Isomorphisms in a Category of Proofs

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My *Aim*

To show how a category of propositions and *classical* proofs can give rise to finely grained hyperintensional notions of sameness of content.

One notion is *very* finely grained (distinguishing p and $p \wedge p$) others are is less finely grained.

One of these notions amounts to equivalence in Richard B. Angell's logic of analytic containment.

My *Motivation*

To apply distinctively
proof theoretical methods
to issues in philosophical logic.

Acknowledgements

Thanks to
Rohan French, Lloyd Humberstone,
Dave Ripley, Shawn Standefer
& the *Melbourne Logic Seminar*
for helpful feedback
on this material.

My Plan

The Category of Classical Proofs

Isomorphisms

More Proofs from A to A

Matching & Logics of Analytic Containment

Matching as Isomorphism

THE CATEGORY OF CLASSICAL PROOFS

There can be different ways to prove the same thing

$$p \wedge q \succ p \vee q$$

Four different derivations,

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R$$

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L$$

$$\dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R$$

$$\dfrac{\dfrac{q \succ q}{q \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L$$

Four different derivations, two *proofs*

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R \quad\approx\quad \dfrac{\dfrac{p \wedge q}{p}}{p \vee q} \quad\approx\quad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L$$

$$\dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R \quad\approx\quad \dfrac{\dfrac{p \wedge q}{q}}{p \vee q} \quad\approx\quad \dfrac{\dfrac{q \succ q}{q \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L$$

Motivating Idea

Proof terms are an *invariant*
for derivations under rule permutation.

δ_1 and δ_2 have the same *term* iff
some permutation sends δ_1 to δ_2.

Four different derivations, two *proof terms*

$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : p \succ y : p}}{\overset{\acute{\wedge} x \curvearrowright y}{x : p \wedge q \succ y : p}}\ \wedge L}{\overset{\acute{\wedge} x \curvearrowright \acute{\vee} y}{x : p \wedge q \succ y : p \vee q}}\ \vee R$$

$$\acute{\wedge} x \curvearrowright \acute{\vee} y$$

$$\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ y : p}}{\overset{x \curvearrowright \acute{\vee} y}{x : p \succ y : p \vee q}}\ \vee R}{\overset{\acute{\wedge} x \curvearrowright \acute{\vee} y}{x : p \wedge q \succ y : p \vee q}}\ \wedge L$$

$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{\grave{\wedge} x \curvearrowright y}{x : p \wedge q \succ y : q}}\ \wedge L}{\overset{\grave{\wedge} x \curvearrowright \grave{\vee} y}{x : p \wedge q \succ y : p \vee q}}\ \vee R$$

$$\grave{\wedge} x \curvearrowright \grave{\vee} y$$

$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{x \curvearrowright \grave{\vee} y}{x : q \succ y : p \vee q}}\ \vee R}{\overset{\grave{\wedge} x \curvearrowright \grave{\vee} y}{x : p \wedge q \succ y : p \vee q}}\ \wedge L$$

Ingredients

- λ terms
- flow graphs
- proof nets

Slogan

A *proof term* for $\Sigma \succ \Delta$
encodes the flow of information
in a proof of $\Sigma \succ \Delta$.

Proof Terms

$$\acute{\wedge} x \curvearrowright \acute{\wedge} \acute{\vee} y \quad \acute{\wedge} x \curvearrowright \acute{\wedge} \grave{\vee} y \quad \acute{\vee} \grave{\wedge} x \curvearrowright \grave{\wedge} \acute{\vee} y \quad \grave{\vee} \grave{\wedge} x \curvearrowright \grave{\wedge} \grave{\vee} y$$

$$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$$

Proof Terms as Graphs on Sequents

$$\acute{\wedge} x \curvearrowright \acute{\wedge}\,\acute{\vee} y \quad \acute{\wedge} x \curvearrowright \acute{\wedge}\,\grave{\vee} y \quad \acute{\vee}\,\grave{\wedge} x \curvearrowright \grave{\wedge}\,\acute{\vee} y \quad \grave{\vee}\,\grave{\wedge} x \curvearrowright \grave{\wedge}\,\grave{\vee} y$$

$$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$$

$$\begin{tikzcd}
p \arrow[d] & \wedge & (q \arrow[d] & \vee & r) \arrow[d] \\
(p & \wedge & q) & \vee & (p \arrow[ull] & \wedge & r)
\end{tikzcd}$$

Finding a Proof Term from a Derivation

$$
\begin{array}{c}
p \succ p \qquad q \succ q \\
\hline
p, q \succ p \wedge q \quad \wedge R
\end{array}
\qquad
\begin{array}{c}
p \succ p \qquad r \succ r \\
\hline
p, r \succ p \wedge r \quad \wedge R
\end{array}
$$

$$
\begin{array}{c}
p, q \succ p \wedge q \qquad p, r \succ p \wedge r \\
\hline
p, q \vee r \succ p \wedge q, p \wedge r \quad \vee L \\
\hline
p, q \vee r \succ (p \wedge q) \vee (p \wedge r) \quad \vee R \\
\hline
p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r) \quad \wedge L
\end{array}
$$

Finding a Proof Term from a Derivation

$$
\begin{array}{c}
p \succ p \qquad q \succ q \qquad\qquad p \succ p \qquad r \succ r \\
\hline
p, q \succ p \wedge q \;\; \wedge R \qquad\qquad p, r \succ p \wedge r \;\; \wedge R \\
\hline
p, q \vee r \succ p \wedge q, p \wedge r \;\; \vee L \\
\hline
p, q \vee r \succ (p \wedge q) \vee (p \wedge r) \;\; \vee R \\
\hline
p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r) \;\; \wedge L
\end{array}
$$

$$
\begin{tikzcd}
p \arrow[d] & q \arrow[d] & r \arrow[d] \\
(p \wedge q) \vee (p \wedge r) & & \\
\end{tikzcd}
$$

Finding a Proof Term from a Derivation

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,\wedge R \qquad \dfrac{p \succ p \qquad r \succ r}{p, r \succ p \wedge r}\,\wedge R}{p, q \vee r \succ p \wedge q, p \wedge r}\,\vee L}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}\,\vee R}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}\,\wedge L
\end{array}
$$

$$
\begin{tikzcd}
 & p \wedge (q \vee r) \arrow[dl] \arrow[dr] & \\
(p \wedge q) \vee (p \wedge r) & &
\end{tikzcd}
$$

Finding a Proof Term from a Derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,\wedge R \qquad \dfrac{p \succ p \qquad r \succ r}{p, r \succ p \wedge r}\,\wedge R}{p, q \vee r \succ p \wedge q, p \wedge r}\,\vee L}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}\,\vee R}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}\,\wedge L
$$

$$
p \wedge (q \vee r) \qquad (p \wedge q) \vee (p \wedge r)
$$

Finding a Proof Term from a Derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,\wedge R \qquad \dfrac{p \succ p \qquad r \succ r}{p, r \succ p \wedge r}\,\wedge R}{p, q \vee r \succ p \wedge q, p \wedge r}\,\vee L}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}\,\vee R}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}\,\wedge L
$$

$$
p \wedge (q \vee r) \qquad (p \wedge q) \vee (p \wedge r)
$$

Finding a Proof Term from a Derivation

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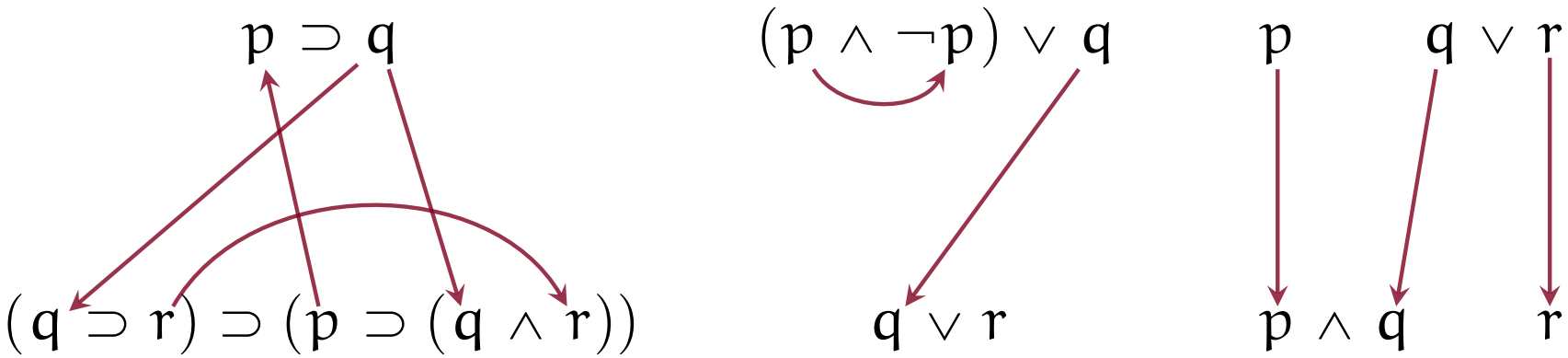
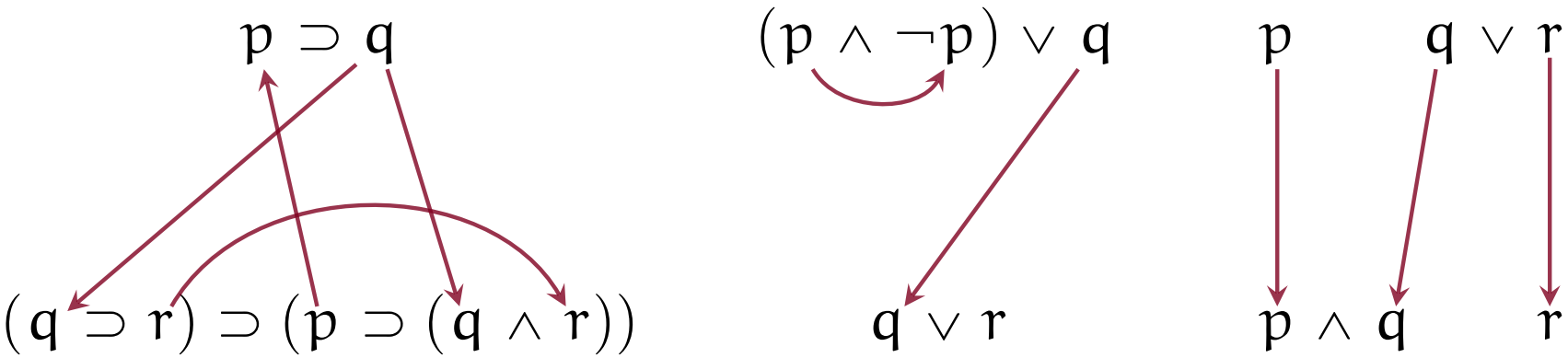
\begin{array}{c}
p \succ p \qquad q \succ q \qquad\qquad p \succ p \qquad r \succ r \\
\hline
p, q \succ p \wedge q \;\; \wedge R \qquad\qquad p, r \succ p \wedge r \;\; \wedge R \\
\hline
p, q \vee r \succ p \wedge q, p \wedge r \;\; \vee L \\
\hline
p, q \vee r \succ (p \wedge q) \vee (p \wedge r) \;\; \vee R \\
\hline
p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r) \;\; \wedge L
\end{array}
$$

$$
\begin{tikzcd}
p \arrow[d] & q \arrow[d] & r \arrow[d] \\
(p \wedge q) \vee (p \wedge r) & & \\
\end{tikzcd}
$$

More Flow Graphs



More Flow Graphs



Links wholly internal to a *premise* or a *conclusion* are called *cups* (⤻) and *caps* (⤺).

Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

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- They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]

Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

- They *typecheck*. [An occurrence of p is linked only with an occurrence of p.]
- They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]
- They must satisfy an "enough connections" condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise $p \vee q$ and conclusion $p \wedge q$ is not connected enough to be a proof term.]

Cut is chaining of proof terms

$$(p \wedge q) \vee (p \wedge r)$$

$$p \wedge (q \vee r)$$

Cut is chaining of proof terms

$$\begin{tikzcd}
p \wedge (q \vee r) \arrow[d] \arrow[dr] \arrow[dr, bend left] \\
(p \wedge q) \vee (p \wedge r)
\end{tikzcd}$$

Cut is chaining of proof terms

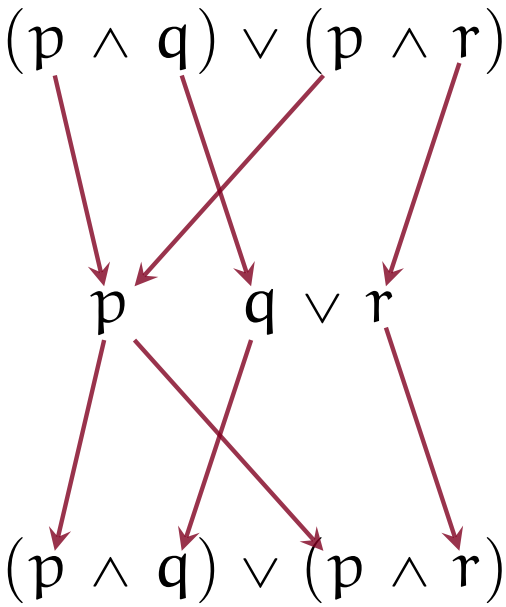
$$
\begin{tikzcd}
(p \wedge q) \vee (p \wedge r) \arrow[d] \arrow[dd, bend right] \arrow[dd, bend left] \arrow[dd, bend right] \\
p \wedge (q \vee r) \arrow[d] \arrow[dr] \arrow[d] \\
(p \wedge q) \vee (p \wedge r)
\end{tikzcd}
$$

The *cut formula* is no longer a premise or a conclusion in the proof term.

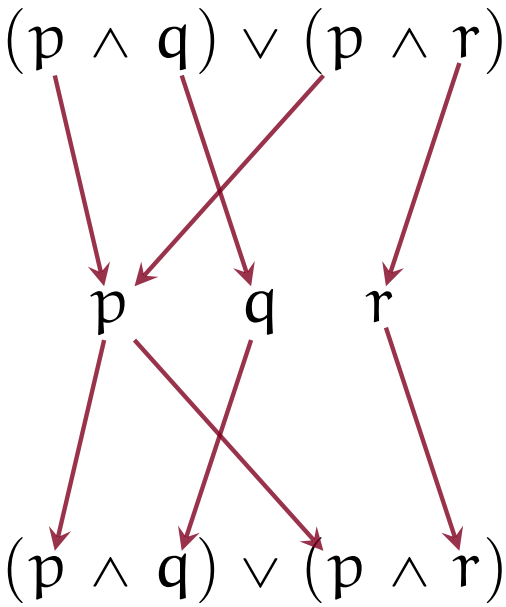
Eliminating Cuts is Local

$$
\begin{tikzcd}
(p \wedge q) \vee (p \wedge r) \arrow[d] \arrow[dd, bend right] \arrow[ddl] \arrow[ddr] \\
p \wedge (q \vee r) \arrow[d] \arrow[dr] \arrow[dr, bend left] \\
(p \wedge q) \vee (p \wedge r)
\end{tikzcd}
$$

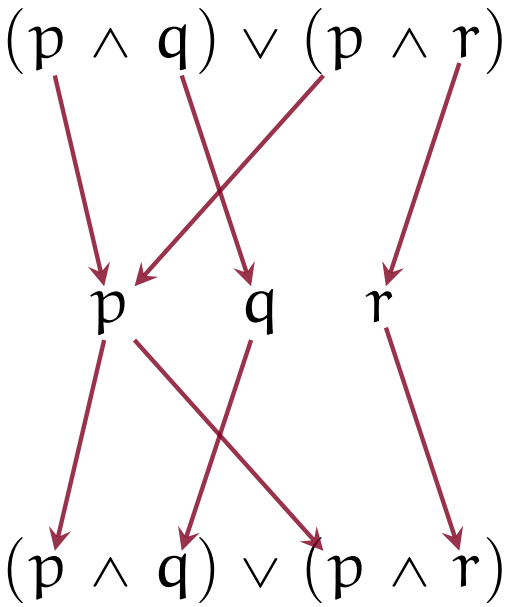
Eliminating Cuts is Local

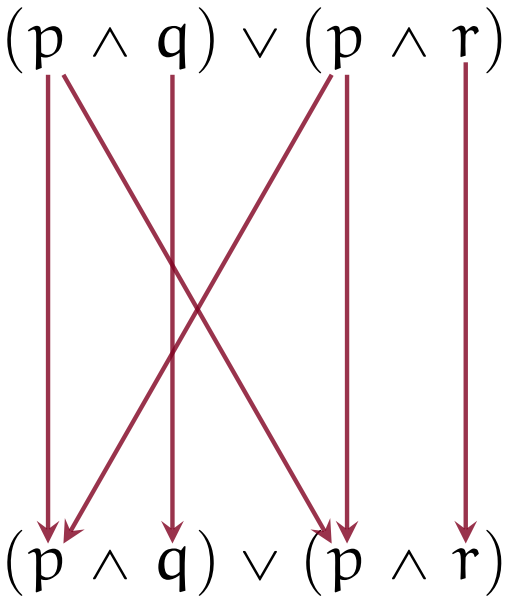


Eliminating Cuts is Local



Eliminating Cuts is Local





Results

- Cut elimination is *confluent* and *terminating*.

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- Cut elimination for proof terms is *local*.

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  [So it can be understood as a kind of *evaluation*.]

- Cut elimination for proof terms is *local*.

  [So it is easily made parallel.]

Cuts with Caps and Cups

$$p \wedge (p \supset q)$$

$$q \wedge (r \vee \neg r)$$

$$\neg p \vee (p \wedge q)$$

Cuts with Caps and Cups

$$\begin{tikzcd}
p \wedge (p \supset q) \arrow[r, bend left] \arrow[ddr] & \\
q \wedge (r \vee \neg r) \arrow[u, bend right] \arrow[ddr] & \\
\neg p \vee (p \wedge q) \arrow[r, bend left] &
\end{tikzcd}
\qquad
\begin{tikzcd}
p \wedge (p \supset q) \arrow[r, bend left] \arrow[dd] & \\
 & \\
\neg p \vee (p \wedge q) \arrow[r, bend left] &
\end{tikzcd}$$

$\mathfrak{C}$ is the Category of Classical Proofs

OBJECTS Formulas — A, B, etc.

ARROWS Cut-Free Proof Terms — $\pi : A \succ B$.

COMPOSITION Composition of derivations with the elimination of *Cut* — If $\pi : A \succ B$ and $\tau : B \succ C$ then $\tau \circ \pi : A \succ C$.

IDENTITY Canonical identity proofs — $Id(A) : A \succ A$.

Identity Proofs

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,\supset L}{p \supset p \succ p \supset p}\,\supset R}{\neg p \vee (p \supset p) \succ \neg p, p \supset p}\,\vee L}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,\vee R
$$

$$\neg p \vee (p \supset p)$$

$$\neg p \vee (p \supset p)$$

Identity Proofs

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,\supset L}{p \supset p \succ p \supset p}\,\supset R}{\neg p \vee (p \supset p) \succ \neg p, p \supset p}\,\vee L}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,\vee R
$$

$$\neg p \vee (p \supset p)$$

$$\neg p \vee (p \supset p)$$

Identity Proofs

$$
\begin{array}{cl}
\dfrac{p \succ p}{p, \neg p \succ}\ \neg L & \dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\ \supset L \\[2ex]
\dfrac{p, \neg p \succ}{\neg p \succ \neg p}\ \neg R & \dfrac{p \supset p, p \succ p}{p \supset p \succ p \supset p}\ \supset R \\[2ex]
\multicolumn{2}{c}{\dfrac{\neg p \succ \neg p \qquad p \supset p \succ p \supset p}{\neg p \vee (p \supset p) \succ \neg p, p \supset p}\ \vee L} \\[2ex]
\multicolumn{2}{c}{\dfrac{\neg p \vee (p \supset p) \succ \neg p, p \supset p}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\ \vee R}
\end{array}
$$

$$
\begin{array}{c}
\neg p \vee (p \supset p) \\
\uparrow \\
\neg p \vee (p \supset p)
\end{array}
$$

Identity Proofs

$$
\begin{array}{ll}
\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{\neg p \succ \neg p}\ \neg R & \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\ \supset L}{p \supset p \succ p \supset p}\ \supset R \\
\multicolumn{2}{c}{\dfrac{\neg p \succ \neg p \qquad p \supset p \succ p \supset p}{\neg p \vee (p \supset p) \succ \neg p, p \supset p}\ \vee L} \\
\multicolumn{2}{c}{\dfrac{\neg p \vee (p \supset p) \succ \neg p, p \supset p}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\ \vee R}
\end{array}
$$

$$
\begin{tikzcd}
\neg p \vee (p \supset p) \\
\neg p \vee (p \supset p) \arrow[u]
\end{tikzcd}
$$

Identity Proofs

$$
\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,\supset L}{p \supset p \succ p \supset p}\,\supset R}{\neg p \vee (p \supset p) \succ \neg p, p \supset p}\,\vee L
\qquad
\dfrac{\neg p \vee (p \supset p) \succ \neg p, p \supset p}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,\vee R
$$

$$
\begin{tikzcd}
\neg p \vee (p \supset p) & \neg p \vee (p \supset p) \\
\neg p \vee (p \supset p) \arrow[u] & \neg p \vee (p \supset p) \arrow[u]
\end{tikzcd}
$$

Identity Proofs

$$
\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,\supset L}{p \supset p \succ p \supset p}\,\supset R}{\dfrac{\neg p \vee (p \supset p) \succ \neg p, p \supset p}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,\vee R}\,\vee L
$$

$$
\begin{tikzcd}
\neg p \vee (p \supset p) & \\
\neg p \vee (p \supset p) \arrow[u] & \arrow[u]
\end{tikzcd}
$$

Identity Proofs

$$
\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,\supset L}{p \supset p \succ p \supset p}\,\supset R}{\dfrac{\neg p \vee (p \supset p) \succ \neg p, p \supset p}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,\vee R}\,\vee L
$$

$$
\begin{tikzcd}
\neg p \vee (p \supset p) \arrow[d, leftrightarrow] & \neg p \vee (p \supset p) \arrow[d, leftrightarrow] & \neg p \vee (p \supset p) \arrow[d, leftrightarrow] \\
\neg p \vee (p \supset p) & \neg p \vee (p \supset p) & \neg p \vee (p \supset p)
\end{tikzcd}
$$

Identity Proofs

In the identity proof from A to A,

- A *positive* occurrence of an atom in the premise linked *to* its mate in the conclusion.
- A *negative* occurrence of an atom in the premise is linked *from* its mate in the conclusion.
- There are no other links.

Identity and Composition in $\mathfrak{C}$

$$
\begin{tikzcd}
(p \wedge \neg q) \vee (p \wedge r) \arrow[d] \arrow[dr] \arrow[dr, bend left] \arrow[dr, bend right] \\
p \wedge (\neg q \vee r) \arrow[d] \\
p \wedge (\neg q \vee r) \arrow[uu, bend left] \arrow[uu, bend right] \arrow[uu] \arrow[uu, bend left] \arrow[uu, bend right]
\end{tikzcd}
\qquad
\begin{tikzcd}
(p \wedge \neg q) \vee (p \wedge r) \arrow[d] \arrow[d] \arrow[d] \arrow[d] \\
(p \wedge \neg q) \vee (p \wedge r) \arrow[d] \arrow[dr] \arrow[dr, bend left] \arrow[dr, bend right] \\
p \wedge (\neg q \vee r) \arrow[uu, bend left] \arrow[uu, bend right] \arrow[uu] \arrow[uu, bend left] \arrow[uu, bend right]
\end{tikzcd}
$$

Identity and Composition in $\mathfrak{C}$

$$\begin{tikzcd}
(p \land \neg q) \lor (p \land r) \arrow[dd] & \\
& \\
p \land (\neg q \lor r) \arrow[uu, bend right] \arrow[uu, bend left] & \arrow[uu, bend right] \arrow[uu, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
(p \land \neg q) \lor (p \land r) \arrow[d] & \arrow[d] & \arrow[d] \\
(p \land \neg q) \lor (p \land r) \arrow[d] & \arrow[u] & \arrow[d] \\
p \land (\neg q \lor r) & \arrow[u] & \arrow[d]
\end{tikzcd}$$

Identity and Composition in $\mathfrak{C}$

$$(p \wedge \neg q) \vee (p \wedge r)$$

$$p \wedge (\neg q \vee r)$$

$$(p \wedge \neg q) \vee (p \wedge r)$$

$$p \wedge (\neg q \vee r)$$

The Category $\mathfrak{C}$ ...

- ... is *symmetric monoidal* and *star autonomous*
- but not *Cartesian*,
- with structural *monoids* and *comonoids*,
- and is enriched in *SLat* (the category of semilattices).

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$$\pi \subseteq \pi' \quad \Rightarrow \quad \pi \circ \tau \subseteq \pi' \circ \tau$$

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$$\pi \subseteq \pi' \quad \Rightarrow \quad \pi \circ \tau \subseteq \pi' \circ \tau$$
$$\tau \subseteq \tau' \quad \Rightarrow \quad \pi \circ \tau \subseteq \pi \circ \tau'$$

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$$\begin{aligned} \pi \subseteq \pi' &\Rightarrow \pi \circ \tau \subseteq \pi' \circ \tau \\ \tau \subseteq \tau' &\Rightarrow \pi \circ \tau \subseteq \pi \circ \tau' \\ \pi \circ (\tau \cup \tau') &= (\pi \circ \tau) \cup (\pi \circ \tau') \end{aligned}$$

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$$\begin{array}{rcl} \pi \subseteq \pi' & \Rightarrow & \pi \circ \tau \subseteq \pi' \circ \tau \\ \tau \subseteq \tau' & \Rightarrow & \pi \circ \tau \subseteq \pi \circ \tau' \\ \pi \circ (\tau \cup \tau') & = & (\pi \circ \tau) \cup (\pi \circ \tau') \\ (\pi \cup \pi') \circ \tau & = & (\pi \circ \tau) \cup (\pi' \circ \tau) \end{array}$$

$\mathfrak{C}$ is just *classical* propositional logic,
in a categorical setting.

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in a categorical setting.

(The sequent calculus plays no essential role here.
You can define proof terms on other proof systems,
e.g. *natural deduction, Hilbert proofs, tableaux, resolution*.)

ISOMORPHISMS

Isomorphisms in Categories

$f : A \to B$ is an *isomorphism* in a category iff
it has an *inverse* $g : B \to A$, where
$g \circ f = id_A : A \to A$ and $f \circ g = id_B : B \to B$.

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If g and g' are both inverses, we have
$g = id_A \circ g = (g' \circ f) \circ g = g' \circ (f \circ g) = g' \circ id_B = g'$,
so any inverse is unique. We can call it f^{-1}.

Why Isomorphisms?

If A and B are isomorphic in a category $\mathcal{C}$,
then what we can do with A (in $\mathcal{C}$)
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but also, on *proofs*.

The distinctions drawn when you analyse how something is *proved* (from premises), are not far from what you want to understand when you ask how something is *made true*.

Isomorphisms in $\mathfrak{C}$

$$p \wedge q \cong q \wedge p$$

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$$\begin{tikzcd}
p \wedge q \arrow[d] \arrow[d] \\
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p \wedge q
\end{tikzcd}$$

Isomorphisms in $\mathfrak{C}$

$$p \wedge q \cong q \wedge p$$

$$\begin{tikzcd}
p \wedge q \arrow[d] \arrow[dr] \\
q \wedge p \arrow[d] \arrow[dr] \\
p \wedge q
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
p \wedge q \arrow[d] \arrow[d] \\
p \wedge q
\end{tikzcd}$$

Isomorphisms in $\mathfrak{C}$

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p \vee q
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
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\end{tikzcd}$$

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$$p \wedge (q \wedge r) \cong (p \wedge q) \wedge r$$

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$$\begin{tikzcd}
p \wedge (q \wedge r) \arrow[d] \arrow[d] \arrow[d] \\
(p \wedge q) \wedge r \arrow[d] \arrow[d] \arrow[d] \\
p \wedge (q \wedge r)
\end{tikzcd}$$

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(p \wedge q) \wedge r \arrow[d] \arrow[d] \arrow[d] \\
p \wedge (q \wedge r)
\end{tikzcd}
\qquad
\begin{tikzcd}
p \wedge (q \wedge r) \arrow[d] \arrow[d] \arrow[d] \\
p \wedge (q \wedge r)
\end{tikzcd}$$

Isomorphisms in $\mathfrak{C}$

$$\neg(p \wedge q) \cong \neg p \vee \neg q$$

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$$\begin{tikzcd}
\neg(p \wedge q) & \\
\neg p \vee \neg q \arrow[u] & \\
\neg(p \wedge q) \arrow[u] &
\end{tikzcd}$$

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$$\begin{tikzcd}
\neg(p \wedge q) & \\
\neg p \vee \neg q \arrow[u] & \\
\neg(p \wedge q) \arrow[u] & \neg(p \wedge q) \arrow[u] \arrow[uu, phantom, "\cong"] & \neg(p \wedge q) \arrow[u] & \neg(p \wedge q) \arrow[u]
\end{tikzcd}$$

Isomorphisms in $\mathfrak{C}$

$$\neg\neg p \cong p$$

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$$\begin{tikzcd}
\neg\neg p \arrow[d] \\
p \arrow[d] \\
\neg\neg p
\end{tikzcd}$$

Isomorphisms in $\mathfrak{C}$

$$\neg\neg p \cong p$$

$$\begin{tikzcd}
\neg\neg p \arrow[d] \\
p \arrow[d] \\
\neg\neg p
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\neg\neg p \arrow[dd] \\
\\
\neg\neg p
\end{tikzcd}$$

Non-isomorphisms in $\mathfrak{C}$

$$p \wedge (q \vee \neg q) \not\cong p$$

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$$p \wedge (q \vee \neg q)$$

$$\begin{tikzcd}
p \arrow[dl] & \\
p \wedge (q \vee \neg q) & q \arrow[l, bend right] \\
& \neg q \arrow[u, bend right]
\end{tikzcd}$$

Non-isomorphisms in $\mathfrak{C}$

$$p \wedge (q \vee \neg q) \not\cong p$$

$$p \wedge (q \vee \neg q) \qquad\qquad p \wedge (q \vee \neg q)$$

$$\begin{tikzcd}
p \arrow[dl] & \\
p \wedge (q \vee \neg q) \arrow[u, bend right] & p \wedge (q \vee \neg q) \arrow[u, bend right]
\end{tikzcd}$$

Non-isomorphisms in $\mathfrak{C}$

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$$\begin{tikzcd}
p \wedge (q \vee \neg q) \arrow[dr] & & p \wedge (q \vee \neg q) \\
& p \arrow[dl] & \\
p \wedge (q \vee \neg q) \arrow[uur, bend left] & & p \wedge (q \vee \neg q) \arrow[ul, bend right]
\end{tikzcd}$$

Non-isomorphisms in $\mathfrak{C}$

$$p \wedge (q \vee \neg q) \not\cong p$$

$$\begin{tikzcd}
p \wedge (q \vee \neg q) \arrow[dr] & \\
& p \arrow[dl] \\
p \wedge (q \vee \neg q) \arrow[r, bend right] & q \vee \neg q
\end{tikzcd}
\qquad
\begin{tikzcd}
p \wedge (q \vee \neg q) \arrow[dd] & \\
& \\
p \wedge (q \vee \neg q) \arrow[r, bend right] & q \vee \neg q
\end{tikzcd}$$

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p \wedge (q \vee \neg q) \arrow[dr] & \\
& p \arrow[dl] \\
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p \wedge (q \vee \neg q) \arrow[u, bend right]
\end{tikzcd}$$

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p \wedge (q \vee \neg q) \arrow[dr] & \\
& p \arrow[dl] \\
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\end{tikzcd}
\qquad
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\end{tikzcd}$$

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$$\begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ p \wedge p & & \end{tikzcd}$$

Non-isomorphisms in $\mathfrak{C}$

$$p \wedge p \not\cong p$$

$$\begin{tikzcd}
p \wedge p \arrow[d] \\
p \arrow[dl] \arrow[dr] \\
p \wedge p & p \wedge p
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
p \wedge p \arrow[dd] \arrow[ddr] \\
\\
p \wedge p & p \wedge p
\end{tikzcd}$$

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$$p \wedge p \not\cong p$$

$$\begin{tikzcd}
p \wedge p \arrow[d] \\
p \arrow[dl] \arrow[dr] \\
p \wedge p & p \wedge p
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
p \wedge p \arrow[dd, shift right] \arrow[dd, shift left] \\
\\
p \wedge p & p \wedge p
\end{tikzcd}$$

Non-isomorphisms in $\mathfrak{C}$

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$$\begin{tikzcd}
p \wedge p \arrow[d, shift right] \arrow[d, shift left] \\
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$$\begin{tikzcd}
(p \wedge q) \vee (p \wedge r) \arrow[d] \arrow[dd, bend right] \arrow[ddr, bend left] \arrow[ddr, bend right] \\
p \wedge (q \vee r) \arrow[d] \arrow[dr] \\
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\end{tikzcd}$$

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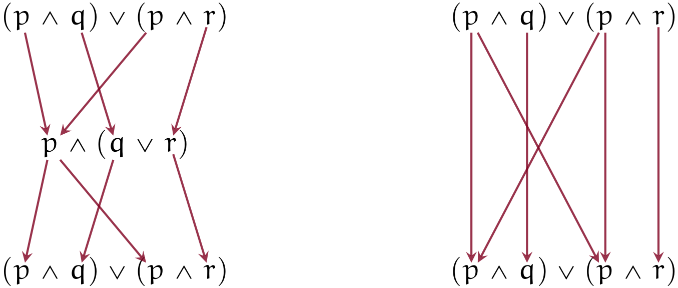
$$p \wedge (q \vee r) \not\cong (p \wedge q) \vee (p \wedge r)$$



Non-isomorphisms in $\mathfrak{C}$

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Occurrence Polarity Condition

If A is isomorphic to B in $\mathfrak{C}$
then each variable occurs positively [negatively] in A
exactly as many times as it occurs positively [negatively] in B.

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If A is isomorphic to B in $\mathfrak{C}$
then each variable occurs positively [negatively] in A
exactly as many times as it occurs positively [negatively] in B.

(This condition is *necessary*, not *sufficient*: $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$.)

Characterising Isomorphisms

A is *isomorphic to* B
iff A and B are equivalent
in the following calculus:

$$A \wedge B \leftrightarrow B \wedge A, \qquad A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C.$$

$$A \vee B \leftrightarrow B \vee A, \qquad A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C.$$

$$\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B, \qquad \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B.$$

$$\neg\neg A \leftrightarrow A. \qquad A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B).$$

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$$\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B, \qquad \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B.$$

$$\neg\neg A \leftrightarrow A. \qquad A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B).$$

This allows for a *negation normal form*, but not DNF or CNF.

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- If A and B are diversified, isomorphic, and in negation normal form, if $l \wedge m$ is a conjunction in A (l and m, literals), a substitution argument (substituting $\top$ and $\bot$ for the *other* atoms) shows that l and m must be conjunctively joined in B, too. The same goes for $l \vee m$.

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- Replace $l \wedge m$ by a new atom in both A and B, and repeat.
- This shows how to reconstruct a proof of equivalence for A and B in the syntactic calculus for isomorphic formulas.

This is a *very* tight constraint

- If A and B are isomorphic, they can play *essentially* the same role in proof.

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- Not even A and $A \wedge A$ are equivalent in *this* sense.
- Yet, A and $A \wedge A$ seem to have identical *subject matter* (insofar as we understand that notion).
- Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?

MORE PROOFS FROM A TO A

$Id(A), Hz(A), Mx(A)$

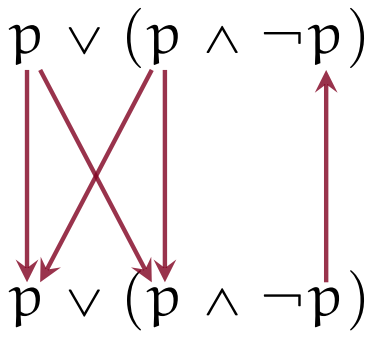
$Id(p \vee (p \wedge \neg p))$

$$\begin{tikzcd}
p \arrow[d] & \vee & (p \arrow[d] & \wedge & \neg p) \\
p & \vee & (p & \wedge & \neg p) \arrow[u]
\end{tikzcd}$$

In $Id(A)$, each occurrence of an atom in the premise
is linked with every its corresponding occurrence of in the conclusion.
Different occurrences of atoms in A *are treated differently.*

$Id(A), Hz(A), Mx(A)$

$Hz(p \vee (p \wedge \neg p))$



In $Id(A)$, each occurrence of an atom in the premise
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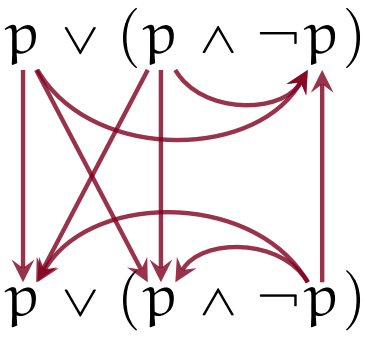
In $Hz(A)$, each positive [negative] occurrence of an atom in the premise
is linked with every positive [negative] occurrence in the conclusion.
We treat occurrences of an atom in A—with the same polarity—equally.

$Id(A), Hz(A), Mx(A)$

$Mx(p \vee (p \wedge \neg p))$



In $Id(A)$, each occurrence of an atom in the premise
is linked with every its corresponding occurrence of in the conclusion.
Different occurrences of atoms in A are treated differently.

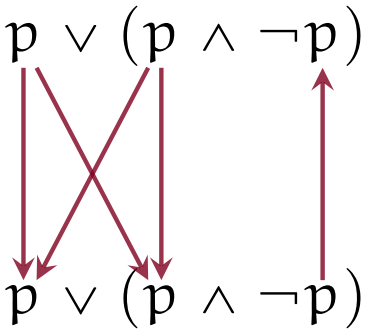
In $Hz(A)$, each positive [negative] occurrence of an atom in the premise
is linked with every positive [negative] occurrence in the conclusion.
We treat occurrences of an atom in A—with the same polarity—equally.

In $Mx(A)$, each syntactically possible linking is included.
We treat all occurrences of an atom in A equally.

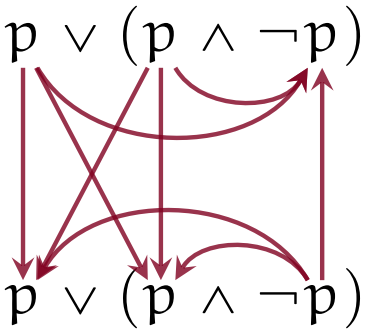
$Hz(A)$, $Mx(A)$, Caps and Cups

Note: $Hz(A)$ is $Mx(A)$ with the caps and cups removed.

$Hz(p \vee (p \wedge \neg p))$



$Mx(p \vee (p \wedge \neg p))$



Erasing Distinctions

Let's look at relations like isomorphism,
but which erase distinctions, up to *Hz* or *Mx*.

Hz-Matching

Let's say that A and B Hz-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$.

Hz-Matching

Let's say that A and B *Hz*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$.

We write "$\approx_{Hz}$" for the *Hz*-matching relation, and we write "$\pi, \pi' : A \approx_{Hz} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Hz*-match between A and B.

Mx-Matching

Let's say that A and B Mx-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$.

Mx-Matching

Let's say that A and B Mx-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$.

We write "$\approx_{Mx}$" for the Mx-matching relation, and we write "$\pi, \pi' : A \approx_{Mx} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a Mx-match between A and B.

Isomorphism $\subseteq$ *Hz*-Matching

If $\pi : A \succ B$ and $\pi^{-1} : B \succ A$, then consider $\pi' = Hz(B) \circ \pi \circ Hz(A)$ and $\tau' = Hz(A) \circ \pi^{-1} \circ Hz(B)$.

These satisfy the *Hz*-matching criteria, $\tau' \circ \pi' = Hz(A)$ and $\pi' \circ \tau' = Hz(B)$.

Proof

$$
\begin{aligned}
Hz(A) &= Id(A) \circ Id(A) \circ Hz(A) \\
&\subseteq Hz(A) \circ Id(A) \circ Hz(A) \\
&= Hz(A) \circ (\pi^{-1} \circ \pi) \circ Hz(A) \\
&= Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \\
&\subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \\
&= (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \\
&= \tau' \circ \pi' \\
&\subseteq Hz(A)
\end{aligned}
$$

Proof

$$
\begin{aligned}
Hz(A) &= Id(A) \circ Id(A) \circ Hz(A) \\
&\subseteq Hz(A) \circ Id(A) \circ Hz(A) \\
&= Hz(A) \circ (\pi^{-1} \circ \pi) \circ Hz(A) \\
&= Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \\
&\subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \\
&= (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \\
&= \tau' \circ \pi' \\
&\subseteq Hz(A)
\end{aligned}
$$

...and similarly, $Hz(B) \subseteq \pi' \circ \tau' \subseteq Hz(B)$

Hz-Matching $\subseteq$ Mx-Matching

If $\pi, \pi' : A \approx_{Hz} B$, then
consider $\tau = Mx(B) \circ \pi \circ Mx(A)$
and $\tau' = Mx(A) \circ \pi' \circ Mx(B)$.

These satisfy the Mx-matching criteria,
$\tau' \circ \pi' = Mx(A)$ and $\pi' \circ \tau' = Mx(B)$.

Proof

$$
\begin{aligned}
Mx(A) &= Id(A) \circ Id(A) \circ Mx(A) \\
&\subseteq Mx(A) \circ Hz(A) \circ Mx(A) \\
&= Mx(A) \circ (\pi' \circ \pi) \circ Mx(A) \\
&= Mx(A) \circ (\pi' \circ Id(B) \circ Id(B) \circ \pi) \circ Mx(A) \\
&\subseteq Mx(A) \circ (\pi' \circ Mx(B) \circ Mx(B) \circ \pi) \circ Mx(A) \\
&= (Mx(A) \circ \pi' \circ Mx(B)) \circ (Mx(B) \circ \pi \circ Mx(A)) \\
&= \tau' \circ \tau \\
&\subseteq Mx(A)
\end{aligned}
$$

Proof

$$
\begin{aligned}
Mx(A) &= Id(A) \circ Id(A) \circ Mx(A) \\
&\subseteq Mx(A) \circ Hz(A) \circ Mx(A) \\
&= Mx(A) \circ (\pi' \circ \pi) \circ Mx(A) \\
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&= (Mx(A) \circ \pi' \circ Mx(B)) \circ (Mx(B) \circ \pi \circ Mx(A)) \\
&= \tau' \circ \tau \\
&\subseteq Mx(A)
\end{aligned}
$$

...and similarly, $Mx(B) \subseteq \pi' \circ \tau' \subseteq Mx(B)$

Mx-Matching $\subseteq$ Logical Equivalence

If $A \approx_{Mx} B$ then there are proofs
$\pi : A \succ B$ and $\tau : B \succ A$.

Matching Relations are Equivalence Relations

REFLEXIVE $Hz(A), Hz(A) : A \approx_{\mathrm{Hz}} A.$

$Mx(A), Mx(A) : A \approx_{\mathrm{Mx}} A.$

Matching Relations are Equivalence Relations

REFLEXIVE $Hz(A), Hz(A) : A \approx_{Hz} A.$

$Mx(A), Mx(A) : A \approx_{Mx} A.$

SYMMETRIC If $\pi, \pi' : A \approx_{Hz} B$, then $\pi', \pi : B \approx_{Hz} A.$

If $\pi, \pi' : A \approx_{Mx} B$, then $\pi', \pi : B \approx_{Mx} A.$

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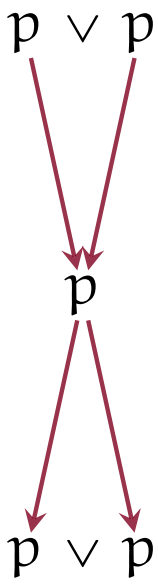
TRANSITIVE If $\pi, \pi' : A \approx_{Hz} B$ and $\tau, \tau' : B \approx_{Hz} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Hz} C$.

If $\pi, \pi' : A \approx_{Mx} B$ and $\tau, \tau' : B \approx_{Mx} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Mx} C$.

More Matchings

$$p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$$



More Matchings

$$p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$$

$$\begin{tikzcd}
 & p \arrow[dl] \arrow[dr] & \\
p \vee p \arrow[dr] & & p \arrow[dl] \\
 & p &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
p \arrow[d] \\
p
\end{tikzcd}$$

More Matchings

$$p \wedge (q \vee r) \approx_{Hz} (p \wedge q) \vee (p \wedge r)$$

$$\begin{tikzcd}
p \wedge (q \vee r) \arrow[d] \arrow[dr] \arrow[dr] & & \\
(p \wedge q) \vee (p \wedge r) \arrow[d] \arrow[dr] & & \\
p \wedge (q \vee r) & &
\end{tikzcd}
\qquad
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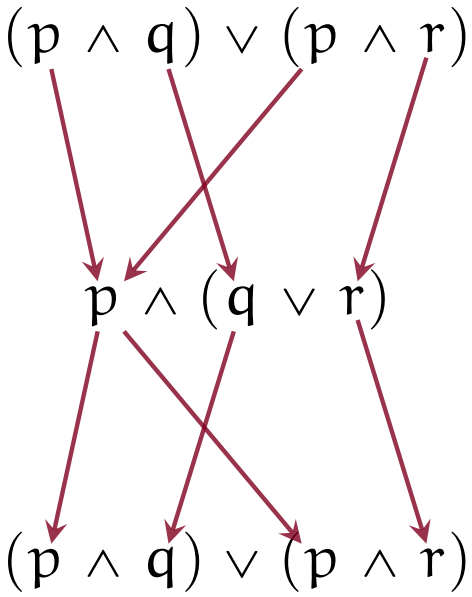

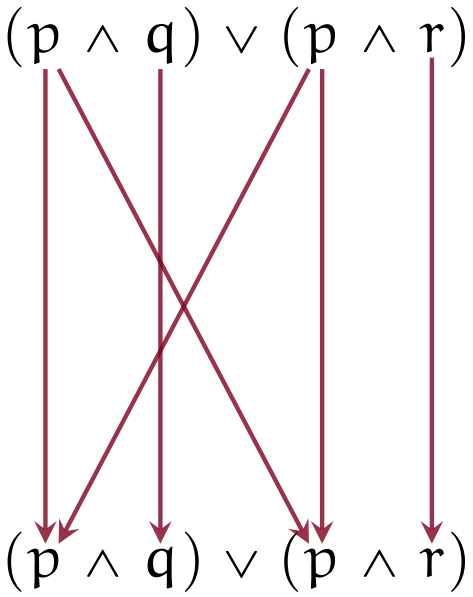

Mx-Matching $\subset$ Logical Equivalence

FACT: If an atom p occurs positively in A but not in B, then A and B do not Mx-match.

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So, in the composition proof from A to A, there is no link from the premise occurrence to the conclusion occurrence. No proof from A to B and back can recreate $Mx(A)$.

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No proof from A to B contains a link from [to] that occurrence to anything in B (since there is no positive [negative] occurrence in B at all).

So, in the composition proof from A to A, there is no link from [to] the premise occurrence to [from] the conclusion occurrence. No proof from A to B and back can recreate $Mx(A)$.

Mx-Matching $\subset$ Logical Equivalence: Examples

FACT: If an atom occurs positively [negatively] in A but not in B then A and B do not Mx-match.

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COROLLARY: $p \not\approx_{Mx} p \wedge (q \vee \neg q)$.

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FACT: If an atom occurs positively [negatively] in A but not in B then A and B do not Mx-match.

COROLLARY: $p \not\approx_{Mx} p \wedge (q \vee \neg q)$.

$$p \wedge \neg p \not\approx_{Mx} q \wedge \neg q.$$

Hz-matching $\subset$ Mx-matching

$$(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{Mx} (p \vee \neg p) \wedge (q \wedge \neg q)$$

$$\begin{tikzcd}
(p \wedge \neg p) \wedge (q \vee \neg q) \arrow[d, bend right] \arrow[d, bend left] \arrow[r, bend left] \arrow[r, bend right] & \\
(p \vee \neg p) \wedge (q \wedge \neg q) \arrow[d, bend right] \arrow[d, bend left] \arrow[r, bend left] \arrow[r, bend right] & \\
(p \wedge \neg p) \wedge (q \vee \neg q) \arrow[r, bend left] \arrow[r, bend right] &
\end{tikzcd}$$

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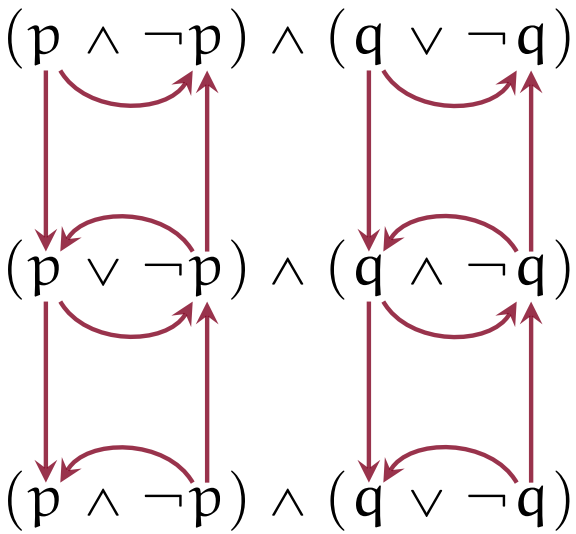

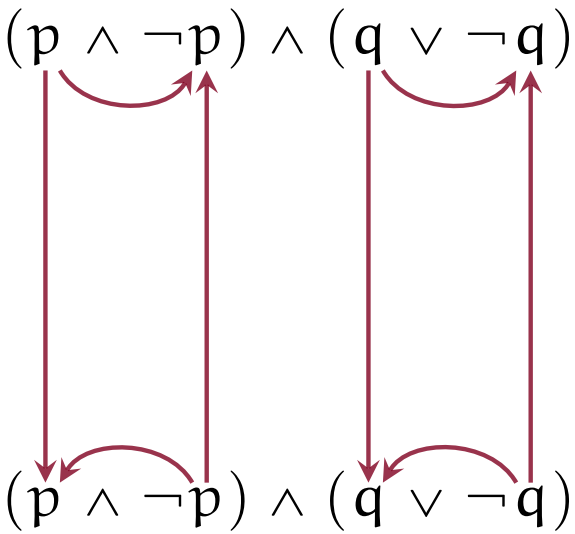

$$\textit{Isomorphism} \subset \textit{Hz-Matching} \subset \textit{Mx-Matching} \subset \textit{Logical Equivalence}$$

So what *are* the *matching* relations?

MATCHING & LOGICS OF ANALYTIC CONTAINMENT

Angell's Logic of Analytic Containment

AC1 $A \leftrightarrow \neg\neg A$

AC2 $A \leftrightarrow (A \wedge A)$

AC3 $(A \wedge B) \leftrightarrow (B \wedge A)$

AC4 $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$

AC5 $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$

R1 $A \leftrightarrow B, C(A) \Rightarrow C(B)$

Angell's Logic of Analytic Containment

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Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$.

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You can define $A \rightarrow B$ as $A \leftrightarrow (A \wedge B)$.

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Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$.

You can define $A \rightarrow B$ as $A \leftrightarrow (A \wedge B)$.

Famously, $A \rightarrow (A \vee B)$ is not derivable in Angell's logic.
We cannot prove $A \leftrightarrow (A \wedge (A \vee B))$.

Extensions of Angell's Logic

- The first degree fragment of *Parry's* Logic of Analytic Containment is found by adding $(A \lor (B \land \neg B)) \to A$ to Angell's Logic.
  - Parry's logic still satisfies this relevance constraint: $A \to B$ is provable only when the atoms in B are present in A.
- *First Degree Entailment* (FDE) is found by adding $A \to (A \lor B)$ to Angell's Logic.
  - FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that $p \lor \neg p$, and $q \land \neg q$ are both non-trivial, and ineliminable.
  - A simple translation encodes FDE inside classical logic. Choose, for each atom p, a fresh atom p', its *shadow*. For each FDE formula A, its translation is the formula A' found by replacing the negative occurrences of atoms p in A by their shadows. An argument is FDE valid iff its translation is classically valid.

$Mx(A, B)$

DEFINITION: $Mx(A, B)$ is the set of all possible linkings which could occur in any proof from A to B.

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{positive atoms in A, negative atoms in B}
to matching {positive atoms in B, negative atoms in A}.

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That is, it contains a link from
{positive atoms in A, negative atoms in B}
to matching {positive atoms in B, negative atoms in A}.

FACT: $Mx(A, B)$ is a proof iff
there is some proof from A to B.

$Mx(A, B)$ examples

p

$Mx(p, q)$

q

$Mx(A, B)$ examples

$Mx(p, q)$

p

q

No links.

$Mx(A, B)$ examples

$Mx(p, q)$

$$\begin{tikzcd} p \\ q \end{tikzcd}$$

No links.

$Mx(p \vee \neg p, p \wedge \neg q)$

$$\begin{tikzcd} p \vee \neg p \arrow[d] \arrow[r, bend left] & \\ p \wedge \neg q & \end{tikzcd}$$

$Mx(A, B)$ examples

$Mx(p, q)$

$$\begin{tikzcd} p \\ q \end{tikzcd}$$

No links.

$Mx(p \vee \neg p, p \wedge \neg q)$

$$\begin{tikzcd} p \vee \neg p \arrow[d] \arrow[r, bend left] & \\ p \wedge \neg q & \end{tikzcd}$$

Not a proof.

$Mx(A, B)$ and matching

LEMMA: If $A \approx_{Mx} B$, then $Mx(A, B)$ and $Mx(B, A)$ are proofs, and $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$

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PROOF: If $\pi, \pi' : A \approx_{Mx} B$, then $\pi \subseteq Mx(A, B)$ and $\pi' \subseteq Mx(B, A)$, so $Mx(A, B)$ and $Mx(B, A)$ are both proofs.

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Since $\pi' \circ \pi = Mx(A)$, we have
$Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A)$,
and similarly, $Mx(B) = Mx(A, B) \circ Mx(A)$,
so $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

Characterising *Mx*-Matching

FACT: If A is classically logically equivalent to B,
and all atoms occurring positively [negatively] in A
also occur positively [negatively] in B,
and *vice versa*, then A and B *Mx*-match
— and conversely.

Proof

If A is logically equivalent to B, then $Mx(A, B)$ and $Mx(B, A)$ are both proofs.

It suffices to show that $Mx(B, A) \circ Mx(A, B) = Mx(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Mx(A, B)$ composed with a link in $Mx(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Mx(A, B)$ and $Mx(B, A)$.

Conversely, if $A \approx_{Mx} B$, we have already seen that A and B must be equivalent, and no atom occurs positively [negatively] in A but not B.

This is *not* Equivalence in Parry's Logic

FACT: A is equivalent to B
in Parry's logic of analytic containment iff
A is classically equivalent to B
and the atoms present in A are present in B
and *vice versa*.

This is *not* Equivalence in Parry's Logic

FACT: A is equivalent to B
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A is classically equivalent to B
and the atoms present in A are present in B
and *vice versa*.

$$(p \wedge \neg p) \wedge q \not\approx_{Mx} (p \wedge \neg p) \wedge \neg q$$

But this pair satisfies Parry's variable sharing criteron.

Open Question

Does the equivalence relation of Mx-matching occur elsewhere in the literature?

$Hz(A, B)$

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That is, it contains a link from
positive atoms in A to corresponding positive atoms in B
and negative atoms in A to corresponding negative atoms in B.

$Hz(A, B)$ examples

$Hz(p \wedge \neg p, q \vee \neg q)$

$Hz(A, B)$ examples

$$p \wedge \neg p$$

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$$q \vee \neg q$$

$Hz(A, B)$ examples

$p \wedge \neg p$

$Hz(p \wedge \neg p, q \vee \neg q)$

No links!

$q \vee \neg q$

$Hz(A, B)$ examples

$p \wedge \neg p$

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$Hz(A, B)$ examples

$Hz(p \wedge \neg p, q \vee \neg q)$

$$p \wedge \neg p$$

$$q \vee \neg q$$

No links!

$Hz(p \wedge \neg p, p \vee \neg p)$

$$\begin{tikzcd} p \wedge \neg p \arrow[d] & \\ p \vee \neg p & p \vee \neg p \arrow[u] \end{tikzcd}$$

$Hz(A, B)$ examples

$Hz(p \wedge \neg p, q \vee \neg q)$

$$p \wedge \neg p$$

$$q \vee \neg q$$

No links!

$Hz(p \wedge \neg p, p \vee \neg p)$

$$\begin{tikzcd} p \wedge \neg p \arrow[d] & \\ & p \vee \neg p \arrow[u] \end{tikzcd}$$

A proof, but not the maximal one.

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$Hz(A, B)$ and FDE

FACT: $Hz(A, B)$ is a proof iff
the argument from A to B is FDE valid.

- From FDE-validity to *Hz*-proof: straightforward induction on an FDE-axiomatisation.
- From the *Hz*-proof $Hz(A, B)$ to FDE-validity. Notice that no negative occurrences of atoms in A or B are linked to any positive occurrences of atoms in A or B. So, there is another *Hz*-proof $Hz(A', B')$ for the FDE translations for A and B.

$Hz(A, B)$ and Hz-matching

LEMMA: If $A \approx_{Hz} B$, then $Hz(A, B)$ and $Hz(B, A)$ are proofs, and $Hz(A, B), Hz(B, A) : A \approx_{Hz} B$

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PROOF: If $\pi, \pi' : A \approx_{Hz} B$, then then since $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$, π and π' are cap- and cup-free, so $\pi \subseteq Hz(A, B)$ and $\pi' \subseteq Hz(B, A)$, so $Hz(A, B)$ and $Hz(B, A)$ are both proofs.

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Since $\pi' \circ \pi = Hz(A)$, we have
$Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$,
and similarly, $Hz(B) = Hz(A, B) \circ Hz(A)$,
so $Hz(A, B), Hz(B, A) : A \approx_{Mx} B$.

Characterising *Hz*-Matching

FACT: If A is FDE-equivalent to B,
and all atoms occurring positively [negatively] in A
also occur positively [negatively] in B,
and *vice versa*, then A and B *Hz*-match
— and conversely.

Proof

If A is FDE-equivalent to B, then $Hz(A, B)$ and $Hz(B, A)$ are both proofs.

It suffices to show that $Hz(B, A) \circ Hz(A, B) = Hz(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Hz(A, B)$ composed with a link in $Hz(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Hz(A, B)$ and $Hz(B, A)$.

Conversely, if $A \approx_{Hz} B$, we have already seen that A and B must be FDE-equivalent, and no atom occurs positively [negatively] in A but not B.

Hz-matching $\equiv$ Angellic Equivalence

FACT: (Fine, Ferguson) A is equivalent to B in Angell's logic of analytic containment iff A is FDE equivalent to B, and any atom occurs positively [negatively] in A iff it occurs positively [negatively] in B.

Hz-matching ≡ Angellic Equivalence

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So, *Hz*-matching *is* equivalence
in Angell's Logic of Analytic Containment.

MATCHING AS ISOMORPHISM

$Hz(A)$ and $Mx(A)$ are Idempotents

- $Hz(A) \circ Hz(A) = Hz(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.

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- $Hz(A) \circ Hz(A) = Hz(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.
- For any category $\mathcal{C}$, if i_A is an idempotent for each object A, we can form a new category $\mathcal{C}_i$ with the same objects as $\mathcal{C}$, and with arrows $i_B \circ f \circ i_A : A \to B$.

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- In this new category, the idempotents i_A are the new identity arrows.

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- In this new category, the idempotents i_A are the new identity arrows.
- So, $\mathfrak{C}_{Hz}$ and $\mathfrak{C}_{Mx}$ are both categories — like $\mathfrak{C}$, but less discriminating, with fewer arrows.

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- In this new category, the idempotents i_A are the new identity arrows.
- So, $\mathfrak{C}_{Hz}$ and $\mathfrak{C}_{Mx}$ are both categories — like $\mathfrak{C}$, but less discriminating, with fewer arrows.
- *Hz*-matching is *isomorphism* in $\mathfrak{C}_{Hz}$.

$Hz(A)$ and $Mx(A)$ are Idempotents

- $Hz(A) \circ Hz(A) = Hz(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.
- For any category $\mathcal{C}$, if i_A is an idempotent for each object A, we can form a new category $\mathcal{C}_i$ with the same objects as $\mathcal{C}$, and with arrows $i_B \circ f \circ i_A : A \to B$.
- In this new category, the idempotents i_A are the new identity arrows.
- So, $\mathfrak{C}_{Hz}$ and $\mathfrak{C}_{Mx}$ are both categories — like $\mathfrak{C}$, but less discriminating, with fewer arrows.
- *Hz*-matching is *isomorphism* in $\mathfrak{C}_{Hz}$.
- *Mx*-matching is *isomorphism* in $\mathfrak{C}_{Mx}$.

$\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$ are nontrivial, nonetheless

$$
\begin{tikzcd}
p \wedge q \arrow[d] & p \wedge q \arrow[d] & p \wedge q \arrow[d] \\
p \wedge q \arrow[d] & p \wedge q \arrow[d] & p \wedge q \arrow[dd] \\
p \vee q \arrow[d] & p \vee q \arrow[d] & \\
p \vee q & p \vee q & p \vee q
\end{tikzcd}
$$

These are each different proofs in $\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$.

$\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$ are nontrivial, nonetheless

$$\begin{tikzcd} p \wedge q \arrow[d] \\ p \vee q \end{tikzcd} \qquad \begin{tikzcd} p \wedge q \arrow[d] \arrow[d] \\ p \vee q \end{tikzcd} \qquad \begin{tikzcd} p \wedge q \arrow[d] \\ p \vee q \end{tikzcd}$$

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- Extending these results to include the units $\top$ and $\bot$ are not difficult. (They were left out only to ease the presentation).
- Relate these results to *models* of logics of content.
- Extend these results to first order logic, and beyond!

THANK YOU!

Thank you!

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or *email* at `restall@unimelb.edu.au`