ISOMORPHISMS IN A CATEGORY OF PROOFS

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I aim to show how a category of propositional formulas and *classical* proofs can give rise to finely grained hyperintensional notions of sameness of content. One notion is *very* finely grained (distinguishing p and $p \wedge p$), others less so. I show that one notion amounts to equivalence in Richard Angell's logic of analytic containment [1].

1 THE CATEGORY OF CLASSICAL PROOFS

Four different *derivations*, and two *proofs*.

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R \quad\approx\quad \dfrac{\dfrac{p \wedge q}{p}}{p \vee q} \quad\approx\quad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L$$

$$\dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R \quad\approx\quad \dfrac{\dfrac{p \wedge q}{q}}{p \vee q} \quad\approx\quad \dfrac{\dfrac{q \succ q}{q \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L$$

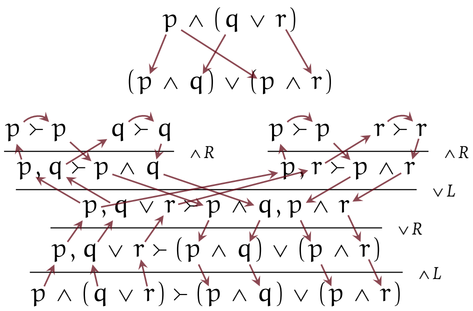
MOTIVATING IDEA: *Proof terms* are an *invariant* for derivations under rule permutation. δ_1 and δ_2 have the same *term* iff some permutation sends δ_1 to δ_2.

$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : p \succ y : p}}{\overset{\acute{\wedge} x \curvearrowright y}{x : p \wedge q \succ y : p}}\,\wedge L}{\overset{\acute{\wedge} x \curvearrowright \acute{\vee} y}{x : p \wedge q \succ y : p \vee q}}\,\vee R \qquad \acute{\wedge} x \curvearrowright \acute{\vee} y \qquad \dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ y : p}}{\overset{x \curvearrowright \acute{\vee} y}{x : p \succ y : p \vee q}}\,\vee R}{\overset{\acute{\wedge} x \curvearrowright \acute{\vee} y}{x : p \wedge q \succ y : p \vee q}}\,\wedge L$$

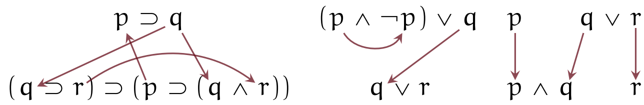
$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{\grave{\wedge} x \curvearrowright y}{x : p \wedge q \succ y : q}}\,\wedge L}{\overset{\grave{\wedge} x \curvearrowright \grave{\vee} y}{x : p \wedge q \succ y : p \vee q}}\,\vee R \qquad \grave{\wedge} x \curvearrowright \grave{\vee} y \qquad \dfrac{\dfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{x \curvearrowright \grave{\vee} y}{x : q \succ y : p \vee q}}\,\vee R}{\overset{\grave{\wedge} x \curvearrowright \grave{\vee} y}{x : p \wedge q \succ y : p \vee q}}\,\wedge L$$

A *proof term* for $\Sigma \succ \Delta$ encodes the flow of information in a proof of $\Sigma \succ \Delta$. They can be represented as directed graphs on sequents [2, 10].

$$\overset{\acute{\wedge} x \curvearrowright \acute{\wedge}\acute{\vee} y \;\; \acute{\wedge} x \curvearrowright \acute{\wedge}\grave{\vee} y \;\; \acute{\vee}\grave{\wedge} x \curvearrowright \grave{\wedge}\acute{\vee} y \;\; \grave{\vee}\grave{\wedge} x \curvearrowright \grave{\wedge}\grave{\vee} y}{x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)}$$



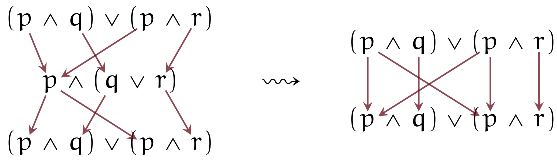
More examples:



Links wholly internal to a *premise* or a *conclusion* are called *cups* ($\smile$) and *caps* ($\frown$).

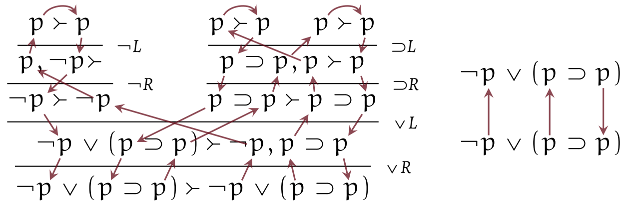
FACTS: Not every directed graph on occurrences of atoms in a sequent is a proof term. ¶ They *typecheck*. [An occurrence of p is linked only with an occurrence of p.] ¶ They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.] ¶ They must satisfy an "enough connections" condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise $p \vee q$ and conclusion $p \wedge q$ is not connected enough to be a proof term.]

Cut is chaining of proof terms, composition of graphs.



Cut elimination is *confluent* and *terminating*. [So it can be understood as a kind of *evaluation*.] ¶ Cut elimination for proof terms is *local*. [So it is easily made parallel.]

$\mathfrak{C}$ *is the Category of Classical Proofs.* OBJECTS: Formulas — A, B, etc. ARROWS: Cut-Free Proof Terms — $\pi : A \succ B$. COMPOSITION: Composition of derivations with the elimination of *Cut* — If $\pi : A \succ B$ and $\tau : B \succ C$ then $\tau \circ \pi : A \succ C$. IDENTITY: Canonical identity proofs — $\mathrm{Id}(A) : A \succ A$.



The category $\mathfrak{C}$ is *symmetric monoidal* and *star autonomous*, but not *Cartesian*, with structural *monoids* and *comonoids*, and is enriched in *SLat* (the category of semilattices) [9]. Being enriched in *SLat* means that proofs terms come ordered by $\subseteq$, and compose under $\cup$, and these interact sensibly with composition.

$$\begin{array}{rcl} \pi \subseteq \pi' & \Rightarrow & \pi \circ \tau \subseteq \pi' \circ \tau \\ \tau \subseteq \tau' & \Rightarrow & \pi \circ \tau \subseteq \pi \circ \tau' \\ \pi \circ (\tau \cup \tau') & = & (\pi \circ \tau) \cup (\pi \circ \tau') \\ (\pi \cup \pi') \circ \tau & = & (\pi \circ \tau) \cup (\pi' \circ \tau) \end{array}$$

$\mathfrak{C}$ is just *classical* propositional logic, in a categorical setting. (The sequent calculus plays no essential role here. You can define proof terms on other proof systems, e.g. *natural deduction*, *Hilbert proofs*, *tableaux*, *resolution*.)

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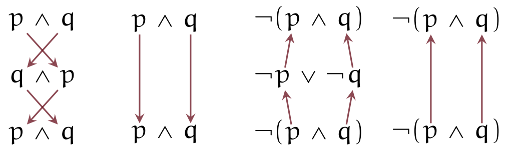
2 ISOMORPHISMS

$f : A \to B$ is an *isomorphism* in a category iff it has an *inverse* $g : B \to A$, where $g \circ f = id_A : A \to A$ and $f \circ g = id_B : B \to B$. (If g and g' are inverses, $g = id_A \circ g = (g' \circ f) \circ g = g' \circ (f \circ g) = g' \circ id_B = g'$, so any inverse is unique. We can call it f^{-1}.)

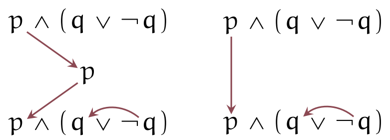
If A and B are isomorphic in a category $\mathcal{C}$, then what we can do with A (in $\mathcal{C}$) we can do with B, too.

If A and B are isomorphic in $\mathfrak{C}$, then they agree not only on *provability*, but also, on *proofs*. The distinctions drawn when you analyse how something is *proved* (from premises), are not far from what you want to understand when you ask how something is *made true*.

Isomorphisms in $\mathfrak{C}$: $p \wedge q \cong q \wedge p$; $\neg(p \wedge q) \cong \neg p \vee \neg q$



Non-isomorphisms in $\mathfrak{C}$: $p \wedge (q \vee \neg q) \not\cong p$; $p \wedge p \not\cong p$; $p \wedge (q \vee r) \not\cong (p \wedge q) \vee (p \wedge r)$; $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$.



Occurrence Polarity Condition: If A is isomorphic to B in $\mathfrak{C}$ then each variable occurs positively [negatively] in A exactly as many times as it occurs positively [negatively] in B. (This condition is *necessary*, not *sufficient*: $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$.)

A is *isomorphic to* B iff A and B are equivalent in the following calculus:

$$\begin{array}{cc} A \wedge B \leftrightarrow B \wedge A, & A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C. \\ A \vee B \leftrightarrow B \vee A, & A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C. \\ \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B, & \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B. \\ \neg\neg A \leftrightarrow A. & A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B). \end{array}$$

(This allows for a *negation normal form*, but not DNF or CNF. These are the equivalences in multiplicative linear logic, where $\wedge$ is understood as $\otimes$, $\vee$ is $\parr$, and $\neg$ is linear negation.)

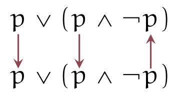
Proof Sketch (Došen and Petrić, 2012 [3]).

If $A \leftrightarrow B$ holds in the calculus, A and B are isomorphic. ¶ A is isomorphic to B iff there are *diversified* A' and B' where A' and B' are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution σ. ¶ A is isomorphic to B iff their negation normal forms are isomorphic. (If A is diversified, so is its negation normal form.) ¶ If A and B are diversified, isomorphic, and in negation normal form, if $l \wedge m$ is a conjunction in A (l and m, literals), a substitution argument (substituting $\top$ and $\bot$ for the *other* atoms) shows that l and m must be conjunctively joined in B, too. The same goes for $l \vee m$. ¶ Replace $l \wedge m$ by a new atom in both A and B, and repeat. ¶ This shows how to reconstruct a proof of equivalence for A and B in the syntactic calculus for isomorphic formulas.

Isomorphism is a very tight constraint: If A and B are isomorphic, they can play *essentially* the same role in proof. ¶ Replacing A by B in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*. ¶ Not even A and $A \wedge A$ are equivalent in *this* sense. ¶ Yet, A and $A \wedge A$ seem to have identical *subject matter* (insofar as we understand that notion). ¶ Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?

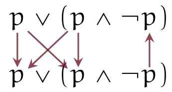
3 MORE PROOFS FROM A TO A

$Id(p \vee (p \wedge \neg p))$



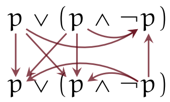
In $Id(A)$, each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion. *Different occurrences of atoms in* A *are treated differently.*

$Hz(p \vee (p \wedge \neg p))$



In $Hz(A)$, each positive [negative] occurrence of an atom in the premise is linked with every positive [negative] occurrence in the conclusion. *We treat occurrences of an atom in* A*—with the same polarity—equally.*

$Mx(p \vee (p \wedge \neg p))$



In $Mx(A)$, each syntactically possible linking is included. *We treat all occurrences of an atom in* A *equally.*

Note: $Hz(A)$ is $Mx(A)$ with the caps and cups removed.

Let's look at relations like isomorphism, but which erase distinctions, up to *Hz* or *Mx*.

Let's say that A and B *Hz*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$. We write "$\approx_{Hz}$" for the *Hz*-matching relation, and we write "$\pi, \pi' : A \approx_{Hz} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Hz*-match between A and B.

Let's say that A and B *Mx*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$. We write "$\approx_{Mx}$" for the *Mx*-matching relation, and we write "$\pi, \pi' : A \approx_{Mx} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Mx*-match between A and B.

Isomorphism $\subseteq$ *Hz-Matching*: If $\pi : A \succ B$ and $\pi^{-1} : B \succ A$, then consider $\pi' = Hz(B) \circ \pi \circ Hz(A)$ and $\tau' = Hz(A) \circ \pi^{-1} \circ Hz(B)$. These satisfy the *Hz*-matching criteria, $\tau' \circ \pi' = Hz(A)$ and $\pi' \circ \tau' = Hz(B)$.

Hz-Matching $\subseteq$ *Mx-Matching*: If $\pi, \pi' : A \approx_{Hz} B$, then consider $\tau = Mx(B) \circ \pi \circ Mx(A)$ and $\tau' = Mx(A) \circ \pi' \circ Mx(B)$. These satisfy the *Mx*-matching criteria, $\tau' \circ \pi' = Mx(A)$ and $\pi' \circ \tau' = Mx(B)$.

Mx-Matching $\subseteq$ *Logical Equivalence*: If $A \approx_{Mx} B$ then there are proofs $\pi : A \succ B$ and $\tau : B \succ A$.

Matching Relations are Equivalences: REFLEXIVE $Hz(A), Hz(A) : A \approx_{Hz} A$. $Mx(A), Mx(A) : A \approx_{Mx} A$. ¶ SYMMETRIC If $\pi, \pi' : A \approx_{Hz} B$, then $\pi', \pi : B \approx_{Hz} A$. If $\pi, \pi' : A \approx_{Mx} B$, then $\pi', \pi : B \approx_{Mx} A$. ¶ TRANSITIVE If $\pi, \pi' : A \approx_{Hz} B$ and $\tau, \tau' : B \approx_{Hz} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Hz} C$. If $\pi, \pi' : A \approx_{Mx} B$ and $\tau, \tau' : B \approx_{Mx} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Mx} C$.

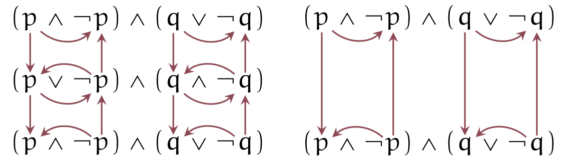
Matchings: $p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$; $p \wedge (q \vee r) \approx_{Hz} (p \wedge q) \vee (p \wedge r)$.

Mx-Matching $\subset$ *Logical Equivalence*: If an atom p occurs positively [negatively] in A but not in B, then A and B do not *Mx*-match.

PROOF: $Mx(A) : A \succ A$ contains the link from [to] that occurrence of p in the premise A to [from] its corresponding occurrence in the conclusion A. ¶ No proof from A to B contains a link from [to] that occurrence to anything in B (since there is no positive [negative] occurrence in B at all). ¶ So, in the composition proof from A to A, there is no link from [to] the premise occurrence to [from] the conclusion occurrence. No proof from A to B and back can recreate $Mx(A)$.

COROLLARY: $p \not\approx_{Mx} p \wedge (q \vee \neg q)$; $p \wedge \neg p \not\approx_{Mx} q \wedge \neg q$.

Hz-matching $\subset$ *Mx-matching*: $(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{Mx} (p \vee \neg p) \wedge (q \wedge \neg q)$.



However, $(p \wedge \neg p) \wedge (q \vee \neg q) \not\approx_{Hz} (p \vee \neg p) \wedge (q \wedge \neg q)$. So:

Isomorphism $\subset$ *Hz-Matching* $\subset$ *Mx-Matching* $\subset$ *Logical Equivalence*

4 MATCHING & LOGICS OF ANALYTIC CONTAINMENT

Angell's Logic of Analytic Containment: [AC1] $A \leftrightarrow \neg\neg A$ [AC2] $A \leftrightarrow (A \wedge A)$ [AC3] $(A \wedge B) \leftrightarrow (B \wedge A)$ [AC4] $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$ [AC5] $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$ [R1] $A \leftrightarrow B, C(A) \Rightarrow C(B)$

Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$. You can define $A \to B$ as $A \leftrightarrow (A \wedge B)$.

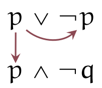
The first degree fragment of *Parry's* Logic of Analytic Containment is found by adding $(A \vee (B \wedge \neg B)) \to A$ to Angell's Logic. (Parry's logic still satisfies this relevance constraint: $A \to B$ is provable only when the atoms in B are present in A.)

First Degree Entailment (FDE) is found by adding $A \to (A \vee B)$ to Angell's Logic. ¶ FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that $p \vee \neg p$, and $q \wedge \neg q$ are both non-trivial, and ineliminable. ¶ A simple translation encodes FDE inside classical logic. Choose, for each atom p, a fresh atom p', its *shadow*. For each FDE formula A, its translation is the formula A' found by replacing the negative occurrences of atoms p in A by their shadows. An argument is FDE valid iff its translation is classically valid.

DEFINITION: $Mx(A, B)$ is the set of all possible linkings which could occur in any proof from A to B. ¶ That is, it contains a link from {positive atoms in A, negative atoms in B} to matching {positive atoms in B, negative atoms in A}.

FACT: $Mx(A, B)$ is a proof iff there is some proof from A to B. (And if so, it is the maximal such proof.)

$Mx(p \vee \neg p, p \wedge \neg q)$ is not a proof:



LEMMA: If $A \approx_{Mx} B$, then $Mx(A, B)$ and $Mx(B, A)$ are proofs, and $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

PROOF: If $\pi, \pi' : A \approx_{Mx} B$, then $\pi \subseteq Mx(A, B)$ and $\pi' \subseteq Mx(B, A)$, so $Mx(A, B)$ and $Mx(B, A)$ are both proofs. ¶ Since $\pi' \circ \pi = Mx(A)$, we have $Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A)$, and similarly, $Mx(B) = Mx(A, B) \circ Mx(A)$, so $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

FACT: If A is classically equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Mx*-match—and conversely.

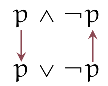
PROOF: If A is logically equivalent to B, then $Mx(A, B)$ and $Mx(B, A)$ are both proofs. ¶ It suffices to show that $Mx(B, A) \circ Mx(A, B) = Mx(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Mx(A, B)$ composed with a link in $Mx(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Mx(A, B)$ and $Mx(B, A)$. ¶ Conversely, if $A \approx_{Mx} B$, we have already seen that A and B must be equivalent, and no atom occurs positively [negatively] in A but not B.

This is *not* Equivalence in Parry's Logic. A is equivalent to B in Parry's logic of analytic containment iff A is classically equivalent to B and the atoms present in A are present in B and *vice versa*. ¶ $(p \wedge \neg p) \wedge q \not\approx_{Mx} (p \wedge \neg p) \wedge \neg q$, but this pair satisfies Parry's variable sharing criterion.

QUESTION: Does the equivalence relation of *Mx*-matching occur elsewhere in the literature?

DEFINITION: $Hz(A, B)$ is the set of all possible linkings which could occur in any proof from A to B, excluding caps and cups. ¶ That is, it contains a link from positive atoms in A to corresponding positive atoms in B and negative atoms in A to corresponding negative atoms in B.

$Hz(p \wedge \neg p, q \vee \neg q)$ contains no links. $Hz(p \wedge \neg p, p \vee \neg p)$ is a proof, but not the maximal one:



FACT: $Hz(A, B)$ is a proof iff A entails B in FDE.

PROOF: From FDE-validity to *Hz*-proof: straightforward induction on an FDE-axiomatisation. ¶ From the *Hz*-proof $Hz(A, B)$ to FDE-validity: Notice that no negative occurrences of atoms in A or B are linked to any positive occurrences of atoms in A or B. So, there is another *Hz*-proof $Hz(A', B')$ for the FDE translations for A and B.

LEMMA: If $A \approx_{Hz} B$, then $Hz(A, B)$ and $Hz(B, A)$ are proofs, and $Hz(A, B), Hz(B, A) : A \approx_{Hz} B$.

PROOF: If $\pi, \pi' : A \approx_{Hz} B$, then then since $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$, π and π' are cap- and cup-free, so $\pi \subseteq Hz(A, B)$ and $\pi' \subseteq Hz(B, A)$, so $Hz(A, B)$ and $Hz(B, A)$ are both proofs. ¶ Since $\pi' \circ \pi = Hz(A)$, we have $Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$, and similarly, $Hz(B) = Hz(A, B) \circ Hz(A)$, so $Hz(A, B), Hz(B, A) : A \approx_{Mx} B$.

FACT: If A is FDE-equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Hz*-match — and conversely.

PROOF: If A is FDE-equivalent to B, then $Hz(A, B)$ and $Hz(B, A)$ are both proofs. ¶ It suffices to show that $Hz(B, A) \circ Hz(A, B) = Hz(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Hz(A, B)$ composed with a link in $Hz(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Hz(A, B)$ and $Hz(B, A)$. ¶ Conversely, if $A \approx_{Hz} B$, we have already seen that

A and B must be FDE-equivalent, and no atom occurs positively [negatively] in A but not B.

FACT: (Ferguson 2016 [4]; Fine 2016 [5]) A is equivalent to B in Angell's logic of analytic containment iff A is FDE equivalent to B, and any atom occurs positively [negatively] in A iff it occurs positively [negatively] in B.

So, *Hz-matching* $\equiv$ *Angellic Equivalence*.

5 MATCHING AS ISOMORPHISM

$Hz(A)$ and $Mx(A)$ are Idempotents: $Hz(A) \circ Hz(A) = Hz(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.

For any category $\mathcal{C}$, if i_A is an idempotent for each object A, we can form a new category $\mathcal{C}_i$ with the same objects as $\mathcal{C}$, and with arrows $i_B \circ f \circ i_A : A \to B$. ¶ In this new category, the idempotents i_A are the new identity arrows. ¶ So, $\mathfrak{C}_{Hz}$ and $\mathfrak{C}_{Mx}$ are both categories — like $\mathfrak{C}$, but less discriminating, with fewer arrows.

Hz-matching is *isomorphism* in $\mathfrak{C}_{Hz}$.

Mx-matching is *isomorphism* in $\mathfrak{C}_{Mx}$.

$\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$ are nontrivial, nonetheless.

$$\begin{tikzcd}[column sep=tiny] p \arrow[d] & \wedge & q \\ p & \vee & q \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] p \arrow[d] & \wedge & q \arrow[d] \\ p & \vee & q \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] p & \wedge & q \arrow[d] \\ p & \vee & q \end{tikzcd}$$

These are each different proofs in $\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$.

6 IN CONCLUSION

¶ These results allow for genuinely hyperintensional distinctions to be drawn, using tools that are native to classical proof theory. Proof theoretical resources indigenous to *classical logic* provide tools for fine-grained hyperintensional distinctions, and some of these tools slice at exactly the same joints as have been discerned using very different techniques. It is encouraging to see how non-classical logics like FDE and Angell's logic of analytic containment arise out of proof theoretical considerations in classical logic. (This is not unprecedented. In Chapter I.3 of *Proof Theory and Logical Complexity*, Girard shows how the sequent calculus, under another guise, gives rise to Kleene's 3-valued logic [6].) Here, we have started with the hyperintensionality of the phrase "… proves that …" and shown this has an underlying logical structure and coherence deeper than the surface syntax of a particular representation system for proofs.

¶ Extending these results to include the units $\top$ and $\bot$ are not difficult. (They were left out only to ease the presentation). In short, we allow for degenerate edges for proofs involving the units. For $\succ \top$ we have a link with $\top$ as the target, but with no source. There are *no* links with $\top$ as a source. So, in the identity arrow from $\top$ to $\top$, there is a degenerate link into the conclusion $\top$, and nothing leaving the premise. The situation is reversed for $\bot$. For $\bot \succ$ we have a link *from* $\bot$ going nowhere. This link features in the identity proof for $\bot \succ \bot$.

As for isomorphisms in the calculus with $\top$ and $\bot$, it turns out that $A \vee \bot \approx A \approx A \wedge \top$, $\neg\top \approx \bot$, and $\neg\bot \approx \top$. However, $A \wedge \bot \not\approx \bot$, in general, since this would violate the variable occurrence condition (which still holds). Nonetheless, $\bot \wedge \bot \approx \bot$ and $\bot \vee \bot \approx \bot$ and $\top \wedge \top \approx \top$.

¶ One open question is how to relate these results to *models* of logics of content. Is there a way to move from the family of different proofs for A (from different premises) to *situations* making A true in any rich sense? An immediate issue to be confronted is that proofs—and proof terms—wear their premises and their conclusions on their face. A proof from A to B is not *also* a proof from a different C to a different D. Even though proof terms abstract away from some of the syntactic details of derivations or proofs, they don't abstract away the *premise* and the *conclusion*.

Situations, even though they can be more local and discriminating than possible worlds (or models assigning a truth value to every formula in the language), generally make more than one thing true. To construct situations from proof terms, we must bridge this gap in some way or other.

¶ Another step to consider is whether we can expand these results to first order logic. Some recent work of Dominic Hughes on unification nets for first order multiplicative linear logic [8] brings to light an important distinction for different approaches to proof terms for predicate logic. It is clear that these two derivations here correspond to the one natural deduction proof, and should have the same proof term:

$$\dfrac{\dfrac{Ft \succ Ft}{Ft \succ \exists xFx}\,\exists R}{\forall xFx \succ \exists xFx}\,\forall L \approx \dfrac{\dfrac{\forall xFx}{Ft}\,\forall E}{\exists xFx}\,\exists I \approx \dfrac{\dfrac{Ft \succ Ft}{\forall xFx \succ Ft}\,\forall L}{\forall xFx \succ \exists xFx}\,\exists R$$

But what about two different derivations going through two different intermediate terms, t_1 and t_2? Girard's proof nets for first order MLL take these to be *different* [7]. There is one clear sense, proof theoretically, that the information flows from $\forall xFx$ to $\exists xFx$ in the same way regardless of which term used, so Hughes' unification nets (which abstract away from the identity of the particular unifiers used) seem well motivated on proof theoretic grounds.

However, when it comes to the metaphysics of grounding and subject matter, it seems that there is good reason allow each object that makes $\exists xFx$ true contribute in its own, individual, way. This much seems clear. Different objects witness quantifiers in different ways, and this should be reflected in the detail of truthmakers. However, the *logic* of such distinctions is yet to be understood clearly. Perhaps tools from proof theory will be able to help clarify some of the options to further explore.

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