DEFINING RULES, PROOFS AND COUNTEREXAMPLES

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My aim: To present an account of *defining rules*, with the aim of explaining these rules they play a central role in *analytic proofs*. Along the way, I'll explain how Kreisel's *squeezing argument* helps us understand the connection between an informal notion of validity and the notions formalised in our accounts of proofs and models, and the relationship between proof-theoretic and model-theoretic analyses of logical consequence.

1 POSITIONS AND BOUNDS

POSITIONS collect together *assertions* and *denials* $[X : Y]$. ¶ *Assertions* and *denials* are moves in a communicative practice. I can *deny* what you *assert*. We can assert or deny the same thing. We can also *retract* assertions and denials. I can try on assertion or denial hypothetically (suppose p — then q…) ¶ Asserting or denying involves *taking a stand* on some matter. ¶ Assertion and denial *clash*. ¶ Ask the question: p? Answering YES amounts to the assertion of p, while answering NO amounts to its denial.

Not all uses of '*no*' have the same force. Consider the difference between these dialogues: ¶ *Greg*: Is Jen in the study? *Lesley*: No. She's outside. (This 'no' is a *strong* denial, adding 'Jen is in the study' to the negative side of the common ground.) ¶ *Greg*: Jen is in the study. *Lesley*: No. She's either in the study or outside. (This 'no' is a *weak* denial, retracting 'Jen is in the study' from the positive side of the common ground.)

Maybe there is a speech act of weak assertion to parallel weak denial, expressed by "perhaps p", which might retract p from the negative side of the common ground [9].

The BOUNDS on positions — (1) IDENTITY: $[A : A]$ is out of bounds. (2) WEAKENING: If $[X : Y]$ is out of bounds, so are $[X, A : Y]$ and $[X : A, Y]$. (3) CUT: If $[X, A : Y]$ and $[X : A, Y]$ are out of bounds, so is $[X : Y]$. ¶ A position that is *out of bounds* is overcommitted [5]. ¶ If a position is not out of bounds, we call it *available*.

On *Cut*: Suppose $[X : Y]$ is available, but $[X, A : Y]$ is out of bounds. Ask the question: A? The answer *no* is forced as a *yes* answer is excluded (given our other commitments in $[X : Y]$).

STRUCTURAL RULES: These govern assertions and denials *as such*.

$$A \succ A \ \ \mathit{Id} \qquad \frac{X \succ A, Y \quad X', A \succ Y'}{X, X' \succ Y, Y'}\ \mathit{Cut}$$

$$\frac{X \succ Y}{X \succ A, Y}\ K \qquad \frac{X \succ Y}{X, A \succ Y}\ K$$

$$\frac{X \succ A, A, Y}{X \succ A, Y}\ W \qquad \frac{X, A, A \succ Y}{X, A \succ Y}\ W$$

The vocabulary of bounds has a significant expressive power.

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AN EXAMPLE: THE COMPARATIVES: Consider the following forms of judgements: Fs, a simple predication of F; $s >_F t$, the comparative *more* F *than*; $s \geqslant_F t$, the comparative *at least as* F *as*. These are plausibly coordinated by the following norms:

strong transitivity:	$s >_F t,\ t >_F u \succ s >_F u$
weak transitivity:	$s \geqslant_F t,\ t \geqslant_F u \succ s \geqslant_F u$
strong irreflexivity:	$s >_F s \succ$
weak reflexivity:	$\succ s \geqslant_F s$
contraries:	$s >_F t,\ t \geqslant_F s \succ$
subcontraries:	$\succ s >_F t,\ t \geqslant_F s$
strength:	$s >_F t \succ s \geqslant_F t$
preservation:	$Fs,\ t \geqslant_F s \succ Ft$

These characterise *totally ordered sets* with an *upwardly closed subset*.

2 DEFINITIONS

How do you define a concept? By showing people how to *use* it. ¶ Definitions come in a number of flavours. One is obvious, and one is less so.

EXPLICIT DEFINITION: Define a concept by showing how you can compose this concept out of more primitive concepts ¶ (x *is a square* $=_{df}$ x *is a rectangle* $\land$ *all sides of* x *are equal in length*) ¶ Concepts given an explicit definition are sharply delimited (contingent on accepting the definition, of course). Logical concepts like conjunction, disjunction, negation, the (material) conditional, the quantifiers, and identity are similarly sharply delimited, but they cannot be given explicit definition. (They are *used* in giving explicit definitions.)

DEFINITION THROUGH A RULE FOR USE: Define a concept by showing it could be added to one's vocabulary, giving rules for interpreting assertions and denials involving that concept. ¶ $[X, A \otimes B : Y]$ is out of bounds iff $[X, A, B : Y]$ is out of bounds.

$$\frac{X, A, B \succ Y}{X, A \land B \succ Y} \qquad \frac{X \succ A, B, Y}{X \succ A \lor B, Y} \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}$$

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \land B, Y} \qquad \frac{X, A \succ Y \quad X, B \succ Y}{X, A \lor B \succ Y}$$

$$\frac{X \succ A, Y}{X, \neg A \succ Y} \qquad \frac{X \succ A|^x_n, Y}{X \succ (\forall x)A, Y} \qquad \frac{X, A|^x_n \succ Y}{X, (\exists x)A \succ Y} \qquad \frac{X, Fs \succ Ft, Y}{X \succ s = t, Y}$$

These *defining rules* can be used to give Gentzen-style *L/R* rules, which suffice to recover the defining rules, given *Id* and *Cut*. ($*Df \leftrightarrow *L/R$) ¶ The *L/R* rules so defined allow us to *simplify Id* to atomic formulas. (*Id* Elimination) ¶ We can transform derivations with *Cut* into derivations without. (*Cut* elimination). ¶ This process is completely systematic. It works for any collection of defining rules. It also works in the absence of *Contraction* or *Weakening*, and also with single conclusion sequents [6, Chapter 6].

The concepts introduced in this way are *uniquely defined* (if you and I follow the same rule, our usages are intertranslatable) and they conservatively extend the original vocabulary (if a position was safe before we added the concept, it's still safe afterwards) [8]. ¶ Concepts defined in this way also play useful dialogical roles. They increase our expressive power. Once we have conjunction, for example, I can disagree with your assertion of A and B without disagreeing with A or disagreeing with B. They are also *subject-matter-neutral*. To use Brandom's terms, and the new concepts *make explicit* some of what was previously merely implicit [3].

3 WHAT PROOFS ARE, AND WHAT THEY DO

Consider a tiny proof, consisting of a single step of *modus ponens*: *If it's Thursday, I'm in Melbourne. It's Thursday*. So, *I'm in Melbourne*. Here, we have two assertions (the premises), a connecting "*so*" and another assertion (the conclusion). This proof *crucially* uses the conditional. If we mean "$\rightarrow$" by that "if," then since $A \rightarrow B \succ A \rightarrow B$, we have $A \rightarrow B, A \succ B$ (using the defining rule for "$\rightarrow$"). ¶ Hence, a position in which I assert "*If it's Thursday, I'm in Melbourne*" and "*It's Thursday*" but I deny "*I'm in Melbourne*" is out of bounds. So, "*I'm in Melbourne*" is undeniable, and the assertion makes explicit what was previously implicit in granting the premises. ¶ We can show that the defining rules shown here give rise to the standard sequent calculus for classical logic [8].[1] The steps in a Gentzen-style derivation for $X \succ Y$ can be grounded in the defining rules as *definitions* of the concepts appealed to in that derivation. ¶ **A PROOF for the sequent $X \succ Y$ shows that the position $[X : Y]$ is out of bounds, by way of defining rules for the concepts used in X and Y.** ¶ Proofs in this sense are *analytic*. ¶ Proofs can contain a mix of assertions and denials. A proof of $A, B \succ C, D$, for example, can be understood as a proof of C from the position $[A, B : D]$, or a refutation of A from the position $[B : C, D]$.

4 COUNTEREXAMPLES & KREISEL'S SQUEEZE

If $X \succ Y$ is *not* derivable, then the position $[X : Y]$ can be enlarged into a *partition* $[X' : Y']$ of the original language, supplemented with a countable collection of new names [7]. (This is one way to understand Henkin's construction in the completeness proof for first order predicate logic.) ¶ Running the *Cut* rule in reverse, if $[X : Y]$ is available, then either $[X, A : Y]$ or $[X : A, Y]$ is available. Consider each sentence in the language in turn, and add it to the left or the right in your position and continue ... ¶ At the limit of this process, we have a partition $[X' : Y']$ making a verdict on each sentence of the language, and we never have a derivation of $X \succ Y$ for any $X \subseteq X'$ and $Y \subseteq Y'$. ¶ Such a partition $[X' : Y']$ can be viewed as giving rise to a *model*, since it satisfies the truth conditions expected of Tarski's models for first order logic, according to which the formulas in X' are *true* and those in Y' are *false*.

- $A \in X'$ iff $\neg A \notin X'$ iff $\neg A \in Y'$,
- $A \wedge B \in X'$ iff $A \in X'$ and $B \in X'$.
- $A \vee B \in X'$ iff $A \in X'$ or $B \in X'$.
- $A \rightarrow B \in X'$ iff $A \in Y'$ or $B \in X'$.
- $(\forall x)A \in X'$ iff $A|^x_n \in X'$ for each name n.
- $(\exists x)A \in X'$ iff $A|^x_n \in X'$ for some name n.

[1] And with a slight modification for the rules for the quantifiers, allowing for "non-denoting" singular terms, you get a sequent calculus for a standard negative free logic. The differences here are not important for the argument I am making.

We can think of a model, then, as the *limit* of a process of filling out a finite starting position. The completeness theorem states that if a sequent $X \succ Y$ is not derivable, then it may be extended by some limit position $[X' : Y']$—a model where each member of X is true and each member of Y is false.[2]

Now we have the resources to answer the following question: Given that the connectives and quantifiers are *defined* in the way given by these rules, is the logic determined by those rules *correct* and *comprehensive*? ¶ This is the question that Kreisel's squeezing argument addresses [4]. ¶ An argument from X to Y is *informally valid* if and only if there is a *clash* involved in asserting each member of X and denying each member of Y.[3] ¶ First, if $X \succ Y$ is *formally derivable*, then it is informally valid. Why? Because the *axiomatic* sequents are informally valid (there is always a clash involved in asserting A and denying A), and the *rules* show how assertions/denials involving complex vocabulary can be understood in terms of assertions/denials involving simpler vocabulary. We understand them to be definitions of those concepts, in the sense of being *rules* for their *use*. So, formal derivations underwrite informal validity for sequents using these concepts. ¶ If $X \succ Y$ is *underivable* then there is some *model* according to which all of X holds and all of Y fails. This model is uniquely determined by the *domain* (the family of names in the extended language), and the verdict it makes on each primitive sentence (of the form $Fn_1 \cdots n_m$). Provided that each primitive sentence is taken to be logically independent of any other (there is no clash involved in asserting Fab and denying Gcd, for example), the model shows *how* there is no clash involved in asserting each member of X and denying each member of Y. Given that you can take any position (assert/deny) on any primitive sentence, without any clash, the model gives you the reassurance that the position $[X : Y]$ is indeed clash-free. ¶ So, with this proviso, that the primitive non-logical vocabulary hides no clashes of its own, *informal* validity coincides exactly with *formal* validity. In other words, informal validity *in virtue of logical form* (*understood as first order logical form*) coincides with *formal validity*. The squeezing argument shows that formal logic is sound and complete for the informal notion.[4]

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[2] This is a little more complex if we include the identity predicate in the language. We need to add not only the stock of fresh names, but a stock of fresh *predicates*, and the domain of the model is not simply the collection of names, but equivalence classes under the relation of identity in the limit position. If $a = b \in X'$, then the names a and b denote the same object in the model, their equivalence class.

[3] So, a single-conclusion sequent $X \succ A$ is informally valid if asserting the premises X makes the conclusion A *undeniable*.

[4] This understanding of Kreisel's argument leaves open that other formal logics—intuitionistic logic, a paraconsistent logic, etc.,—may be sound and complete for *other* intuitive notions of logical validity. We had to fix the particular conception of validity (there is a clash between asserting the premises and denying the conclusion of the argument) to get the argument off the ground. One can be a pluralist about validity and have different formal notions corresponding to different informal validity concepts [1, 2].

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