DEFINING RULES, PROOFS AND COUNTEREXAMPLES

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My aim: To present an account of defining rules, with the aim of explaining these rules they play a central role in analytic proofs. Along the way, I’ll explain how Kreisel’s squeezing argument helps us understand the connection between an informal notion of validity and the notions formalised in our accounts of proofs and models, and the relationship between proof-theoretic and model-theoretic analyses of logical consequence.

1 POSITIONS AND BOUNDS

Positions collect together assertions and denials [X : Y]. Assertions and denials are moves in a communicative practice. I can deny what you assert. We can assert or deny the same thing. We can also retract assertions and denials. I can try on assertion or denial hypothetically (suppose p — then q…) Asserting or denying involves taking a stand on some matter. Assertion and denial clash. Ask the question: p? Answering yes amounts to the assertion of p, while answering no amounts to its denial.

Not all uses of ‘no’ have the same force. Consider the difference between these dialogues: Greg: Is Jen in the study? Lesley: No. She’s outside. (This ‘no’ is a strong denial, adding ‘Jen is in the study’ to the negative side of the common ground.) Greg: Jen is in the study. Lesley: No. She’s either in the study or outside. (This ‘no’ is a weak denial, retracting ‘Jen is in the study’ from the positive side of the common ground.)

Maybe there is a speech act of weak assertion to parallel weak denial, expressed by “perhaps p”, which might retract p from the negative side of the common ground [9].

The bounds on positions — (1) Identity: [A : A] is out of bounds. (2) Weakening: If [X : Y] is out of bounds, so are [X, A : Y] and [X : A, Y]. (3) Cut: If [X, A : Y] and [X : A, Y] are out of bounds, so is [X : Y]. A position that is out of bounds is overcommitted [5]. If a position is not out of bounds, we call it available.

On Cut: Suppose [X : Y] is available, but [X, A : Y] is out of bounds. Ask the question: A? The answer no is forced as a yes answer is excluded (given our other commitments in [X : Y]).

Structural rules: These govern assertions and denials as such.

\[ A \rightarrow A \text{ Id} \]

\[ X \rightarrow A, Y \]

\[ X, X' \rightarrow Y, Y' \]

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they conservatively extend the original vocabulary (if a position
was safe before we added the concept, it’s still safe afterwards) [8].
¶ Concepts defined in this way also play useful dialogical roles.
They increase our expressive power. Once we have conjunction,
for example, I can disagree with your assertion of A and B
without disagreeing with A or disagreeing with B. They are
also subject-matter-neutral. To use Brandom’s terms, and the new
concepts make explicit some of what was previously merely
implicit [3].

3 WHAT PROOFS ARE, AND WHAT THEY DO

Consider a tiny proof, consisting of a single step of modus po-
nens: If it’s Thursday, I’m in Melbourne. It’s Thursday. So, I’m in Mel-
bourne. Here, we have two assertions (the premises), a connect-
ing “so” and another assertion (the conclusion). This proof cru-
циально uses the conditional. If we mean “—” by that “if,” then since
A → B → A → B, we have A → B, A → B (using the defining rule
for “—”). ¶ Hence, a position in which I assert “If it’s Thursday, I’m
in Melbourne” and “It’s Thursday” but I deny “I’m in Melbourne”
is out of bounds. So, “I’m in Melbourne” is undeniable, and the asser-
tion makes explicit what was previously implicit in granting the
premises. ¶ We can show that the defining rules shown here give
rise to the standard sequent calculus for classical logic [8]. The
steps in a Gentzen-style derivation for X → Y can be grounded in
the defining rules as definitions of the concepts appealed to in that
derivation. ¶ A PROOF for the sequent X → Y shows that the po-
tition [X : Y] is out of bounds, by way of defining rules for the
concepts used in X and Y. ¶ Proofs in this sense are analytic. ¶
Proofs can contain a mix of assertions and denials. A proof of
A, B → C, D, for example, can be understood as a proof of C from
the position [A, B : D], or a refutation of A from the position
[B : C, D].

4 COUNTEREXAMPLES & KREISEL’S SQUEEZE

If X → Y is not derivable, then the position [X : Y] can be enlarged
into a partition [X′ : Y′] of the original language, supplemented
with a countable collection of new names [7]. (This is one way to
understand Henkin’s construction in the completeness proof for
first order predicate logic.) ¶ Running the Cut rule in reverse, if
[X : Y] is available, then either [X, A : Y] or [X : A, Y] is available.
Consider each sentence in the language in turn, and add it to the
left or the right in your position and continue … ¶ At the limit
of this process, we have a partition [X′ : Y′] making a verdict on
each sentence of the language, and we never have a derivation of
X → Y for any X ⊆ X′ and Y ⊆ Y′. ¶ Such a partition [X′ : Y′]
can be viewed as giving rise to a model, since it satisfies the
truth conditions expected of Tarski’s models for first order logic,
according to which the formulas in X′ are true and those in Y′ are
false.

A ∈ X′ iff ¬A ∈ Y′, A ∧ B ∈ X′ iff A ∈ X′ and B ∈ X′.
A ∨ B ∈ X′ iff A ∈ X′ or B ∈ X′.
A → B ∈ X′ iff A ∈ Y′ or B ∈ X′.
(∀x)A ∈ X′ iff A[b/n] ∈ X′ for each name n.
(∃x)A ∈ X′ iff A[b/n] ∈ X′ for some name n.

We can think of a model, then, as the limit of a process of filling
out a finite starting position. The completeness theorem states
that if a sequent X → Y is not derivable, then it may be extended
by some limit position [X′ : Y′]—a model where each member of X
is true and each member of Y is false.²

Now we have the resources to answer the following question:
Given that the connectives and quantifiers are defined in the way
given by these rules, is the logic determined by those rules correct
and comprehensive? ¶ This is the question that Kreisel’s squeeze-
ing argument addresses [4]. ¶ An argument from X to Y is infor-
мally valid if and only if there is a clash involved in asserting each
member of X and denying each member of Y. ¶ First, if X → Y is
formally derivable, then it is informally valid. Why? Because the
axiomatic sequents are informally valid (there is always a clash in-
volved in asserting A and denying A), and the rules show how as-
sertions/denials involving complex vocabulary can be understood
in terms of assertions/denials involving simpler vocabulary. We
understand them to be definitions of those concepts, in the sense
of being rules for their use. So, formal derivations underwrite in-
formal validity for sequents using these concepts. ¶ If X → Y is un-
derivable then there is some model according to which all of X holds
and all of Y fails. This model is uniquely determined by the domain
(the family of names in the extended language), and the verdict it
makes on each primitive sentence (of the form Fr1, …, nm). Pro-
vided that each primitive sentence is taken to be logically inde-
pendent of any other (there is no clash involved in asserting Fab
and denying Gcd, for example), the model shows how there is no
clash involved in asserting each member of X and denying each
member of Y. Given that you can take any position (assert/deny)
on any primitive sentence, without any clash, the model gives you
the reassurance that the position [X : Y] is indeed clash-free. ¶ So,
with this proviso, that the primitive non-logical vocabulary hides
no clashes of its own, informal validity coincides exactly with for-
mal validity. In other words, informal validity in virtue of logical
form (understood as first order logical form) coincides with formal
validity. The squeezing argument shows that formal logic is sound
and complete for the informal notion.⁴

REFERENCES

http://consequently.org/writing/pluralism.

²This is a little more complex if we include the identity predicate in the lan-
guage. We need to add not only the stock of fresh names, but a stock of fresh predi-
cates, and the domain of the model is not simply the collection of names, but equi-
ivalence classes under the relation of identity in the limit position. If a = b ∈ X′,
then the names a and b denote the same object in the model, their equivalence class.
³So, a single-conclusion sequent X → A is informally valid if asserting the
premises X makes the conclusion A undeniable.
⁴This understanding of Kreisel’s argument leaves open that other formal
logics—intuitionistic logic, a paraconsistent logic, etc.—may be sound and com-
plete for other intuitive notions of logical validity. We had to fix the particular
conception of validity (there is a clash between asserting the premises and deny-
ing the conclusion of the argument) to get the argument off the ground. One can
be a pluralist about validity and have different formal notions corresponding to
different informal validity concepts [1, 2].


