Proofs, and what they're good for

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THE UNIVERSITY OF MELBOURNE

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My Aim

To explain the nature of *proof*, from the perspective of a *normative pragmatic account of meaning*, using the formal tools of *proof theory*.

Outline

Motivation

Background

What Proofs Are

How Proofs Work

MOTIVATION

Example Proof 1

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Example Proof 1 (the formal structure)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Da \succ Da \qquad \dfrac{Ba \succ Ba \quad La \succ La}{Ba \vee La \succ Ba, La}\,\vee L}{Da \supset (Ba \vee La), Da \succ Ba, La}\,\supset L}{(\forall x)(Dx \supset (Bx \vee Lx)), Da \succ Ba, La}\,\forall L \qquad Da \succ Da}{(\forall x)(Dx \supset (Bx \vee Lx)), Da \succ Ba, Da \wedge La}\,\wedge R}{(\forall x)(Dx \supset (Bx \vee Lx)), Da \succ Ba, (\exists x)(Dx \wedge Lx)}\,\exists R}{(\forall x)(Dx \supset (Bx \vee Lx)) \succ Da \supset Ba, (\exists x)(Dx \wedge Lx)}\,\supset R}{(\forall x)(Dx \supset (Bx \vee Lx)) \succ (\forall x)(Dx \supset Bx), (\exists x)(Dx \wedge Lx)}\,\forall R}{(\forall x)(Dx \supset (Bx \vee Lx)) \succ (\forall x)(Dx \supset Bx) \vee (\exists x)(Dx \wedge Lx)}\,\vee R
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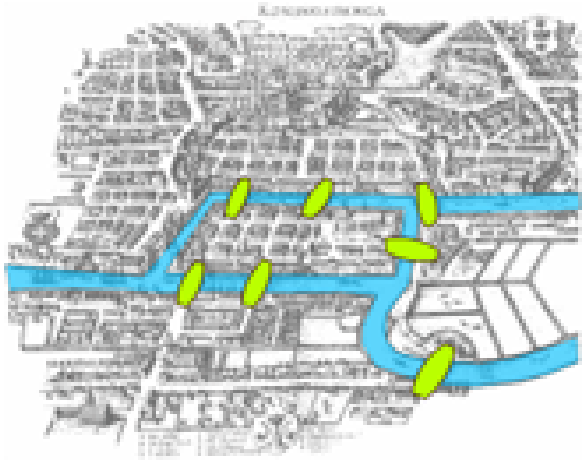
Example Proof 2 (The Bridges of Königsberg)



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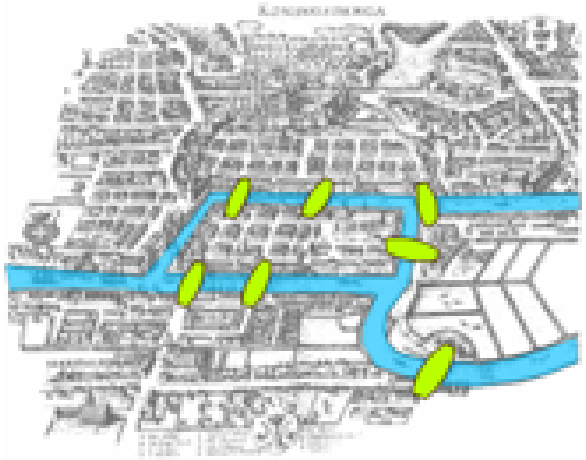

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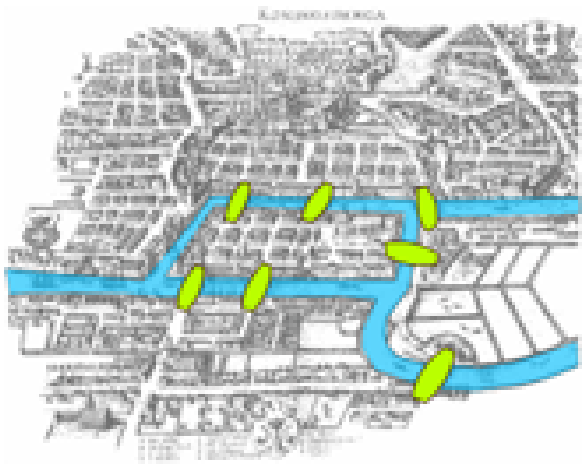

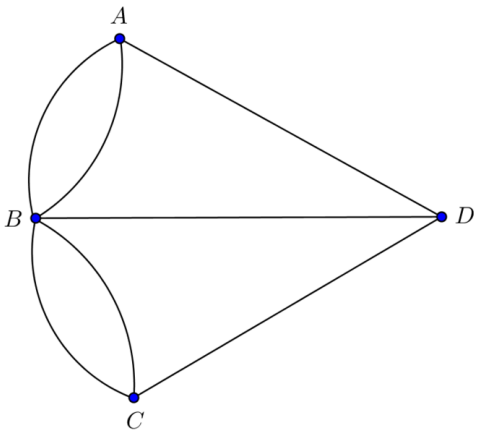

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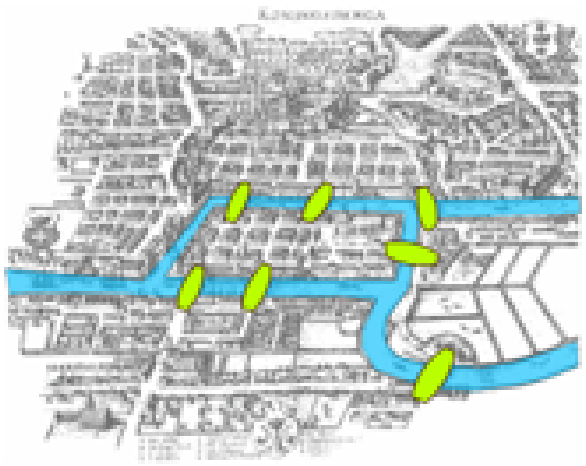

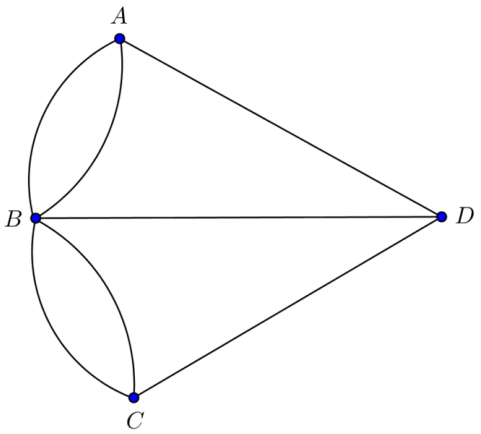

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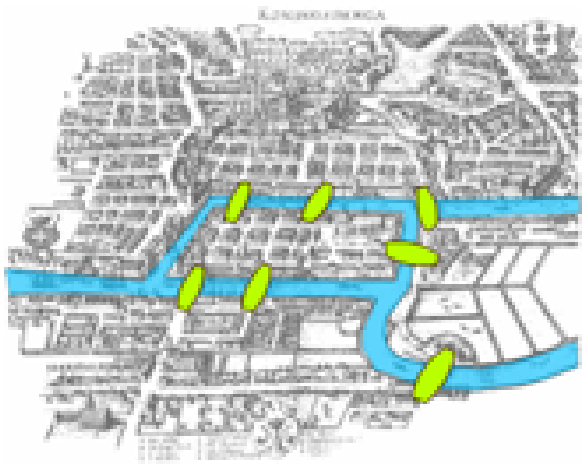

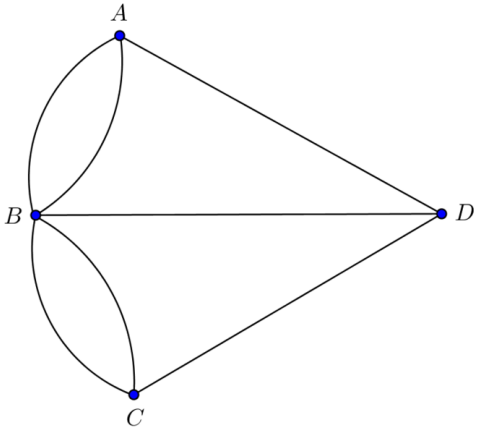

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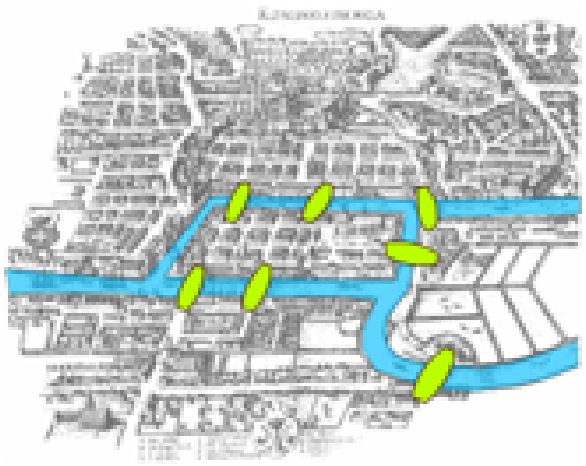

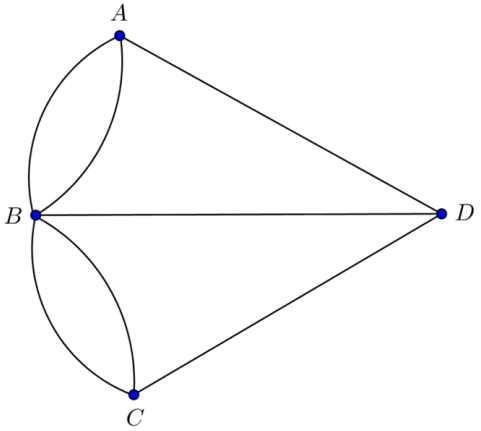

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But what I say here can be extended to proof relying on other concepts.

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- What *grounds* the necessity in the connection between the premises and the conclusion?
- (Notice that these are important questions for proofs in first order predicate logic, as much as for proof more generally.)

BACKGROUND

Positions ...

Assertions and *Denials*

$$[X : Y]$$

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They are connected to other speech acts, too, like imperatives, interrogatives, recognitives, observatives, *etc*.

Norms for Assertion and Denial

Assertions and denials take a *stand* (*pro* or *con*) on something.

DENIAL clashes with assertion.
ASSERTION clashes with denial.

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- A position that is OUT OF BOUNDS doesn't succeed in taking a stand.

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Logical concepts are similarly sharply delimited,
but they cannot all be given explicit definitions.

Definition through a rule for use

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$$\frac{X, A, B \vdash Y}{\overline{\overline{X, A \wedge B \vdash Y}}} \wedge Df$$

What about when to *deny* a conjunction?

$$\dfrac{X \vdash A, Y \qquad \dfrac{X \vdash B, Y \qquad \dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\,Id}{A, B \vdash A \wedge B}\,\wedge Df}{X, A \vdash A \wedge B, Y}\,Cut}{X \vdash A \wedge B, Y}\,Cut$$

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So, we have

$$\dfrac{X \vdash A, Y \qquad X \vdash B, Y}{X \vdash A \wedge B, Y}\,\wedge R$$

Definitions for other logical concepts

$$\frac{X \vdash A, Y}{\overline{\overline{X, \neg A \vdash Y}}}\ \neg Df \qquad \frac{X, A \vdash B, Y}{\overline{\overline{X \vdash A \supset B, Y}}}\ \supset Df \qquad \frac{X \vdash A, B, Y}{\overline{\overline{X \vdash A \vee B, Y}}}\ \vee Df$$

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$$\frac{X \vdash Fa, Y}{X \vdash (\forall x)Fx, Y}\ \forall Df \qquad \frac{X, Fa \vdash Y}{X, (\exists x)Fx \vdash Y}\ \forall Df \qquad \frac{X, Gb \vdash Gc, Y}{X \vdash b = c, Y}\ \forall Df$$

(Where a and G are not present in X and Y.)

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- Are *subject matter neutral*. (They work wherever you assert and deny—and have singular terms and predicates.)
- In Brandom's terms, they *make explicit* some of what was implicit in the practice of assertion and denial.

WHAT PROOFS ARE

A Tiny Proof

If it's Thursday, I'm in Melbourne.

It's Thursday.

Therefore, I'm in Melbourne.

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$$[\textit{It's Thursday} \supset \textit{I'm in Melbourne}, \textit{It's Thursday} : \textit{I'm in Melbourne}]$$

(This is out of bounds.)

The *Undeniable*

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The *stance* (*pro* or *con*)
on *I'm in Melbourne* was already made.

Proofs

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A *proof* of $X \vdash Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.

In this sense, proofs are *analytic*.

They apply, given the definitions, independently of the positions taken by those giving the proof.

What Proofs Prove

A proof of $A, B \vdash C, D$ can be seen

as a *proof* of C from $[A, B : D]$,

and a *refutation* of A from $[B : C, D]$,

and *more*.

HOW PROOFS WORK

Observation 0: Proofs are *analytic*

These proofs are grounded in the *rules defining* the concepts used in them.

Observation 1: *Specification* outstrips *Recognition*

Our ability to *specify* concepts and consequence
far outstrips our ability to *recognise* that consequence.

Peano Arithmetic and Goldbach's Conjecture

SUCCESSOR AXIOMS:

PA1: $\forall x \forall y (x' = y' \supset x = y)$;

PA2: $\forall x (0 \neq x')$.

ADDITION AXIOMS:

PA3: $\forall x (x + 0 = x)$;

PA4: $\forall x \big(x + y' = (x + y)'\big)$.

MULTIPLICATION AXIOMS:

PA5: $\forall x (x \times 0 = 0)$;

PA6: $\forall x \forall y (x \times y' = (x \times y) + x)$.

INDUCTION SCHEME:

PA7: $(\phi(0) \wedge \forall x (\phi(x) \supset \phi(x'))) \supset \forall x \phi(x)$.

GOLDBACH'S CONJECTURE:

GC: $\forall x \exists y \exists z (\text{Prime } y \wedge \text{Prime } z \wedge 0'' \times x = y + z)$

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Verifying a putative proof is straightforward.
Checking that something *has* a proof is not so easy.

Are we logically omniscient?

Suppose that $\textsc{pa} \vdash \textsc{gc}$
(but we don't possess that proof)
and that we *know* PA.

Do we know GC?

In a weak sense of ‘know’, *yes*, we do know GC

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- It's a logical consequence of what we know.
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- There is no epistemic possibility (no circumstance consistent with our knowledge) that leaves GC out.

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- Do we *believe* GC?
- If we believed it, do we believe it *in the right way*?
- There is evidence for GC (its proof from PA, for example), but if that evidence plays no role in our belief...

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- This *follows from* the concepts of consequence and truth.

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- However, given plausible (less minimal) assumptions concerning warrant, we can show that (*for example*) if p is a proof for $A, B \vdash C$ then $x : A, y : B \vdash p(x, y) : C$.
- Here, p transforms warrants for the premises into warrant for the conclusion.

Observation 3: Proofs Can Preserve Warrant

- The account of consequence does not use the concept of *warrant*.
- However, given plausible (less minimal) assumptions concerning warrant, we can show that (*for example*) if p is a proof for $A, B \vdash C$ then $x : A, y : B \vdash p(x, y) : C$.
- Here, p transforms warrants for the premises into warrant for the conclusion.
- This works only for *categorical*, *conclusive* warrants (*grounds*), not for *defeasible* warrants.

A Caveat on Defeasible Warrants

Consider the “Lottery Paradox.”

A Caveat on Defeasible Warrants

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\begin{aligned}
[\ & (\exists x)(Tx \wedge Wx), \\
& (\forall x)(Tx \equiv (x = t_1 \vee x = t_2 \vee \cdots \vee x = t_{1\,000\,000})) \\
& \qquad\qquad\qquad\qquad : \ Wt_1, Wt_2, \ldots, Wt_{1\,000\,000}\ ]
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\end{aligned}
$$

We have a very high degree of confidence in each part.
Each component is highly probable.
But the whole position is out of bounds.

Observation 4: Achilles and the Tortoise

"Well, now, let's take a little bit of the argument in that First Proposition—just *two* steps, and the conclusion drawn from them. Kindly enter them in your note-book. And in order to refer to them conveniently, let's call them A, B, and Z:—

(A) Things that are equal to the same are equal to each other.

(B) The two sides of this Triangle are things that are equal to the same.

(Z) The two sides of this Triangle are equal to each other.

Readers of Euclid will grant, I suppose, that Z follows logically from A and B, so that any one who accepts A and B as true, *must* accept Z as true?"

"Undoubtedly! The youngest child in a High School—as soon as High Schools are invented, which will not be till some two thousand years later—will grant *that*."

"And if some reader had *not* yet accepted A and B as true, he might still accept the *sequence* as a *valid* one, I suppose?"

Observation 4: Achilles and the Tortoise

"No doubt such a reader might exist. He might say 'I accept as true the Hypothetical Proposition that, *if* A and B be true, Z must be true; but, I *don't* accept A and B as true.' Such a reader would do wisely in abandoning Euclid, and taking to football."

"And might there not *also* be some reader who would say 'I accept A and B as true, but I *don't* accept the Hypothetical'?"

"Certainly there might. *He*, also, had better take to football."

"And *neither* of these readers," the Tortoise continued, "is *as yet* under any logical necessity to accept Z as true?"

"Quite so," Achilles assented.

"Well, now, I want you to consider *me* as a reader of the *second* kind, and to force me, logically, to accept Z as true."

"A tortoise playing football would be—" Achilles was beginning

"—an anomaly, of course," the Tortoise hastily interrupted. "Don't wander from the point. Let's have Z first, and football afterwards!"

"I'm to force you to accept Z, am I?" Achilles said musingly. "And your present position is that you accept A and B, but you *don't* accept the Hypothetical—"

"Let's call it C," said the Tortoise.

"—but you *don't* accept

(C) If A and B are true, Z must be true."

"That is my present position," said the Tortoise.

"Then I must ask you to accept C."

Our Analysis

$$A, B \vdash Z$$

Our Analysis

$$A, B \vdash Z$$

or

$$A, A \supset Z \vdash Z$$

Our Analysis

$$A, B \vdash Z$$

or

$$A, A \supset Z \vdash Z$$

This *doesn't* mean when I accept A
and I accept $A \supset Z$,
I ought to also accept Z.

Our Analysis

$$A, B \vdash Z$$

or

$$A, A \supset Z \vdash Z$$

This *doesn't* mean when I accept A
and I accept $A \supset Z$,
I ought to also accept Z.

However, if I assert A and $A \supset Z$ then Z is *undeniable*.

Deviant Use

If I assert A and *if* A *then* Z and *deny* Z,
then I am using '*if* ...*then*' in a way that
deviates from the defining rule for $\supset$,
or I am explicitly contradicting myself.

Deviant Use

If I assert A and *if* A *then* Z and *deny* Z, then I am using '*if* ...*then*' in a way that deviates from the defining rule for $\supset$, or I am explicitly contradicting myself.

$$\frac{A \supset B \vdash A \supset B}{A \supset B, A \vdash B} \supset Df$$

Upshot

An account of the logical concepts given in terms of defining rules governing assertions and denials

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An account of the logical concepts given in terms
of defining rules governing assertions and denials

helps explain how (*first order predicate logic*) proof works,

how possessing a proof can expand our knowledge,
while proofs make explicit what is implicit in what we know.

THANK YOU!

https://consequently.org/presentation/2016/proofs-and-what-theyre-good-for-melb/

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