LECTURE 1 | INFERENTIALISM FOR EVERYONE

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Practice run for the 2025 Wendy Huang Lectures, NTU Taiwan

I have three aims for this set of lectures.

1. To give an account of what is so distinctive about *logic*, insofar as logical notions have a grip on whatever can be *said* or *thought*.
2. To clarify the connections between logic and *semantics*, the theory of meaning.
3. The byproduct of these two aims is that I get to provide a philosophical motivation and non-technical introduction to the more technical work I have done in my forthcoming manuscript *Proof, Rules, and Meaning*. https://consequently.org/w/prm

Today's topic is the fundamentals of *proof*, *rules*, and *meaning*. I aim to give an opinionated introduction to INFERENTIALIST SEMANTICS. ¶ In the second lecture, I will show how this inferentialist semantics applies to issues of predication, quantification, and modality, and thereby provides some distinctive insight into 'possible worlds' models for quantified modal logics. ¶ Then in the final lecture, I show how we might relate inferentialist semantics to truth-conditional accounts of meaning for natural languages.

1.1 LOGIC & SEMANTICS / PROOFS & MODELS

What has logic to do with semantics?

- Logics give an account of the meanings of particular concepts: the logical constants.
- Logical concepts (equivalence, entailment, contradiction, etc.) are useful when accounting for the significance of what is said.
- Logical tools provide frameworks in which semantic theories are formulated. (In the 20th and 21st Centuries, truth-conditional semantic theories are chief among these.)

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Kurt Gödel's *completeness theorem* is, by one measure, the greatest result in 20th Century Logic. The completeness theorem, connecting proofs and models, shows that the concept of logical validity has *two sides*.

Kurt Gödel (1906–1978)

VALIDITY (BY WAY OF PROOFS): $A \vdash B$ says that *there is a proof* from A to B.

VALIDITY (BY WAY OF MODELS): $A \vDash B$ says that *there is no counterexample* to the argument from A to B. (A counterexample is a *model* that says 'true' to the premise and 'false' to the conclusion.)

SOUNDNESS THEOREM: If $A \vdash B$ then $A \vDash B$. (If *there is a proof* then *there is no counterexample*.)

COMPLETENESS THEOREM: If $A \vDash B$ then $A \vdash B$. (If *there is no counterexample* then *there is a proof*.)

A soundness theorem states there is no *overlap* between proofs and counterexamples, and a completeness theorem states that there is no *gap*.

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The soundness and completeness theorems show that the concept of validity can be understood both in terms of models and in terms of proofs. It would be surprising if models were significant for semantics and proofs were not.

However, theorists have, by and large, taken model theory—formulated as it is, in terms of mappings from linguistic items to an extra-linguistic structure, so easily interpreted as a *reference* relation between language and world—as properly *semantics* while relegating proof theory to mere *syntax*.

My aim in this series of lectures is to *explain* and to *exploit* the semantic significance of proof. I will provide a guide to a rigorous *inferentialist* semantics, and then I will *use* that semantics to give a fresh perspective on possible worlds models and truth-conditional semantics for natural language.

1.2 QUESTIONS & ANSWERS / NATURAL DEDUCTION

We start with the first topic—giving an account of what is distinctive about *logic*, insofar as logical notions apply to whatever can be said or thought. What *can* be thought or said? ¶ We start, as usual, considering *declarative* thoughts and sentences: the expressions that we use to make *claims*.

Though, with Nuel Belnap, we should remember that *declaratives are not enough* [2].

There are many ways to study declaratives. We won't start by looking at the *structure* declaratives might have. First, there are some generic features of the class of declarative judgements *as such* which can be isolated, before we turn to their structure.

Nuel Belnap (1930–2024)

When I make a declarative claim, or think a declarative thought, *I take a stand on some issue*.

This is an issue about which others, at least potentially, might *disagree*. If I take it that Queensland will win the Sheffield Shield next year, you might very well disagree. Or, you might not be so confident as me and withhold judgement, or you might not be in a position to even formulate the thought or understand the claim.

See `https://en.wikipedia.org/wiki/Sheffield_Shield`

The items we take to be the constituents of our proofs as premises and conclusions are the kinds of things we can claim, and question, and disagree about, whatever their structure. ¶ How are we to characterise this *disagreement*?

One way is to look at a part of speech paired with the declarative: the polar question. For any claim that A, there is the polar question $A^?$, asking whether A.

To make the claim that A, I take it, has the same upshot as answering *yes* to the question $A^?$. We'll call that speech act *asserting* A. ¶ To say *no* to $A^?$ is to *deny* A.

For more on polar questions, their answers, and how they relate to assertion and denial see my paper 'Questions, Justification Requests, Inference, and Definition' [10].

The answers *yes* and *no* to $A^?$ *disagree*. They take opposing sides on the issue. ¶ The answer *no* stands opposed to *yes*, and conversely, *yes* stands opposed to *no*.

This opposition is not *causal*—my saying *no* does not *prevent* your saying *yes*, or even prevent *my* saying *yes*—it is, in some sense, *normative*.

Given that the aim of this exercise is to settle issues, if I have ruled out one answer to the polar question $A^?$ the answer that remains is the other.

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None of this is to say anything about *proof*, and properly logical notions, such as entailment, contradiction, and equivalence.

A proof is a specific species of a wider genus, of answers to *justification requests*. These are about as fundamental to the notion of making a *claim* as are polar questions. Whenever an assertion is made, a possible response for a hearer is to demur and ask for justification.

If I say that Queensland will win the Sheffield Shield, you may respond: 'Really!? Why?' This indicates that you don't accept my claim, and you ask me to support it.

To *prove* A, given some background context (in which certain given claims are taken for granted) is to *show* that A, in a certain rigorous gap-free sense. Some things follow from this thin characterisation:

'Gap-free' in *what* sense, exactly? Good question. Hold that thought for now. This will be clarified as we go along.

First, proofs can be chained together.

If we have a proof of A from some context $\mathcal{C}$, and we can prove B from $\mathcal{C}', A$ (another context, in which we take A for granted, among other things), then we can prove B from the context $\mathcal{C}, \mathcal{C}'$ by proving B from $\mathcal{C}', A$ (thereby meeting the request to justify B, but incurring the cost of appealing to A), and then using the proof from $\mathcal{C}$ to justify the use of A, on the basis of the background assumptions in $\mathcal{C}'$.

The two proofs π and π' are chained together like this:

$$\begin{array}{c} \mathcal{C} \\ \vdots\,\pi \\ \mathcal{C}',\ A \\ \vdots\,\pi' \\ B \end{array}$$

Proofs, whatever they are, can be chained together. Whatever we take for granted in some context can be justified in another. Proofs, understood in this way, lead from the background context to the conclusion, in a step-by-step fashion.

Chaining proofs together is of little use unless we have some proofs to begin with. ¶ This is the other point at which the background context $\mathcal{C}$ can play a role. If the formula A is already taken for granted in $\mathcal{C}$, we need do nothing else to justify it *in that context*. It is simply taken as given.

We depict the proof of A in a context in which A is taken as given, like this:

$$A$$

The assumptions in a proof are depicted as the (unbracketed—see later) statements that have nothing else written above them to justify them. In the proof of B depicted above, the formulas in $\mathcal{C}$ and $\mathcal{C}'$ are the assumptions.

Using the notation from before, writing $\mathcal{C} \vdash A$ to represent the existence of a proof of A from the context $\mathcal{C}$, we have the following two facts:

- If A is in $\mathcal{C}$ then $\mathcal{C} \vdash A$
- If $\mathcal{C} \vdash A$ and $\mathcal{C}', A \vdash B$ then $\mathcal{C}, \mathcal{C}' \vdash B$.

These are purely *structural* features of proofs and provability.

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What about the polarity between assertion and denial? ¶ A context in which I say *yes* and I say *no* to the same claim is, in some sense, defective. ¶ We will represent *showing* that a context $\mathcal{C}$ is defective by expanding our notion of proof like so.

$$\frac{A \qquad \cancel{A}}{\sharp}$$

A proof ending in '$\sharp$' shows that our context is self-undermining. More generally, if I have a proof from a starting context $\mathcal{C}$ to A, I can extend this into a proof that leads us from $\mathcal{C}, \cancel{A}$ to $\sharp$. We can use the rule to extend a proof in this way:

$$\frac{\begin{array}{c}\mathcal{C}\\ \vdots\,\pi\\ A\end{array} \qquad \cancel{A}}{\sharp}\,\lightning$$

A proof that ends in $\sharp$ does not 'prove $\sharp$'. It is a demonstration that the starting context is out of bounds. The fundamental idea is that a context containing A (*taking* A *for granted*) and $\cancel{A}$ (*ruling* A *out*) is out of bounds.

So, using the turnstile notation, we have $A, \cancel{A} \vdash \sharp$. The context consisting of an assertion and denial of A is defective, or out of bounds.

In a context in which A is granted, adding a *denial* of A is ruled out. Equally, in a context in which A is denied, going on to *grant* A is ruled out. ¶ On the other hand, if I *try* to deny A, and it turns out that this is ruled out, then the context implicitly settles the question $A^?$: it is undeniable. ¶ This grounds the following rule:

$$\frac{\begin{array}{c}\mathcal{C}\ [\cancel{A}]^i\\ \vdots\,\pi\\ \sharp\end{array}}{A}\downarrow^i$$

If the proof π reduces the context $\mathcal{C}$—with the temporary addition of the denial of A—to absurdity, then the context $\mathcal{C}$—now with that denial *discharged*—settles A positively, since ruling it out is excluded.

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We can think of the contexts $\mathcal{C}$ in which certain claims are ruled in and other claims are ruled out as forming the different POSITIONS we are able to formulate using the conceptual resources of the language at hand. ¶ In any language equipped with a notion of proof like this, we will say that a position $\mathcal{C}$ is OUT OF BOUNDS if there is a proof from $\mathcal{C}$ to $\sharp$. (So, $A, \cancel{A}$ is, at the very least, out of bounds.) ¶ We will say that a position is AVAILABLE if it is not out of bounds.

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That's the bacgkround context in which we formulate proofs. We have said nothing yet about the structure of the judgements that populate our proofs. It is to the behaviour of *connectives* that we now turn.

Here are the basic inference rules governing conjunction, from Gentzen's pioneering work on natural deduction [4], popularised and systematised by Dag Prawitz [6].


Gerhard Gentzen (1909–1945)

$$\frac{A \quad B}{A \wedge B}{\wedge}I \qquad \frac{A \wedge B}{A}{\wedge}E \qquad \frac{A \wedge B}{B}{\wedge}E$$

The inference rules for conjunction govern *conjunction*, and nothing else. We are told what it takes to *prove* a conjunction (namely, both the conjuncts) and what you can prove *from* the conjunction (namely, either conjunct). ¶ These rules look as basic as one could hope for. If I am asked why it is the case that $A \wedge B$, and I'm given a reason for A and a reason for B, I don't seem to need anything *else*, above A and above B to justify $A \wedge B$. ¶ Conversely, if I am already in a position to grant the conjunction $A \wedge B$, I need look for no *further* justification to answer the question A, or B. These inferences seem immediate, direct, and as gap-free as one could want.

Similar rules can be given for the conditional, and for negation.

$$\frac{\begin{array}{c}[A]^i\\ \vdots\\ B\end{array}}{A \to B}{\to}I^i \qquad \frac{A \to B \quad A}{B}{\to}E \qquad \frac{\begin{array}{c}[A]^i\\ \vdots\\ \sharp\end{array}}{\neg A}{\neg}I^i \qquad \frac{\neg A \quad A}{\sharp}{\neg}E$$

We can combine these rules, using the background structural setting of proofs described above, to formulate extended proofs. ¶ The first is a proof from $\neg\neg p$ to p. The second, from $\neg(p \to q)$ to $p \wedge \neg q$.

In the second proof, note that the rightmost $\to I$ inference is *vacuous*: zero instances of the assumption p are discharged in the step from q to $p \to q$. Similarly, the first $\downarrow$ inference is vacuous, in that the discharged denial q occurs zero times in the context. This proof system is unapologetically *irrelevant*.

$$\cfrac{\cfrac{\neg\neg p \qquad \cfrac{\cfrac{[p]^1 \quad [\cancel{p}]^2}{\sharp}\lightning}{\neg p}{\neg}I^1}{\sharp}{\neg}E}{p}{\downarrow^2} \qquad \cfrac{\cfrac{\cfrac{\neg(p \to q) \qquad \cfrac{\cfrac{\cfrac{[p]^1 \quad [\cancel{p}]^2}{\sharp}\lightning}{q}{\downarrow}}{p \to q}{\to}I^1}{\sharp}{\neg}E}{p}{\downarrow^2} \qquad \cfrac{\cfrac{\neg(p \to q) \qquad \cfrac{[q]^3}{p \to q}{\to}I}{\sharp}{\neg}E}{\neg q}{\neg}I^3}{p \wedge \neg q}{\wedge}I$$

These rules suffice for classical propositional logic [9], given the bilateral setting in which the context can contain both positive judgements (A) and negative ones ($\cancel{A}$).

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It is compelling to think of the introduction and elimination rules for logical concepts as, in some sense, *definitions* of the concepts they govern. ¶ Someone who says, I grant A and grant B but I'm not sure about $A \wedge B$, is, in some sense, missing the point, and not grasping the concept '$\wedge$'. ¶ It would be a mistake to look for *another premise* to add to A and to B to justify the inference to $A \wedge B$ if our interlocutor does not make the inference on their own. ¶ Facility with the conjunction introduction and elimination rules seems like as a criterion for understanding the concept '$\wedge$', and some such criterion is useful, because it is not as if that understanding can be simply taken as a given.

Taking certain inference rules to be definitional in this sense provides one possible response to the Tortoise's behavior in Lewis Carroll's classic paper [3, 10].

Can we take this thought seriously? Can we treat inference rules as *definitions* of the concepts they govern? ¶ If we could get away with this idea, then it clarifies one of the questions we left hanging: A proof is gap-free in the sense that each of the inference steps is *definitional* of the concepts they govern. ¶ Similarly, the purely structural rules $\downarrow$ and $\lightning$ constrain the field of contents upon which the connective rules operate. ¶ These proofs have no *gaps* because mere competence with the concepts involved underwrites each inference step.

1.3 RULES & DEFINITIONS / INFERENCE & MEANING

Taking introduction and elimination rules as *definitions* is a compelling thought, but it cannot be the end of the matter. As Arthur Prior showed in 1960 [7], if we could simply specify introduction and elimination rules for a putative concept and declare them to be *definitions*, we could prove anything we like. ¶ Prior introduced tonk with two rules:

Arthur Prior (1914–1969)

$$\frac{A}{A \text{ tonk } B}\,\text{tonk}I \qquad \frac{A \text{ tonk } B}{B}\,\text{tonk}E$$

These rules trivialise. From any premise A, you can prove any conclusion B.

$$\cfrac{\cfrac{A}{A \text{ tonk } B}\,\text{tonk}I}{B}\,\text{tonk}E$$

So we must either (1) explain the difference between definitions that are OK to adopt and those that aren't, or (2) say more about what counts as a definition (and find a way to exclude Prior's rules for tonk from the fold, while retaining the rules for connectives like $\wedge$), or (3) give up the 'definition' game entirely. ¶ I choose option (2).

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Here is one proposal for what a definition does for us: It shows us how extend a starting language $\mathcal{L}_1$ with a new item of vocabulary, to form an extended language, $\mathcal{L}_2$, in which we can use the new item in some determinate way.

An *abbreviative* definition is a paradigm case. If $\mathcal{L}_1$ is a language in which I can say 'x has sides all of the same length', then I can extend the language with the term 'equilateral', which is defined as meaning 'having sides all of the same length'. ¶ To adopt the definition is not only to take it that x is equilateral if and only if the sides of x have the same length, but *also*, to grant that a demonstration that x's sides have the same length conclusively answers the question of whether x is equilateral, and *vice versa*. ¶ Someone who took it as given that x's sides were all of the same length, but still took it to be an open question whether x is equilateral would thereby show that either they have not *adopted* the definition, or at least, they have not *mastered* it.

In the language of *proofs*, introducing a newly defined expression E in terms of previously introduced vocabulary D, can be seen as adopting this *invertible* rule:

$$\frac{D}{\overline{E}}\,\text{E}Df$$

In any context at all, from D you can infer E, and *vice versa*. ¶ Here, $\mathcal{L}_2$ is formed by taking the language $\mathcal{L}_1$ (with all its inference rules taken as given), adding the vocabulary item E, and constraining it by imposing the two-way rule E*Df*.

Abbreviative definitions have two useful features. ¶ They are CONSERVATIVE: positions in the original language ($\mathcal{L}_1$) that were available beforehand remain available afterward from the point of view of $\mathcal{L}_2$. ¶ The new vocabulary of $\mathcal{L}_2$ introduces a new rule, and scope for formulating different proofs in $\mathcal{L}_2$, even concerning $\mathcal{L}_1$-vocabulary alone. However, it does not introduce new facts about what is *provable* in the $\mathcal{L}_1$ language, since any $\mathcal{L}_2$-proof may be rewritten as an $\mathcal{L}_1$-proof.

In any $\mathcal{L}_2$-proof, replace every E by D, cut out the E*Df* steps, which are then redundant, and the result is an $\mathcal{L}_1$-proof.

Abbreviative definitions are also UNIQUELY DETERMINED: The newly introduced expression E is *fixed*, in the sense that if we happened to add the term *twice* (E_1 and E_2, say), using definitions of the same form, then the two defined terms are equivalent, in the sense that we can infer one from the other, in any context. ¶ This means that our definition is not merely a schematic specification which could be instantiated in different non-equivalent ways.

Consider the traditional axioms and rules governing the necessity modality in the modal logic S5. These can be instantiated in different ways in the one language. $\Box_1$ and $\Box_2$ could be two different necessity operators, each equally satisfying the traditional axioms and rules for S5. These do not define that modality, but at most, define what it is for an operator to be *an* S5 *necessity*.

CONSERVATIVITY and UNIQUENESS are useful features. A *conservative* definition is safe. (It is not like substantial claim that rules out some positions. No $\mathcal{L}_1$ position becomes out of bounds that wasn't already out of bounds. Every *available* position remains available. ¶ A *uniquely defining* rule means that the concept is as completely delineated as the prior vocabulary. It introduces no new 'give' into the system that wasn't already there.

Definitions that are conservative and uniquely defining (whether abbreviative or not) can be used, safely, as ways to introduce new concepts, given a background context. They close off no positions that were already open, and they give us new capacities we might otherwise not have had. These are the desiderata used by Nuel Belnap in his response to Prior [1], and I endorse this approach and take myself to be elaborating and developing it.

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The inference rules for conjunction, the conditional, and negation, given above are *not* abbreviative definitions. They are also not necessarily conservative, unless we are careful to specify the commitments in our starting language. ¶ Using the conjunction rules alone, we can construct a proof from p, q to p, like so:

$$\dfrac{\dfrac{p \quad q}{p \wedge q}\,{\wedge I}}{p}\,{\wedge E}$$

If our original language $\mathcal{L}_1$ did not tell us that there was a proof from p, q to p, then the addition of these conjunction rules would be non-conservative. ¶ However, our account of proof, motivated in the previous chapter, took it as given that the assumption p on its own, counts not only as a proof from p to p, but also as a proof of p in any context in which p is given, including the context p, q. So, for *us* there remains some chance that these conjunction rules might well be conservative.

If, on the other hand, you wish to explore a more *relevant* notion of consequence, where we admit $p \vdash p$ and exclude $p, q \vdash p$, on the grounds that the q is not used in the derivation of p, then while *these* conjunction rules are unhelpful, other options are open to you [9].

These introduction and elimination rules look rather too much like Prior's rules for tonk for us to be comfortable with them. We will see, though, that with a small tweak, we can see how they are much *closer* to abbreviative definitions than they first appear.

Any rules for $\wedge$, which introduce it to a vocabulary that has no connectives at all, *cannot* be abbreviative. The context $p, q, \cancel{p \wedge q}$ cannot be available in the extended language, while $p, \cancel{p \wedge q}$ and $q, \cancel{p \wedge q}$ are available. There is no formula in the basic language just containing atomic formulas, that (asserted or denied) clashes with p, q together but does not clash with them individually.

Consider these introduction and elimination rules for the conditional:

$$\dfrac{\begin{matrix}[A]^i \\ \vdots \\ B\end{matrix}}{A \to B}\,{\to}I^i \qquad \dfrac{A \to B \quad A}{B}\,{\to}E$$

These do not look anything like *abbreviative definitions*. To get closer to the form of an abbreviative definition, let's make the context explicit: The introduction rule says that to derive $A \to B$ from a context $\mathcal{C}$, you can add A to the context and derive B. The elimination rule tells us that if we can derive $A \to B$ in some context, and A in another, we can derive B in a context that shares the assumptions in the two original contexts. We have:

$$\dfrac{\mathcal{C}, A \succ B}{\mathcal{C} \succ A \to B}\,{\to}I \qquad \dfrac{\mathcal{C} \succ A \to B \quad \mathcal{C}' \succ A}{\mathcal{C}, \mathcal{C}' \succ B}\,{\to}E$$

If, in the *elimination* rule we restrict our attention to the case where $\mathcal{C}'$ is a context in which A alone is assumed, then the two rules are in fact two directions of the one transition:

$$\dfrac{\mathcal{C}, A \succ B}{\overline{\mathcal{C} \succ A \to B}}\,{\to}Df$$

This, at least, is a two-way rule, and one that can be treated in just the same way as an abbreviative definition. ¶ If we have a language $\mathcal{L}_1$ *without* the conditional $\to$ in its vocabulary, we can add the conditional, governed by $\to$*Df* as its defining rule. ¶ This rule is clearly *uniquely defining* (if we happen to apply the defining rule twice, to define two conditionals $\to_1$ and $\to_2$, in any context $\mathcal{C}$, we can prove $A \to_1 B$ if and only if we can prove B from $\mathcal{C}, A$, which holds if and only if we can prove $A \to_2 B$ from $\mathcal{C}$. So, the two conditionals are interchangable.) ¶ That the rule is *conservative* over $\mathcal{L}_1$ requires a rather more work to demonstrate, and the details depend a little on how the starting language $\mathcal{L}_1$ is formulated. The details of exactly what is formally provable is detailed in Chapter 6 of *Proof, Rules, and Meaning*. Suffice to say, *any* family of rules, for a logical connective, whether $\to$*Df*, or of a form such as these:

A "form such as these" is not too difficult to state: What is below the double line is a general position in which an arbitrary instance of a formula with the newly defined n-ary connective dominant, and above the line are some number of sequents consisting of that position, to which only the constituents of that formula are added.

$$\dfrac{\mathcal{C}, A \succ \sharp}{\overline{\mathcal{C} \succ \neg A}}\,{\neg}Df \qquad \dfrac{\mathcal{C} \succ A \quad \mathcal{C} \succ B}{\overline{\mathcal{C} \succ A \wedge B}}\,{\wedge}Df \qquad \dfrac{\mathcal{C}, A \succ C \quad \mathcal{C}, B \succ C}{\overline{\mathcal{C}, A \vee B \succ C}}\,{\vee}Df$$

conservatively extends a base language satisfying the basic structural rules (and perhaps already containing some concepts already given by concepts governed by defining rules from among this family) with the newly introduced concepts. ¶ We can see these rules as showing how to *add* these concepts to our vocabulary, defining exactly how to evaluate claims involving the newly defined concepts.

The putative rules for *tonk* do not have the form of a defining rule, and cannot be transformed into such a form, since they do not even conservatively extend a basic language containing only atomic vocabulary.

If we selected either tonk*I* or tonk*E* to be strengthened into an invertible rule, then the result would of course be safe. However, the first choice would render A tonk B a needlessly complicated way to say A, and the second, a needlessly complicated way to say B.

Viewing the logical concepts as introduced by defining rules, we have a ready answer to how it is that proofs can be gap-free. ¶ Furthermore, the logical concepts used in these proofs are conservative—so they are free to add, in that no pre-existing positions are ruled out by their addition—and uniquely defined. The concepts are as sharply delineated as possible, given the antecedently given field of contents, governed by the structural rules.

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Treating these concepts as given by defining rules does not mean that whenever someone uses the everyday concepts, 'and', 'if', 'or', and 'not' they must be given by these rules, or that the everyday concepts agree with the sharply defined concepts $\wedge$, $\to$, $\vee$ and, $\neg$. ¶ However, insofar as there is unclarity about what we might mean by 'and', 'if', 'or' and 'not', it is always an option that we might clarify our intentions by using the sharply defined concepts, and settling on using the defining rules as definitional for them.

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In what sense do defining rules act as *semantics* for the connectives they govern? In one sense—as giving rules for how to interpret, and to use these concepts—this is completely straightforward. The defining rules are rules for *use* of the governed concepts. ¶ None of this is to do semantics in the model-theoretic sense. We have said nothing about truth, or about truth-in-a-model.

1.4 POSITIONS & LIMITS / MODELS & TRUTH

Nonetheless, there is a natural way to approach models from this proof-first perspective. ¶ Given any available position $\mathcal{C}$, we can say that A is *true-in-$\mathcal{C}$* if $\mathcal{C} \vdash A$, and that A is *false-in-$\mathcal{C}$* if $\mathcal{C}, A \vdash \sharp$. If the connectives are given by the defining rules, we have the following facts immediately:

- $A \wedge B$ is true-in-$\mathcal{C}$ iff A is true-in-$\mathcal{C}$ and B is true-in-$\mathcal{C}$.
- $A \wedge B$ is false-in-$\mathcal{C}$ *if* A is false-in-$\mathcal{C}$ or B is false-in-$\mathcal{C}$.
- $A \vee B$ is true-in-$\mathcal{C}$ *if* A is true-in-$\mathcal{C}$ or B is true-in-$\mathcal{C}$.
- $A \vee B$ is false-in-$\mathcal{C}$ iff A is false-in-$\mathcal{C}$ and B is false-in-$\mathcal{C}$.
- $A \to B$ is true-in-$\mathcal{C}$ *if* A is false-in-$\mathcal{C}$ or B is true-in-$\mathcal{C}$.
- $A \to B$ is false-in-$\mathcal{C}$ iff A is true-in-$\mathcal{C}$ and B is false-in-$\mathcal{C}$.
- $\neg A$ is true-in-$\mathcal{C}$ iff A is false-in-$\mathcal{C}$.
- $\neg A$ is false-in-$\mathcal{C}$ iff A is true-in-$\mathcal{C}$.

Note the *if* in the disjunctive clauses.

Our finite available positions are incomplete. Positions may be ordered by extension in a natural way, by adding more assertions or denials. As a position is extended, it makes more things true and more things false. ¶ It is possible to extend the notion of a position (finite, for the moment) to include infinite positions. Infinite positions (in a given language $\mathcal{L}$) can settle all issues, either positively or negatively. ¶ It is not difficult to show (given the basic structural rules governing positions) that any available finite position $\mathcal{C}$ may be extended, systematically, into *limit* position $\mathcal{C}^*$, settling all issues, one way or another [8]. ¶ In any limit position $\mathcal{C}^*$, the truth conditions given above hold, except that the disjunctive conditions hold as *biconditionals*, rather than conditionals.

A limit position for $\mathcal{L}$ determines a two-valued boolean valuation for the language. In the case that $\mathcal{L}$ is a purely *formal* propositional language, governed by the basic structural rules and the defining rules for the propositional connectives, the limit positions are *exactly* the two-valued boolean valuations, and the usual soundness and completeness result for propositional logic follows as an immediate consequence. ¶ Here, the connectives are *defined* in terms of their inference rules, and their behaviour in *models* arises out of that behaviour. Models are secondary, and proofs are primary.

Limit positions and truth-in-a-limit-position are the inferentialist analogue of models and truth-in-a-model. However, there is more to semantics than truth-in-a-model. We would like to give an account of *truth* [5], and how it relates to truth-in-a-model. What can we say about about truth *per se*? Here, we must step from talking *about* the language we are modelling, to actually *using* it. ¶ Here is how to specify the limit position C^* which counts as a model of truth. For each sentence A in the language $\mathcal{L}$, if A, then you add A to the context, affirmatively. Otherwise, you add its denial, $\cancel{A}$. ¶ Of course, this does not give us an *algorithm* for determining a single position, any more than we have an algorithm for determining what is true in our language $\mathcal{L}$. We have merely *clarified* how truth *tout court* is related to truth-in-a-position.

This is the point at which the sentence A is *used*, and not merely mentioned.

If $\mathcal{L}$ contains context-sensitive expressions, here, we assume that the context of use for asserting A when checking for the position is the same context of use *in* the position so evaluated. Matters are more subtle if we wish to include positions containing judgements with different contexts of use, and context-sensitive expressions.

There is more to be said about the relationship between inference rules, inferentialist semantics in general, and truth-conditional theories of meaning, but this is enough for us to start with.

FOR NEXT TIME

ONE QUESTION: A common criticism about inferentalism, structuralism, and conceptual role semantics is that the semantic relations are all language-internal, and are disconnected from *reality*. This criticism is understandable, but mistaken. In the next lecture I will sketch how an inferentialist semantics can take account of language–world relations.

ONE CHALLENGE: As if that were not enough, I will also given an account of how inferentialist conditions can shed light on the meanings of modal concepts.

ONE INSIGHT: The answers provided for both issues will involve saying more about the background context governing judgements *as such*. Once that is clarified, the details of the definitions of the quantifiers and modal operators—and their interactions—will be relatively straightforward.

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