding the target. Balaguer writes

> ... if all of the mathematical objects that possibly *could* exist actually *do* exist, as FBP dictates, then all (consistent) mathematical descriptions and singular terms will refer, and any (consistent) representation of a mathematical object that someone could construct will be an *accurate* representation of an actually existing mathematical object. (page 43)

Here it is clear that we have a very different inference from that present in other statements of FBP. In the original statements of FBP the inference is from having a *logically possible object* to having an actual object. In this use of FBP we need something rather different. We need to infer from a *consistent description* or *consistent representation* to an actual object. In fact, a cursory examination of Balaguer's use of FBP in the arguments for the success of platonism against the epistemological argument appears to reveal that this brand of full-bloodedness is all he ever needs. To continue the analogy of target shooting, aiming in a particular direction seems to correspond with having a (consistent) *theory* or a (consistent) *description* of a mathematical object. What is required is that the target extends out to that direction. We do not exercise our faculties by grasping *possible* mathematical items. We exercise our faculties by constructing theories and describing what we hope to be mathematical objects. Plenitudinous platonism is simply the thesis then that there are such objects corresponding to these theories or descriptions.

Let's see how this works in practice. A mathematician asserts that there is an abelian group of order 4. That is, there is a set of four objects, equipped with a two-place operation, satisfying the abelian group axioms. FBP tells us that if this list of conditions is *consistent*, then there is a mathematical object with this structure. Of course, there is no *single* object picked out by this description, as there are many abelian groups of order 4. The integers mod 4 under addition ($\mathbb{Z}_4$), the set of rotational symmetries of a square, or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Now some of these structures are isomorphic to others, and others are not. There is no question, even in the case of isomorphism (between $\mathbb{Z}_4$ and the set of rotational symmetries of a square) of identity between the two mathematical objects, for a platonist. $\mathbb{Z}_4$ is constituted by the collection of numbers $\{0, 1, 2, 3\}$ under modular addition. The set of rotational symmetries of a square is constituted by the set of possible rotations of a square. Mathematical theories may, obviously, be *multiply realised*. In fact, mathematical theories are *essentially* multiply realised, since for any model of a theory it is simple to construct isomorphic models which do equally well at modelling the theory.

This talk of *modelling* gives us one way to make precise Balaguer's use of FBP. The *weak* version of FBP is the following claim:

Any consistent mathematical theory has a *model*. (5)

The notion of 'model' here is the Tarskian notion of satisfaction holding between a model and the sentences of some language. This gives a precise meaning to Balaguer's claims, and there is no chance of any inconsistency following from this form of platonism as we had to attempt to avoid in the first section of this paper. For the inference is not from consistency of some theory to the *truth* of the theory, but from consistency of a mathematical theory to its having a *model*. This certainly delivers a kind of *platonism*, for given a consistent theory (say, a set theory such as ZF) the theory has a model. There is a collection of objects, equipped with a two-place relation, which satisfies the conditions laid down by the theory. We have a genuine *ontology* arising out of merely consistent mathematical theory. The only degree of *weakness* in this form of platonism results from our unwillingness to agree that the mathematical theory *truly describes* the model. In all other respects, we have the full-blooded platonism Balaguer wishes.

However, we can note that this platonism comes at a much lower price. For this 'weak FBP´ (5) is not particularly full-blooded at all. All we require is a completeness theorem for the notion of consistency in use in the statement of FBP. For most of us, this completeness theorem is available. Defenders of first-order logic have completeness with respect to Tarski's models of predicate logic. Some proponents of full second-order logic *define* consistency of a theory *as* the possession of a model by that theory [Shapiro, 1991], though of course this notion is not recursively definable by any proof theory. Furthermore, (5) is a consequence of the the completeness theorem for consistency:

Any consistent theory has a model. (6)

There is nothing especially *mathematical* about the theories in question here. We have reduced this weak form of FBP to simple facts of *logic*.

It follows that this 'platonism' is much less full-blooded than Balaguer's, because almost *every* platonist counts agrees that every consistent mathematical theory has a model. In particular, any platonist who satisfies these conditions can agree with (5).

- Consistency is coextensive with consistency in first-order logic, or with some other logic for which the downward Löwenheim–Skolem theorem holds.
- There are infinitely many things.

If the downward Löwenheim–Skolem theorem holds and then any consistent theory has a countable model, and this can be provided by the infinitely many things which are at hand. So, platonists like Quine for whom logic is first-order, and for whom the natural numbers are metaphysically respectable, can agree with (5). *This* form of platonism is certainly too weak to count as plenitudinous.[7]

3 Just Right?

We have seen that (5) is too weak to count as *plenitudinous*, as any platonist with the downward Löwenheim–Skolem theorem and an ω-sequence can agree. The culprit is the use of *modelling* in its statement. For plenitudinous platonism to do what Balaguer desires, we need not that consistent mathematical theories be *modelled* by some structure but that

$$\text{Consistent mathematical theories } \textit{truly describe} \text{ some structure.} \qquad (7)$$

But given this stronger notion, we must face the same problem from the first section of this paper.[8] I will complete the story by showing how (7) is false. The first dilemma we have already seen. The sentence:

$$(\exists x)(x = 2 \wedge \text{Queensland lost the Sheffield Shield in 1995})$$

is consistent, it is a description of a mathematical object, and it does not truly describe *anything*. There is no object satisfying this description, because the (non-mathematical) world does not comply. It *could* have complied, had things turned out differently (so the description is certainly consistent) yet there is no object truly described by this description.

So, we are left with attempting to weaken (7) so that theories making reference to non-mathematical objects do not interfere. But *this* calls an important part of Balaguer's argument into question. He appeals to a theory (an intuitive theory, but a theory nonetheless): the *full conception of natural numbers* FCNN which includes not only the standard mathematical truths about the natural numbers, but also such claims as "three is not a chair." (page 80). This has non-mathematical vocabulary, and is certainly not a part of a *mathematical theory* in the traditional sense. If FBP applies to the theory FCNN, we must assume that the theories in the scope of FBP contain more than purely mathematical vocabulary. But Balaguer has given us no assurance that he can avoid falsehood in the way sketched above. Not all consistent mathematical theories are true of some structure, but some are. Which theories are? If FBP does not apply to theories such as FCNN then we need some *other* reason to conclude that fcnn truly describes some mathematical structure. What kind of reason could that be?

Even if we restrict our claim to mathematical theories which use *purely* mathematical vocabulary (eliminating FCNN from consideration) the problem with (7) remains. This is best illustrated with Balaguer's own discussion

[7] Balaguer agrees with this point here. Weak Platonism will not do. We need our mathematical theories to be *about* mathematical structures. They are not simply modelled by those structres. See, for example, Chapter 3, footnote 10, on page 190.

[8] This notion is due to Balaguer himself [Balaguer, 1998, page50]: "*all* consistent purely mathematical theories truly describe some collection of mathematical objects." Specifying what counts as "true description" is a difficult matter, but the exact nature of its boundary need not concern us here.

of the multiple realisability of mathematical theories. Balaguer exploits the way in which mathematical theories are multiply modelled to give a new epistemology of mathematics. He explicitly endorses *non-uniqueness platonism* (NUP), but he doesn't go so far as to say that any mathematical objects can play any role in any structure:

> FBP–NUP-ists admit that there might be many 3s, but they do *not* admit that one of these 3s could be a "4" in another ω-sequence that satisfied FCNN. Now, of course, *all* of these 3s appear in the "4 position" in other ω-sequences, but none of these other ω-sequences satisfy FCNN, because they all have objects in the "4 position" that have the property *being* 3. And we know that these objects have the property *being* 3 because (a) by hypothesis, they appear in the "3 position" in ω-sequences that do satisfy FCNN, and (b) they couldn't do this without having the property *being* 3, because it is built into FCNN that the number 3 has the property *being* 3. (page 89)

This section of the argument shows how FBP in the form of (7) is used. An object satisfies the 3 role in FCNN, and as a result, it is *true* of that object that it has the property *being* 3, for that is the property ascribed to it by FCNN. It is not sufficient that the object be "3 in the model." The object truly *is* three.

But this gives us our final inconsistency. For FBP as expressed by (7) is *inconsistent* with non-uniqueness, at least for any mathematical theories expressed in a language with an identity relation with anything like a standard semantics.[9] For what Balaguer does with the property *being* 3 is also available with *identity*. Suppose, then, that we have two structures *satisfying* the Peano Axioms (we do not require all of FCNN). If these structures are different, they must have different elements somewhere. Without loss of generality, let's suppose that the "3" in one structure differs from the "3" in the other, and call the first 'x' and the second 'y.' If the Peano Axioms are *true* of both structures, then not only does x have the property of *being* 3, it is true that $x = 3$. Similarly, it is true that $y = 3$. It follows that we must have $x = y$, contrary to our assumption that x and y differ.

So, if mathematical theories are to be couched in anything like standard logic, and if *multiple realisability* is to be saved, then *true description* must go. Plenitudinous platonism cannot say that the properties ascribed to objects in any consistent mathematical theory are genuinely borne by those objects, lest the theory give rise to a plenitude of contradictions. As a result, full-blooded platonism, if it is to be consistent, cannot be the theory Balaguer takes it to be.

References

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[9] Classical first-order logic will do, as usual.

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