Assertions, Questions, Answers & the Common Ground

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EXPRESS Workshop · JUNE 2019 · *Amsterdam*

MY AIM: To better understand the speech acts of *assertion* and *denial*, their relationship to other speech acts, and the connections between these speech acts and logical notions, including the classical sequent calculus.[1] MY PROMPT: To revisit some themes from my 2005 paper "Multiple Conclusions" [26]. MY FOCUS: The behaviour of two kinds of speech acts: *polar* (yes/no) *questions* and *justification requests*.

1. ASSERTION AND DENIAL

In "Multiple Conclusions" [26], I interpreted valid sequents of the from $X \succ Y$ as enjoining us to not assert each member of X and deny each member of Y, or as claiming that a position $[X : Y]$ in which such a combination of assertions and denials has been made is *out of bounds*. With that understanding in place, rules for logical connectives and quantifiers like these—

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ \wedge\mathit{Df} \qquad \frac{X \succ A, B, Y}{X \succ A \vee B, Y}\ \vee\mathit{Df}$$

$$\frac{X \succ A, Y}{X, \neg A \succ Y}\ \neg\mathit{Df} \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}\ \to\mathit{Df}$$

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\ \forall\mathit{Df} \qquad \frac{X, A(n) \succ Y}{X, \exists x A(x) \succ Y}\ \exists\mathit{Df}$$

$$\frac{X, Fa \succ Fb, Y \quad X, Fb \succ Fa, Y}{X \succ a = b, Y}\ {=}\mathit{Df}$$

Terms & conditions apply: the singular term n (in $\forall/\exists Df$) and the predicate F (in $=Df$) do not appear below the line in those rules.

—can be understood as *definitions* of the concepts introduced, because they give us the conditions under which an assertion (or a denial) involving that concept is out of bounds, in terms of the bounds governing the prior vocabulary [27]. The connection *between* assertion and denial and the bounds is given by the structural rules:[2]

$$X, A \succ A, Y\ \mathit{Id} \qquad \frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\ \mathit{Cut}$$

$\succ \quad \prec$

One advantage of such an approach is the connection with our *practices* of assertion and denial. But to make this connection, we should address our understanding of assertion (and denial). What *is* assertion? Unsurprisingly, there's a very large literature on this.[3] One large strand in this literature involves characterising assertion as a rule-governed activity, which can be understood in terms of the norms operating on assertions. These approaches can be helpfully divided into three emphases:

NORMS FOR *ME*: These are norms for the *source* of an assertion. Aim to say what is *true*! Only say what you *know*! Be prepared to *back it up* when requested!

NORMS FOR *YOU*: These are norms governing the *target* of an assertion. To assert that p is allow others to point back to the speaker to vouch for it, to entitle others to reassert what you have asserted.

NORMS FOR *US*: These are norms governing the *shared space* formed by the conversation. To assert p is to bid for the common ground to be updated with certain information. This common ground is public and shared.

I will take for granted the notion that these three perspectives are each important ways to understand the function of assertion (while sidestepping the active debate over whether any or each of these norms *characterise* assertion, and whether any of these perspectives is more fundamental than the others). I will take it that if some speech act is governed by these norms, then it is a good candidate for counting as assertion, and if it is *not* so governed, it does not count as assertion.[4]

To attend to the *common ground* for a moment, see Robert Stalnaker's helpful characterisation of the common ground when giving an account of presupposition:

> To presuppose something is to take it for granted, or at least to act as if one takes it for granted, as background information as *common ground* among the participants in the conversation. What is most distinctive about this propositional attitude is that it is a social or public attitude: one presupposes that ϕ only if one presupposes that others presuppose it as well. — "Common Ground" [37]

While Stalnaker himself characterises the common ground as a set of possible worlds (all and only those worlds compatible with what has been presupposed) [35, 36, 37], I will not make that assumption here. Our aim is to understand the semantics of logical concepts such as conjunction, disjunction, negation and the quantifiers, and the dynamics of proof and inference and their interaction with the common ground, so taking a perspective that erases the distinction between logically equivalent formulations of assertions is to blunt our instruments

**Email*: restall@unimelb.edu.au. Thanks to Lloyd Humberstone, Dave Ripley and Shawn Standefer for helpful comments on this material. ¶ A draft of the full paper will be available at `https://consequently.org/writing/assertion-denial-common-ground/`.

[1] As Nuel Belnap has taught us, *declaratives are not enough* [1].

[2] Other structural rules, such as exchange and contraction are elided here becausethe LHS and RHS of a sequent are *sets*, for which association, order and repetition are not recorded.

[3] This selection of readings merely scratches the surface [2, 3, 10, 23, 35, 40].

[4] For example, observatives—such as "Lo, a rabbit!", appropriately expressed when I have seen or otherwise sensed a rabbit—are like assertions, but are not themselves assertions, because they do not license the hearer to repeat the speech act. If I hear you say "Lo, a rabbit!", if I have not myself seen it, I am at most licensed to say "She has seen a rabbit", rather than expressing the observative for myself [16]. In this way, "Lo, a rabbit!" stands to "I have seen a rabbit" rather like "Ow!" (expressing pain) stands to "That hurt!" (which is an assertion, and can be appropriately repeated by another). Similarly, if I ask you to pass the salt, this does not, in and of itself, license anyone else to ask you to pass the salt, whether to me, or to them.

before using them.[5] For us, it will matter a great deal that we can presuppose, say, the axioms of geometry, and take those claims as a part of the common ground of a mathematical discourse, without also taking all the consequences of those claims *for granted* in the same way. The point of Euclid's *Elements* was to derive consequences of those first principles, and in doing this, you do not first take those consequences for granted. A crucial feature shared between different understandings of the common ground is the notion of a *scoreboard*, which is used to keep track of the status of some rule-governed activity, in this c ase, a conversation [18]. This model of conversation and information update has proved incredibly fertile in semantics, in giving an account of the dynamics of discourse and the behaviour of many different linguistic phenomena, such as presupposition, illocutionary mood, parentheticals, and not-at-issue content [9, 25, 34]. A great deal can be learned in the shift to a dynamic perspective, understanding speech acts as moves that update the scoreboard, rather than as static representation [39, 41].

Let's turn to the relationship between *assertion* and *denial*. In "Multiple Conclusions" [26], I did not say a much about the relationship between assertion and denial other than that they clash, and this is an issue which deserves clarification [6, 15]. The characterisation in terms of a clash doesn't help with distinguishing denial from other speech acts which also clash or conflict (in some way) with assertion. *Retraction*—taking back an assertion—is, in *some* sense, incompatible with assertion, too. Retraction (taking back an assertion) aims at reversing the effect of an assertion, so it clashes with assertion in some sense.[6] An assertion that p and a retraction of that assertion are aimed at incompatible outcomes (that p be in the common ground). But to take back the assertion that p is not the same as to *deny* p in the sense in question in "Multiple Conclusions". To clarify the difference between denial and retraction, we turn to related speech acts, *polar questions*.

2. POLAR QUESTIONS

Polar (yes/no) questions, and their answers, are distinct speech acts with their own norms.[7] The polar question "is it the case that p?" (abbreviated "p?" in what follows) raises an *issue*. There are two ways to *settle* the issue so raised: positively (to say YES) and negatively (to say NO). These two answers to a polar question *clash* in the sense that if I say YES and you say NO to the same polar question p? then we DISAGREE: there is no shared position incorporating both of our answers.

Other responses to p?, such as *maybe*, *I don't know*, or *I think so* are acceptable *responses* to p?, but they don't answer the question, or settle the issue.

Looking back on the norms of assertion, we see that it is quite plausible that settling answers are governed by these very same norms. These answers are appropriately governed by norms of truth and knowledge, they entitle hearers to reassert (to repeat the same answer),[8] and they add to the common ground in the same way as do other assertions.

If settling answers to polar questions count as assertions, what do they assert? Presumably, a YES to p? asserts p, while the NO to p asserts $\neg p$. However, I prefer to think of a YES to p? as ruling p *in*, while a NO to p? rules p *out*. This answer to p? is a *denial* of p in the sense considered in "Multiple Conclusions."[9] So, we can think of the common ground as constituting a *position*, a pair of sets of issues $[X : Y]$ where each issue in X has been ruled *in* (this is the POSITIVE COMMON GROUND) while every issue in Y has been ruled *out* (this is the NEGATIVE COMMON GROUND).

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Let's turn to the relationship between assertion and denial when it comes to declarative utterances in general, rather than answers to polar questions, in particular. Consider these two short dialogues, between Abelard and Eloise, concerning their young son, Astralabe,[10] who is hiding from them.

(1) ABELARD: Astralabe is in the study.
ELOISE: No, he's in the kitchen.

(2) ABELARD: Astralabe is in the study.
ELOISE: No, he's *either* in the kitchen *or* the study.

Eloise's "no" is clearly *negative* in both dialogues, but there seems to be some kind of difference between them. Eloise's "no" in (2) is not, by my lights, a *denial* of Astralabe's claim, in the sense of settling the issue of Astralabe's presence in the study *negatively*. The "no" is better understood as bid a for Abelard's claim to be *retracted* from the common ground (if the claim managed to enter it in the first place), or as a bid to block its entry into the common ground. The distinction can be clarified if we consider the related dialogues where Abelard is instead asking a question:

(3) ABELARD: Is Astralabe in the study?
ELOISE: No, he's in the kitchen.

(4) ABELARD: Is Astralabe in the study?
ELOISE: *No, he's *either* in the kitchen *or* the study.
ELOISE: *Maybe*, he's *either* in the kitchen *or* the study.

Here, we can clearly see the difference between Eloise's distinct responses. The "no" in (3) settles the issue negatively. In (4) a "no" is not appropriate, because it *would* have settled the issue negatively, but this is not what Eloise intends to do. She is able to give a *partial* answer to the question (hence her "*maybe*"). The "no" is not appropriate here if

[5] David Lewis is helpfully explicit concerning the price of such an assumption. "Likewise, the known proposition that I have two hands may go unrecognised when presented as the proposition that the number of my hands is the least number n such that every even number is the sum of n primes. (Or if you doubt the necessary existence of numbers, switch to an example involving equivalence by logic alone.) These problems of disguise shall not concern us here. Our topic is modal, not hyperintensional, epistemology" [19, p. 552]. In contrast, *our* topic is the semantics of logical connectives, so identifying propositions up to logical equivalence is to work with a coarseness of grain that obscures exactly the features we wish to clarify. So, we will not take the approach of identifying the common ground with a set of worlds. (Thanks to Lloyd Humberstone for reminding me of Lewis' formulation.) This does not mean, of course, that the coarser perspective is to be rejected. Much can be gained by *ignoring* the differences we must attend to here. There is a place for acting fast and loose with propositions, but *this* is not that place.

[6] Of course, a retraction also *requires* an earlier assertion, so there is another sense in which they are compatible.

[7] This is only a brief introduction to some of the issues around polar questions and their answers. This is a rich and interesting literature on its own. I have found papers by Bruce, Farkas, Humberstone and Roelofsen particularly helpful here [7, 8, 14, 32].

[8] Consider some potentially troubling cases: a child asks her mother "can I stay up late tonight?"—a polar question. The mother answers YES, thereby giving permission. You, not the child's parent are not in a position to give permission yourself, but you *are* allowed to repeat the answer to the question, if asked. You can say "yes, you can stay up late tonight", even though you are *not* thereby also giving permission.

[9] Nothing important for us on the distinction. If you wish to think of denials of p as assertions of $\neg p$, that is consistent with the line developed here. However, I prefer to keep track of what has been ruled *in* and what has been ruled *out* separately. In this way, we can directly model practices in which the issues to be addressed are not themselves closed under negation. The issues (contents) to be added to the common ground can be identified with those issues that the language community can raise and consider.

[10] Abelard and Eloise (often simply abbreviated $\forall$ and $\exists$) are characters who often appear in modern treatments of dialogues and games in logic [13, e.g., pp. 23, 24]. Peter Abelard (1079–1142) was a medieval French theologian and logician. Héloïse d'Argenteuil (?–1164) was a French theologian and abbess. They had a troubled relationship, which could variously be described as a forbidden love affair between peers or as sexual abuse of a younger student by an older teacher. Before being forcibly separated, they had a child, whom Héloïse named Astralabe. For more on the life, thought and correspondence of Peter Abelard and Héloïse d'Argenteuil, consult Constant Mews' *Abelard and Heloise* [24]. For more on Astralabe's curious name, and what we know of his life, start on page 13 of William Levitan's *Abelard and Heloise: The Letters and Other Writings* [17].

its function would be to block the addition of the issue (that Astralabe is in the study) to the positive common ground, because in dialogue (4) the issue is *not* in the positive common ground. Abelard raised the issue, he did not bid to settle it positively, unlike his assertion in dialogue (2). This distinction motivates the following understanding of the difference between settling an issue negatively (which I will call STRONG DENIAL) and retraction or blocking (which I will call WEAK DENIAL).[11]

- To *strongly deny* p is to bid to add p to the *negative common ground*.
- To *weakly deny* p is to *block* the addition of p to the *positive common ground*, or to bid for its *retraction* if it is already there.

So, strong *or* weak denials of p are appropriate responses to an assertion of p, because the assertion of p is a bid to add p to the positive common ground. A strong denial of p is one way to settle the question p? — this is generally an appropriate response. On the other hand, a weak denial of p is not generally an appropriate response to the polar question p?, as the polar question does not place p in the positive common ground, and the question is inappropriate if p is already in the positive common ground, so there is no p to block or retract.

We have strong and weak *denial*. Clearly there is room for strong and weak *assertion*, too. To complete the picture, we have:

- STRONG DENIAL: add to the *negative common ground*.
- STRONG ASSERTION: add to the *positive common ground*.
- WEAK DENIAL: retract (or block) from the *positive common ground*.
- WEAK ASSERTION: retract (or block) from the *negative common ground*. — "Perhaps p."

This is one way to understand the relationship between assertion and denial, and how to distinguish strong denial from other negative speech acts.

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This view of the common ground and this understanding of the role of weak denial as blocking or retracting from the positive common ground means that we have reason to understand the common in a *very* finely grained way. Consider the following dialogue, where Abelard is being tutored by Eloise in geometry. He is reasoning about a triangle with interior angles of 40, 60 and 80 degrees. He adds up the angles, and notices that they sum to 180°...

(5) ABELARD: The interior angles of triangles sum to 180°.
ELOISE: No, *this* triangle's interior angles sum to 180°. Can you prove the general case?

Here, Eloise seems to block from the common ground (to weakly deny) a logical consequence of claims that are already in the common ground (that is, the axioms of geometry), for the same general reason as for other weak denials—Abelard has jumped to conclusions too fast, without sufficient evidence. Understanding the common ground as a set of worlds would make this analysis impossible.[12]

It's for this reason that I concur with the analysis given in "Multiple Conclusions" that the validity of a sequent $X \succ Y$ is to be understood as ruling out the *strong* assertion of each member of X and the *strong* denial of each member of Y. Here, Eloise is being well within her rights to strongly assert the axioms of geometry (they have been ruled in to the positive common ground) and to weakly deny—at this juncture—Abelard's hasty claim that the interior angles of triangles sum to 180°.

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Before moving on to the next section, we should confirm that the structural rules for the sequent calculus (*Identity*, *Weakening* and *Cut*) make sense, given this characterisation of strong assertion and strong denial.

Identity and *Weakening* together form the axiom $X, A \succ A, Y$, according to which any position $[X, A : A, Y]$ in which the issue A has been both asserted and denied is out of bounds. This is another way to understand the claim that the two ways to settle the issue A? conflict with one another. There is no available position that takes both sides of that disagreement.[13] The *Cut* rule, on the other hand

$$\frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\, Cut$$

can be understood as saying that there are no *quandries* if a position $[X : Y]$ is available (not out of bounds). If $[X : Y]$ is available, then given the polar question A?, at least one of the answers YES or NO is *available*. While the *Cut* rule understood in this way is not without its critics [5, 29, 30, 31], at the very least, the notion that if something is undeniable (in the sense that its strong denial is out of bounds) then asserting is a matter of making explicit the commitments we have already implicitly undertaken, has a certain natural appeal.[14]

3. POSITIONS AND RULES

Recall the defining rules:

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\, \wedge Df \qquad \frac{X \succ A, B, Y}{X \succ A \vee B, Y}\, \vee Df$$

$$\frac{X \succ A, Y}{X, \neg A \succ Y}\, \neg Df \qquad \frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y}\, \rightarrow Df$$

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\, \forall Df \qquad \frac{X, A(n) \succ Y}{X, \exists x A(x) \succ Y}\, \exists Df$$

$$\frac{X, Fa \succ Fb, Y \quad X, Fb \succ Fa, Y}{X \succ a = b, Y}\, {=}Df$$

Terms & conditions apply: the singular term n (in $\forall/\exists Df$) and the predicate F (in $=Df$) do not appear below the line in those rules.

These can be understood as kinds of *definitions*, showing how to treat assertions or denials of the defined concepts in terms of assertions or denials of their components. Using these rules, together with the structural rules, we can derive complex sequents:

$$\dfrac{\dfrac{\neg p \succ \neg p}{\succ p, \neg p}\, \neg Df}{\succ p \vee \neg p}\, \vee Df \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\, \neg Df}{p \wedge \neg p \succ}\, \wedge Df$$

[11] Here I follow Luca Incurvarti and Julian Schlöder, though my understanding of the relationship between strong and weak speech acts differs from theirs, as will become clear [15, 33].

[12] Understanding the common ground as a pair $[X : Y]$ of claims ruled in and claims ruled out is much closer then to the analysis of *commitment stores* in dialogue, from the work of Hamblin [11, 12] and Mackenzie [20, 21, 22] dating from the 1970s, as well as more recent work on the common ground in some more recent work in the semantics of questions, for which a fine-grained conversational slate is required [7].

[13] So, if you are tempted to say "yes and no" to a polar question A?, the strategy is to disambiguate. To find two closely related issues A_1 and A_2, where you say YES to one and NO to the other.

[14] Consider this extract from the *Meno*:

> SOCRATES: ... if he always possessed this knowledge he would always have known; or if he has acquired the knowledge he could not have acquired it in this life, unless he has been taught geometry; for he may be made to do the same with all geometry and every other branch of knowledge. Now, has any one ever taught him all this? You must know about him, if, as you say, he was born and bred in your house. MENO: And I am certain that no one ever did teach him. SOCRATES: And yet he has the knowledge? MENO: The fact, Socrates, is *undeniable*.

When Meno says that it is *undeniable* that the boy has knowledge, he is not saying something *weaker* than the claim that the boy has knowledge. To grant that is undeniable that p is to grant no less than (and often rather more than) to grant p.

$$
\dfrac{\dfrac{\dfrac{\dfrac{p, q \vee r \succ p \wedge q, q \vee r}{p, q \vee r \succ p \wedge q, r, q}\,{\vee Df} \qquad \dfrac{p \wedge q, q \vee r \succ p \wedge q, r}{q, p, q \vee r \succ p \wedge q, r}\,{\wedge Df}}{p, q \vee r \succ p \wedge q, r}\,{Cut}}{p, q \vee r \succ (p \wedge q) \vee r}\,{\vee Df}}{p \wedge (q \vee r) \succ (p \wedge q) \vee r}\,{\wedge Df}
$$

A derivation of $X \succ Y$ shows that the position $[X : Y]$ is out of bounds. This raises a number of questions [38].

- Derivations do not have the same *shape* as proofs. (Where is the *conclusion* in $p \vee q \succ p, q$?)
- A derivation of an endsequent $X \succ A$ does not tell you to *infer* A *from* X — it merely tells you to not assert all members of X and deny A.

Let's make this problem sharp by reminding ourselves of the problem of Achilles and the Tortoise [4]:

> "Well, now, let's take a little bit of the argument in that First Proposition—just *two* steps, and the conclusion drawn from them. Kindly enter them in your note-book. And in order to refer to them conveniently, let's call them *A*, *B*, and *Z*:—
>
> (*A*) Things that are equal to the same are equal to each other.
>
> (*B*) The two sides of this Triangle are things that are equal to the same.
>
> (*Z*) The two sides of this Triangle are equal to each other.
>
> Readers of Euclid will grant, I suppose, that *Z* follows logically from *A* and *B*, so that any one who accepts *A* and *B* as true, *must* accept *Z* as true?"
>
> "Undoubtedly! The youngest child in a High School—as soon as High Schools are invented, which will not be till some two thousand years later—will grant *that*."
>
> "And if some reader had *not* yet accepted *A* and *B* as true, he might still accept the *sequence* as a *valid* one, I suppose?"

It seems that the sequent calculus and its derivation of $A, A \to Z \succ Z$ does not address the Tortoise's deviant behaviour here. The tortoise never asserts A and $A \to Z$ while denying Z. The Tortoise merely refrains from accepting A and $A \to Z$ as counting as a *reason for* Z. That is, the Tortoise does not accept A and $A \to Z$ (already accepted into the positive common ground) as satisfying the *justification request* for the assertion of Z.

4. JUSTIFICATION REQUESTS

Justification requests are another kind of speech act, alongside assertions (and denials) and polar questions and their answers. A justification request is another kind of imperative, querying an assertion (or a denial), temporarily blocking it from entering the common ground, until some kind of justification can be provided for it.[15] Here are some example justification requests in dialogue, together with different responses.

(6) ABELARD: Astralabe is in the kitchen.
ELOISE: *Really*?
ABELARD: I saw him there five minutes ago.
ELOISE: OK.

(7) ABELARD: Astralabe is in the kitchen.
ELOISE: *Really*?
ABELARD: I saw him there five minutes ago.
ELOISE: *Are you sure*? He's been in the study with me for the last half hour.

(8) ABELARD: Astralabe is in the kitchen.
ELOISE: *Really*?
ABELARD: I saw him there five minutes ago.
ELOISE: Yes, but he was in the study two minutes ago.

Recall the way that assertions permit their hearers to call on the asserter to vouch for their assertion. Justification requests are requests for the asserter to do just that. A justification request for a (strong) assertion (or a strong denial) is an attempt to block the addition to the common ground until a *reason* is given. This reason is a number of assertions or denials, which must be *granted* (included in the common ground) in order for the request to be met. Granting the reason is a *necessary* but not a *sufficient* condition for the justification request to be satisfied and for the original assertion to be added to the common ground.

In our example dialogues we see three different ways that is may proceed. In (6), Eloise's justification request (her "*Really*?") is answered by Abelard, Eloise accepts this answer, and the original assertion is added to the common ground. In (7), Abelard's answer is met by another justification request, followed by another assertion which seems to undercut Abelard's claim. Unless further justification is given, Abelard's original assertion is blocked from the common ground, because the assertion forming the answer to the justification request is not granted. In (8), the original claim is also not added to the common ground, but Abelard's answer is nonetheless granted (Eloise's "Yes" indicates that she concedes Abelard's assertion), but Eloise does not take this to be a sufficient answer to the original justification request.

With this simple account of justification requests in mind, we turn to the dialogue between Achilles and the Tortoise. The Tortoise violates a norm of some kind or other: she grants A and $A \to Z$ (they are accepted into the positive common ground) but she does not take them to answer the justification request for the assertion of Z. This is clearly deviant behaviour—behaviour that displays a deficient grasp of the meaning of the conditional. But what kind of norm is violated here? We could just grant this as a *sui generis* condition on understanding logical concepts: that *modus ponens* be accepted, to the extent that A and $A \to Z$ are to be accepted as meeting a justification request for Z.[16] However, this would seem to be rather *ad hoc*, and it would give us no particular insight either into the logical connectives or our ways of grasping their meaning.

Instead, I will start with the notion that the sequent rules given above ($\wedge Df$, $\to Df$, etc.) *define* the connectives involved, and we will consider whether there is any connection between *definitions* and justification requests. Consider the following dialogue between Achilles and the Tortoise:

ACHILLES: So … this is an *equilateral* triangle.

TORTOISE: I'm sorry, I don't follow, my heroic friend. I've not heard that word before: what does '*equilateral*' mean?

ACHILLES: Oh, that's easy to explain. '*Equilateral*' means having sides of the same length. An *equilateral* triangle is a triangle with all three sides the same length.

TORTOISE: OK. That sounds good. You may continue with your reasoning.

ACHILLES: Well, as I was saying, the sides of this triangle are all one cubit in length, so it is an equilateral triangle.

TORTOISE: Perhaps you will forgive me, Achilles, but I still don't follow. I grant to you that the sides of this triangle all have the same length. I fail to see, however, that it *follows* that it is an equilateral triangle. Could you explain why it is?

This dialogue clearly pinpoints a norm that the Tortoise violates, and this norm connects definitions and justification requests:

> If I accept the definition $A =_{df} B$, then I should accept granting A as meeting a justification request for the assertion of B and ruling out A as meeting a justification request

[15] For the literature on justification requests, resolution requests and related dialogical moves, see especially the pioneering literature of Charles Hamblin [11, 12] and Jim Mackenzie [20, 21, 22].

[16] Or, slightly more generally, that there is some kind of relation of *immediate consequence* where the premises of an immediate consequence answer a justification request for its conclusion [20].

for B's denial and *vice versa*. A failure to accept this is a sign that I have not mastered the definition.

This norm seems to points the way to a more general way that defining rules like $\wedge Df$, $\rightarrow Df$ and the like can link up with justification requests and their answers. For example, the defining rule $\wedge Df$ conjunction can be understood as giving rise to the following norm:

> It is a mistake to grant A and grant B and to look for something *more* to discharge a justification request for an assertion of $A \wedge B$, if you take $\wedge Df$ as a *definition*.

Similarly, the $\rightarrow Df$ gives rise to this norm:

> It is a mistake to rule A in and rule B out and to look for something more to discharge a justification request for a denial of $A \rightarrow B$ if you accept $\rightarrow Df$ as a *definition*.

This goes only some of the way to a general account of the connection between defining rules, derivations and justification requests. More work must be done to show why granting A and $A \rightarrow Z$ is enough to meet a justification request for an assertion of Z. To do this, consider the following *focussed* derivation.[17]

$$\dfrac{A \rightarrow Z \succ \boxed{A \rightarrow Z}}{A \rightarrow Z, A \succ \boxed{Z}}\ {\rightarrow}Df$$

Read the *premise* $A \rightarrow Z \succ \boxed{A \rightarrow Z}$ as telling us that in a position in which $A \rightarrow Z$ has already been ruled in, we have an answer to the justification request for $A \rightarrow Z$'s assertion. Then, applying $\rightarrow Df$ we see why we have an answer to the request concerning Z's assertion, in any context in which $A \rightarrow Z$ and A have both been ruled in. (In granting $A \rightarrow Z$ and A we have settled Z positively. Its denial is ruled out, since to assert A and deny Z amounts to denying $A \rightarrow Z$.)

What we have done here in miniature, we can extend out totally generally, by showing that focussed derivations give us ways to meet justification request, given that we accept the definitions given in the defining rules. The guiding principle is the following SLOGAN:

> A derivation of $X \succ \boxed{A}, Y$ shows us how to meet a justification request for the assertion of A in any available position extending $[X : Y]$.
>
> A derivation of $X, \boxed{A} \succ Y$ shows us how to meet a justification request for the denial of A in any available position extending $[X : Y]$.

This slogan is plausible, in a sequent system with the following *focussing rules*. First, the structural rules:

$$X, A \succ \boxed{A}, Y \qquad X, \boxed{A} \succ A, Y$$

$$X, A \succ A, \boxed{B}, Y \qquad X, A, \boxed{B} \succ A, Y$$

$$\dfrac{X \succ \boxed{A}, Y \quad X, A \succ \boxed{B}, Y}{X \succ \boxed{B}, Y}\ Cut \qquad \dfrac{X \succ \boxed{A}, Y \quad X, A, \boxed{B} \succ Y}{X, \boxed{B} \succ Y}\ Cut$$

$$\dfrac{X \succ \boxed{B}, A, Y \quad X, \boxed{A} \succ Y}{X \succ \boxed{B}, Y}\ Cut \qquad \dfrac{X, \boxed{B} \succ A, Y \quad X, \boxed{A} \succ Y}{X, \boxed{B} \succ Y}\ Cut$$

$$\dfrac{X, A, \boxed{A} \succ Y}{X, \boxed{A} \succ Y}\ W \qquad \dfrac{X \succ A, \boxed{A}, Y}{X \succ \boxed{A}, Y}\ W$$

Notice that in the presence of focussing, we need to include contraction (W) rules, even though the LHS and RHS of a sequent is still a *set*.

These structural rules suffice for us to *swap* the focus from one point to another in a sequent, for example, like this:

$$\dfrac{\dfrac{X \succ \boxed{A}, B, Y \quad X, A \succ A, \boxed{B}, Y}{X \succ A, B, \boxed{B}, Y}\ Cut}{X \succ A, \boxed{B}, Y}\ W$$

So all four of these *Swap* rules are derivable:

$$\dfrac{X \succ \boxed{A}, B, Y}{X \succ A, \boxed{B}, Y}\ Swap \qquad \dfrac{X, \boxed{A}, B \succ Y}{X, A, \boxed{B} \succ Y}\ Swap$$

$$\dfrac{X, \boxed{A} \succ B, Y}{X, A \succ \boxed{B}, Y}\ Swap \qquad \dfrac{X, A \succ \boxed{B}, Y}{X, \boxed{A} \succ B, Y}\ Swap$$

The defining rules for connectives can be focussed in the following way:

$$\dfrac{X, \boxed{A}, B \succ Y}{X, \boxed{A \wedge B} \succ Y}\ \wedge Df \qquad \dfrac{X, A, \boxed{B} \succ Y}{X, \boxed{A \wedge B} \succ Y}\ \wedge Df$$

$$\dfrac{X \succ \boxed{A}, B, Y}{X \succ \boxed{A \vee B}, Y}\ \vee Df \qquad \dfrac{X \succ A, \boxed{B}, Y}{X \succ \boxed{A \vee B}, Y}\ \vee Df$$

$$\dfrac{X \succ \boxed{A}, Y}{X, \boxed{\neg A} \succ Y}\ \neg Df \qquad \dfrac{X, A \succ \boxed{B}, Y}{X \succ \boxed{A \rightarrow B}, Y}\ {\rightarrow}Df \qquad \dfrac{X, \boxed{A} \succ B, Y}{X \succ \boxed{A \rightarrow B}, Y}\ {\rightarrow}Df$$

and with these rules, any derivation of a sequent $X \succ A, Y$ can be transformed into a focussed derivation of $X \succ \boxed{A}, Y$, and any derivation of a sequent $X, A \succ Y$ can be transformed into a focussed derivation of $X, \boxed{A} \succ Y$. In this way, a sequent derivation of $X \succ A, Y$ can be understood as providing instructions for meeting a justification request for an assertion of A in any available position involving $[X : Y]$.

In this way, classical sequent derivations can be used to answer justification requests, using the structural rules (basic conditions connecting assertion and denial *as such* to justification requests) and the defining rules for connectives. These defining rules manipulate the common ground in ways that are familiar from natural deduction, but are less the focus of theories of the common ground. Consider $\rightarrow Df$:

$$\dfrac{X, A \succ \boxed{B}, Y}{X \succ \boxed{A \rightarrow B}, Y}\ {\rightarrow}Df \qquad \dfrac{X, \boxed{A} \succ B, Y}{X \succ \boxed{A \rightarrow B}, Y}\ {\rightarrow}Df$$

These rules read as follows: to prove $A \rightarrow B$ (to meet a justification request for the assertion of $A \rightarrow B$), you can *rule* A *in* (that is, *suppose* it, or freely add it to the positive common ground) and now prove B. That is, you now meet a justification request for the assertion of B, under the scope of the assumption of A. Or, another method is to *rule* B *out* (suppose it, that is freely add it to the negative common ground), and then refute A. That is, you now meet a justification request for the denial of A, under the scope of that assumption. Here the common ground is manipulated in ways that are familiar to us: we grant something *for the sake of the argument*, and then discharge that granting, once we have concluded that part of our reasoning—that is, when we have discharged the justification request.

Let's spell this out with a longer example, showing how the different structural and connective rules correspond to dialogue steps and shifts in the common ground. Start with this focussed derivation of the classical sequent $\succ ((p \rightarrow q) \rightarrow p) \rightarrow p$, *Peirce's Law*.

$$\dfrac{\dfrac{\dfrac{\dfrac{\boxed{p} \succ p, q}{\succ p, \boxed{p \rightarrow q}}\ {\rightarrow}Df \quad \dfrac{(p \rightarrow q) \rightarrow p \succ \boxed{(p \rightarrow q) \rightarrow p}}{(p \rightarrow q) \rightarrow p, p \rightarrow q \succ \boxed{p}}\ {\rightarrow}Df}{(p \rightarrow q) \rightarrow p \succ p, \boxed{p}}\ Cut}{(p \rightarrow q) \rightarrow p \succ \boxed{p}}\ W}{\succ \boxed{((p \rightarrow q) \rightarrow p) \rightarrow p}}\ {\rightarrow}Df$$

[17] A focussed derivation is a derivation in which every sequent has a formula *in focus*. I will mark the focussed formula in a sequent with a shaded box, like $\boxed{\text{this}}$.

We use this derivation to guide a dialogue, with an aim to meet a justification request for an assertion of an instance of Peirce's Law. Here is one example:

ELOISE: $((p \to q) \to p) \to p$.

ABELARD: *Really*? I can never understand conditionals that are deeply left-associated. Why on earth is that true?

ELOISE: Let's grant $(p \to q) \to p$. I'll now show p.

ABELARD: OK, granted.

CG: *The common ground is now* $[(p \to q) \to p : \ ]$.

ELOISE: To show p, let's first rule it out, and if we can show p *then*, it follows regardless.

ABELARD: If you think that'll help, I'll let you grant it. (It seems like ruling p out would make it harder to prove, not easier.)

CG: $[(p \to q) \to p : p]$

ELOISE: Now, given that p is ruled out, we can prove $p \to q$, since if we also rule q out, we have a refutation of p.

ABELARD: I grant that.

CG: $[(p \to q) \to p, p \to q : p]$ (That was using $p \succ p, q$, and then discharging the p, the left branch of the derivation, using $\to Df$.)

ELOISE: Now we've granted $(p \to q) \to p$ and $p \to q$. You can see where we're going now, can't you? It follows that p.

ABELARD: I see that. This is *modus ponens*.

CG: $[(p \to q) \to p, p \to q, p : p]$

ABELARD: But hang on!? I'm feeling a bit queasy now. Haven't we just asserted p *and* denied it? Aren't we out of bounds?

ELOISE: That's right. But remember, I have ruled p out merely for the sake of the argument. We've managed to show that p is unavoidable: even when we tried to deny it, it came back. So, we can get rid of that assumption. We've shown p.

CG: $[(p \to q) \to p, p : \ ]$ (We remove the denial of p and the other things derived under the scope of that assumption—here, just the assertion $p \to q$—leaving the final p.)

ABELARD: Phew. That feels better. But what were we trying to prove? I forgot.

ELOISE: Remember: you asked me about my claim that $((p \to q) \to p) \to p$. I said I'd prove it by assuming $(p \to q) \to p$ and showing that p is true.

ABELARD: That's right, I remember now.

ELOISE: But we've done it! Notice, we've shown that p.

ABELARD: You're right. We granted $(p \to q) \to p$, and using this, we showed that p. It follows that *if* $(p \to q) \to p$ *then* p.

ELOISE: Which was what you asked me to prove.

CG: $[((p \to q) \to p) \to p : \ ]$.

What goes for this derivation can go for *any* focussed derivation. It follows, that we have answers to our original worries about the relationship between the multiple conclusion classical sequent calculus and proofs and *inference*.

- If we understand a *conclusion* of a *proof* the meeting of a justification request, we can see why this kind of conclusion is *single*.
- Since both assertions and denials can be the target of a justification request, this single conclusion can be in the *right* or the *left* of a sequent.
- The making of an *inference* is a (possibly preemptive) answer to a justification request.
- A derivation of a sequent $X \succ A, Y$ $[X, A \succ Y]$ can be transformed into a *procedure* for meeting a justification request for an assertion of A [denial of A] in any available position, appealing only what is granted in $[X : Y]$, and to the defining rules used in that derivation.

But we can do more than answer *those* original concerns. This characterisation of defining rules also have a clearer grasp of the value of derivations, and the role of proof in expanding our knowledge. Having a proof allows us to *do* something that we cannot do without it.

- The bounds, by themselves, can transcend our grasp.
- Is $[PA : GC]$ out of bounds? Who knows? The bounds in the language of first order predicate logic are undecidable.
- Derivations provide one way we can *grasp* complex bounds and *enforce* them.
- The *negative* view of the bounds is seen in the clash between assertion and denial, and the *positive* view of the bounds is found in the answers we can give to justification requests. *Both* have their role in characterising norms governing our speech acts.
- Adopting *defining rules* is one way to be *very* precise about the norms governing the concepts so defined, combining *safety* (we have conservative extension results, showing that the expansion of our language with concepts given by defining rules is conservative, given some natural conditions [27, 28]), *univocity* (these rules introduce *uniquely defined* concepts) and *expressive power* (concepts like the connectives and quantifiers allow us to enrich our language into something much more expressive than languages without the resources to express any such concepts).

I stand by the analysis of the classical sequent calculus I gave over 16 years ago when first presenting "Multiple Conclusions", but the perspective we can gain on these issues with years of hindsight, and by taking a different vantage point, moving beyond considering only assertions and denials to a wider cluster of speech acts, including polar questions and justification requests, allows the resulting story to be both richer and deeper.

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