LECTURE 2 | EXISTENCE & MODALITY

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Practice run for the 2025 Wendy Huang Lectures, NTU Taiwan

In the last lecture, I introduced inferentialist semantics, and I showed how you might understand the traditional propositional logical constants (conjunction, disjunction, negation, and the material conditional) as being *definable* by way of invertible inference rules. ¶ These inference rules have distinctive properties. The defining rule for a concept can be read as instructions for how to introduce that concept into a vocabulary that lacks it, in a way that is conservatively extending and uniquely defining. ¶ This gives us a way to make sense of proofs using those concepts as gap-free. The inferential transitions in those proofs are definitionally analytic. ¶ Proofs, in this sense, are at least as semantically significant as models.

In today's lecture, our aim to extend this account to *quantifiers* and *modal operators*. ¶ As in the last lecture, my exploration is sparked by an insight from Arthur Prior.

Arthur Prior (1914–1969)

> … possible worlds, in the sense of possible states of affairs are not really individuals (just as numbers are not really individuals). To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i.e. if something else were the case … We understand 'truth in states of affairs' because we understand 'necessarily'; not vice versa. [5]

I agree with Prior that possibility and necessity are—in some sense—more conceptually fundamental than possible worlds. ¶ This raises two issues. (1) How *can* we grasp the concepts of possibility and necessity? (2) How is it that those concepts have a structure that makes possible worlds models appropriate for them?

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Before we explore issues of how our language might in some sense be understood as describing a range of non-actual possibilities, let's start with the down-to-earth issue of how we might understand the phenomenon of our thought and talk about things in the world we actually inhabit.

2.1 SAYING THINGS ABOUT THINGS

We can make claims *of* things. ¶ I can not only say that Clark Kent can fly. I can say *of* Clark Kent, that *he* can fly.

If I say of that man (who happens to *be* Clark, unbeknownst to me) that *he* can fly, then I am saying, *of Clark Kent* that he can fly.

I will record the distinction between the two kinds of judgements in our formal grammar, distinguishing Fx (the *de re* judgement of x that it's F), which uses a *variable* x, from Ft (the *de dicto* judgement that t is F), where t is a term, such as a name or description, that might feature in our conceptual lexicon. ¶ With the *de re* judgement Fx, the *context of use* picks out what is being described. Think of the roles of demonstrative expressions, like 'this', 'it', or 'they'. ¶ With the *de dicto* judgement the rules of the language—concerning names, function terms, descriptions, etc.—play some role in selecting the individual, if any, that is named.

One feature relating *variables* and *proof* is SUBSTITUTION. One assumption I will make concerning the semantic rules of our vocabulary is that the specific identity of which variable in use does not matter to the rules of proof. If we have a proof featuring '*this*', it remains a proof if '*this*' is replaced everywhere by '*that*'. ¶ Formally: Given any proof Π of A from background commitments $\mathcal{C}$, if we replace each free x in the proof by y, the result, Π^x_y is a proof of A^x_y from the background commitments $\mathcal{C}^x_y$. ¶ (All that is required to verify this is to check that each of the *rules* is closed under substitution of one variable for another. This is indeed the case for each of the structural rules and defining rules considered so far, and it is a plausible constraint on any putative lexical rule for an item of our vocabulary.) ¶ The constraint means that these object variables are

DETAILS: Any standard definition of substitution in first-order languages will suffice to make the details precise.

In other words, these variables are appropriately *generic*. If a language contains pronouns (e.g. personal pronouns) that are bound by semantic rules not governing more general pronouns, then the narrower class is not inferentially generic, and the general substitution principle will not hold.

each *inferentially general*: no rule holds of one variable that does not hold of the others in this class of variables.

Do we want this substitution principle to hold for *terms*? Can we replace a variable in a proof with an arbitrary *term*? If a derivation holds concerning x, would it hold when the term t is substituted for the variable x? ¶ There are reasons to hesitate. Our language might well contain items in the category of singular terms that do not take a value. There is a number that is $\frac{5}{2}$, but there is no number that is $\frac{1}{0}$. There is no variable that takes $\frac{1}{0}$ as a value, because $\frac{1}{0}$ is not a number, or a thing at all. '$\frac{1}{0}$' is a term that fails to denote.

REMEMBER: I am interested in logic as it applies to whatever we can think or say. We can make claims in languages with non-denoting terms.

In a language with terms that might not denote, but in which *variables* may be assumed to always take values—since we interpret Fx as saying *of the item* x that it's F—we need to mark the distinction between a term denoting and a term failing to denote. ¶ Our positions can rule claims in (by assertion) and rule claims out (by denial). It is natural to extend this to say that *terms* can be ruled out, by rejecting them as non-denoting, and ruled *in*, by taking them to denote. ¶ I will represent this by allowing terms (s, t, etc.) and slashed terms ($\not s$, $\not t$, etc.) in our positions, alongside declarative judgements.

Then, we can define an existence predicate, as follows:

$$\frac{\mathcal{C} \succ t}{\mathcal{C} \succ E!t}\ E!Df$$

To say that t exists is to say that the term 't' denotes. ¶ Since variables are assumed to always take values, we impose $\succ x$ as an axiom, for each variable x.

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For substituting variables by terms, then, a weaker principle follows, which takes a into account the differences between variables and other terms: a free variable may be replaced in a proof by a term, at the cost of incurring the assumption that the term denotes. The following rule is admissible:

$$\frac{\mathcal{C} \succ A}{t, \mathcal{C}^x_t \succ A^x_t}\ Subst^x_t$$

To verify this for a family of rules you must check that each inference rule is closed under substitution of terms for variables, possibly with the added extra premise to the effect that the term in variable denotes. Appeals to the axiom $\succ x$ are replaced by the assumption that t denotes (giving $t \succ t$) and the rest of the proof proceeds as before.

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Given the category of singular terms, the class of variables in that category, and the behaviour of substitution, we are now in a position to define the quantifiers, using invertible defining rules. ¶ The universal and existential quantifiers may be introduced very simply:

$$\frac{\mathcal{C} \succ A}{\mathcal{C} \succ \forall xA}\ \forall Df \qquad \frac{\mathcal{C}, A \succ C}{\mathcal{C}, \exists xA \succ C}\ \exists Df$$

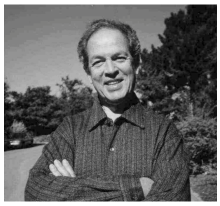
In these rules we have the proviso that the variable x bound by the quantifier does not occur free in among the assumptions $\mathcal{C}$ or the conclusion C (in the existential quantifier rule). ¶ To show that *everything* has some feature, we show that x has that feature, while appealing to nothing about x. This is enough to show that *everything* has this feature, since the proof applies to any *thing* at all. ¶ To use the claim that *something* has a given feature as a premise in your reasoning, use the claim that x has this feature, while making no assumptions about what this x is, and not making a claim about x in your conclusion. This reasoning is enough to show your conclusion for it applies, regardless of what item it is that has the feature in question.

The usual universal quantifier *elimination* rule in natural deduction does not contain the restriction on the assumption context. From $\forall xA$ you can infer A^x_y for *any* y you like, no matter whether it is free or bound in the background assumption context. This follows from the defining rule. $\forall xA \succ \forall xA$ is given, and so, since x is not free in $\forall xA$ (it's bound by the quantifier) we can apply the defining rule to give $\forall xA \succ A$. By *substitution* we therefore have $\forall xA \succ A^x_y$, replacing each free x by y in the conclusion, since the substitution makes no difference to $\forall xA$, since there, each x is bound.

In fact, an appeal to the substitution principle is redundant if formulas identified up to α equivalence. If your formal grammar defines $\forall xFx$ to be *the same formula as* $\forall yFy$ then $\forall yFy \succ Fx$ is a result of applying $\forall Df$ to $\forall yFy \succ \forall yFy$.

Given our assumption that other singular terms may fail to denote, we cannot necessarily conclude $\forall x A \succ A^x_t$. A universally quantified claim only applies to all *things*, and some terms may fail to denote. What does follow is $t, \forall x A \succ A^x_t$, by term substitution. Granted that the term t denotes, if everything has a given feature, t does.

The resulting logic is a simple *free* logic, formulated by Sol Feferman [2, 7]. ¶ If we add the extra assumption that terms denote, this is no more and no less than classical first-order predicate logic.



Solomon Feferman (1928–2016)

I have left out details concerning the behaviour of predicates and function symbols. To get Feferman's negative free logic, we add rules of the form $Ft \succ t$ and $ft \succ t$, constraining predicates to apply to terms only when they denote, and functions having defined values only when their inputs are defined.

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It is reasonably straightforward that the quantifiers, so defined, satisfy the conservative extension and uniqueness criteria. ¶ For CONSERVATIVE EXTENSION, the argument proceeds in the same way as for the propositional constants. (We pass from the invertible defining rule to a pair of left and right sequent rules, and show that the *Cut* rule can be eliminated, and since the remaining rules satisfy the subformula property, if anything can be proved, in the reformulated system, it has a proof that uses only the logical concepts occurring in the item to be proved. So, adding new connectives or quantifiers does not render anything in the old vocabulary provable that was unprovable before that addition.) ¶ For UNIQUENESS, since the quantifiers are defined by an invertible rule, we can pass from $\forall_1 x A$ to $\forall_2 x A$ through the old language, as before.

However, it is worth reflecting on how uniqueness is obtained, since it is well known that there are multi-sorted first-order languages. It is straightforward to have a language with a quantifier that ranges over *objects* (say) and another that ranges over *locations* (say), where predicates are sorted (some apply to objects or relate objects to objects; others apply to or relate places; and yet others relate objects and places). We have object variables and location variables, and as a result we have *distinct* object quantifiers and place quantifiers. ¶ None of this contradicts the uniqueness result given here, since the uniqueness that we have shown depends on the grammatical categorisation of the language under examination. In the derivation showing that $\forall_1$ and $\forall_2$ are equivalent quantifiers, we assume that 1-variables are 2-variables, and *vice versa*. ¶ The meaning of the quantifier expressions given inferentially depends not only on the structural rules governing the field of issues being reasoned about (as we saw in the first lecture), but they *also* require fixing on a grammatical category of singular terms.

This will become a live issue in the next lecture, where we consider what is involved in quantifying over *possible worlds*.

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I have given inference rules for the quantifiers that agree with the standard truth conditions. In none of this discussion have I given a semantics featuring a domain of objects over which the quantifiers range. ¶ Since the inference rules use a notion of substitution, and since substitution in formulas can be defined purely *syntactically*, you might think that this is a substitutional account of the quantifiers [1, 4], according to which a quantified expression $\forall x Fx$ [$\exists x Fx$] is true if and only if every [some] substitution instance of Fx is true. ¶ This is a mistake, and not only because we have allowed for non-denoting terms. Even if we imposed the condition that every term denotes, the truth of every instance Ft, Ft', Fx, Fy, etc., would not be enough to guarantee the truth of $\forall x Fx$. ¶ There is no way to prove the conclusion $\forall x Fx$ from the premises Ft, Ft', Fx, Fy, etc., no matter *how* many terms of the language are included. So, whatever $\forall x Fx$ says, it must say something more than what is said by the instances taken together. This inferentialist semantics is not substitutional in *that* sense.

Modulo a choice for whether to allow non-denoting terms.

Take a model in which there are at least two objects, one of which has feature F and another of which does not, and let's interpret our language so that every term denotes the F-object. Furthermore, assign the value of each variable, *that* object, too. Here, the premises are satisfied, but the conclusion $\forall x Fx$ is not.

What this means for models, domains, and how we *should* interpret the quantifiers, I will leave for Section 4. ¶ Before we get there, we must consider modality.

2.2 SUPPOSING THINGS WERE OTHERWISE

To add quantifiers, we exploited the structure of a language, isolating a class of singular terms. For modal logic, instead, we will return to the kinds of speech acts we can use in our thought and talk. ¶ When we *modalise*—considering matters concerning what is possible and what is necessary—we consider not only what *is* the case, but what *might be*. We do not just assert and deny and ask questions. We *suppose*.

We have already seen supposition, when considering the defining rules for the material conditional and for negation. To prove $A \to B$, we suppose A and derive B. To prove $\neg A$, we suppose A and reduce the result to absurdity. ¶ In these cases, supposition is simple. To suppose A we temporarily add A to the current discursive context, to see what additional consequences we might draw out. (If the current context already rules A out, then this is good news when the aim is to prove $\neg A$.)

Recall, 'absurdity' here is marked in proof by '♯' the sign that we have reached a dead-end, a self-undermining position.

Not all acts of supposition merely add the supposed item to the current context. ¶ When we plan for the future, it is common for us to wonder what might happen, and to take various outcomes into account. We might plan for rainy weather tomorrow and for sunshine. ¶ We might (for the moment) suppose that it *does* rain, and consider this alternative, and then suppose that it *doesn't* rain, and consider what we might do under those circumstances. ¶ That is not very different to ordinary supposition, because we typically do not have very reliable information about the future. However, we can also make suppositions in this way *about the past*. We can reflect, in just the same way, about what would have happened *had it not rained*, or *had it rained*.

There is also a characteristically *epistemic* kind of supposition, arising when we attempt to rationally manage disagreement. You and I might disagree over p, and I might say: suppose you're right, and that it's actually the case that p . . . Here, I do not suppose p counterfactually in the subjunctive sense important for planning. Here, I consider p as part of an alternative account of *how things actually are* [3, 6]. We will not consider this epistemic supposition and the associated modality here, but the techniques under consideration apply equally well to epistemic modality.

To suppose *it hadn't rained yesterday* is to do something quite distinctive. It's not just to add the claim 'it didn't rain yesterday' to my current commitments (or to the common ground in the conversation). That would be a short step to absurdity, in a context where we grant that it *did* rain yesterday. ¶ Instead, this kind of *modal* supposition asks us to reason with this supposed content in a different way. It is isolated from our current commitments (about *how things actually went*), but we reason in the same way that we would when we flatly reason about how things actually turn out. ¶ I will not focus on the detail of how to counterfactually reason about local circumstances. My focus here is on the idealised modal operators of *possibility* ($\Diamond$) and *necessity* ($\Box$) broadly understood.

In particular, I won't concern myself about what from our actual commitments we import to the other context, when we suppose p in that local counterfactual sense, of changing the current circumstances only so far as to admit p.

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The key notion for us is that in modal reasoning we can understand 'supposing things were otherwise' and 'considering alternative possibilities' in a straightforward way. We apply our thought and talk not just to a flat position consisting of assertions and denials reflecting our current commitments, but to a richer structure, which might look like this:

$$[_{@}\, p, q, \cancel{r} \mid p, s, \cancel{r} \mid r, \cancel{p}, \cancel{q}\,]$$

Such a richer position represents the commitment of someone who has granted p and q and denied r (that is their '*actual*' commitment, as marked by the '@') and who has admitted as *possible* two other alternative outcomes: p and s without r or r without p and q. ¶ Such enriched positions, with more than one ZONE, arise when we engage in counterfactual reasoning, and plan in the face of uncertainty. ¶ There is no conflict between asserting p in one zone and denying it in another, because we are considering alternatives. ¶ In such a context, the semantics of modal operators comes to light. To say that something is necessary is to preserve that commitment across alternatives. So, a position like this is defective:

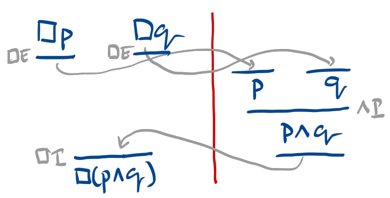
$$[\cdots \mid \Box A \cdots \mid \cancel{A} \cdots \mid \cdots]$$

To grant (somewhere) that A is *necessary* and to deny A (elsewhere) is to contradict yourself. ¶ Similarly, this position is also out of bounds:

At least, to grant $\Box A$ in some zone and to deny A in another is to contradict yourself if the notion of necessity in question encompasses the kinds of alternatives represented by the zones in our position.

$$[\cdots \mid \cancel{\Diamond A} \cdots \mid A \cdots \mid \cdots]$$

This motivates a particular kind of modal reasoning. In natural deduction, it could be represented like this:



This reasoning presumes that the semantic rules for conjunction apply in *other* zones, as much as they do in our 'home' zone. That is the constraint that we will impose throughout. We take our definitions to apply to the language as we use it *across supposition boundaries*. (How *else* could we employ our concepts in subjunctive reasoning?)

The left side of the red line corresponds to our starting zone, and the right side contains the commitments we undertake under the scope of supposing things had gone otherwise. ¶ This kind of graphical representation is difficult to typeset. We could distinguish different zones of our position with tags. Here, 'a' labels the steps taken in the starting zone, while 'b' labels the steps taken in the other zone.

$$\dfrac{\dfrac{\Box p \cdot a}{p \cdot b}\,{\scriptstyle \Box E} \qquad \dfrac{\Box q \cdot a}{q \cdot b}\,{\scriptstyle \Box E}}{\dfrac{p \land q \cdot b}{\Box(p \land q) \cdot a}\,{\scriptstyle \Box I}}\,{\scriptstyle \land I}$$

The defining rules for the modal operators then take a form reminiscent of the quantifier rules, but which arise out of the richer structure of the zones in our positions, rather than quantifying over *objects* to which we might be able to *refer*.

$$\dfrac{\mathcal{C} \succ A \cdot i}{\mathcal{C} \succ \Box A \cdot j}\,{\scriptstyle \Box Df} \qquad \dfrac{\mathcal{C}, A \cdot i \succ C}{\mathcal{C}, \Diamond A \cdot j \succ C}\,{\scriptstyle \Diamond Df}$$

An *actuality* operator may be defined with a rule defining $@A \cdot a$ to have the same effect as $A \cdot @$. Granting $@A$ in any zone of the discourse has the same effect as granting A in the *actual* zone. This, in combination with the rules for $\Box$ and $\Diamond$, is a simple family of rules for the standard logic S5@.

Here, the important side condition in this rule is that the zone tag i in the premise is absent from the assumptions $\mathcal{C}$ (and the conclusion C in the $\Diamond$ rule). ¶ Our demonstration concerning A is *arbitrary*, no matter what else we have supposed in that counterfactual zone. So, whatever we have concluded applies *generally*.

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These defining rules are CONSERVATIVELY EXTENDING and UNIQUELY DEFINING, for the same reasons as we saw for the quantifiers, except now the reasoning depends on the framework of supposition. ¶ The uniqueness result requires coordination on what counts as the relevant kind of *modal supposition*.

2.3 BUT WHAT IF *THIS* THING HADN'T EXISTED?

Now let's combine our quantifier rules with the modal rules. ¶ Here is a natural proof that results, when combining our rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x \Box Fx \cdot a]^1}{\Box Fx \cdot a}\,{\scriptstyle \forall E}}{Fx \cdot b}\,{\scriptstyle \Box E}}{\forall x Fx \cdot b}\,{\scriptstyle \forall I}}{\Box \forall x Fx \cdot a}\,{\scriptstyle \Box I}}{\forall x \Box Fx \to \Box \forall x Fx \cdot a}\,{\scriptstyle \to I}$$


Ruth Barcan Marcus (1921–2012)

We can prove the infamous BARCAN FORMULAS, named after Ruth Barcan Marcus, a pioneer of the study of first-order modal logic. ¶ We have shown that if everything is necessarily F, then indeed, it is necessary that everything is F.

» «

There are reasons to resist the Barcan formulas. It is natural to think that there could have been things that don't actually exist, and that some of what actually exists could have not existed. ¶ If everything that *happens* to exist is necessarily an F, it only follows that it is necessary that everything (that happens to have existed in those circumstances) is also an F only if there could be no other things than those things that happen to exist. ¶ Our quantifier rules, as given in Section 2.1, make no distinction between what exists and what doesn't, because our background context makes the assumption that variables always have values, and so, they always (and in any context) denote existing objects.

This assumption can be rejected, and it *should* be rejected if you take it to be coherent to suppose, of a given item, that *it* might not have existed. ¶ This points to an ambiguity in the notions of a variable *being defined* and *having a value*. When we take a given item x (say *me*) and suppose it *had* x *not existed*, then the variable x is *is defined*, but we are considering a circumstance where in which the value of that variable does not exist. ❡ This motivates revising the variable rule. Instead of requiring that variables have values that are present in *every* zone, we require only that their values be present in *some* zone.

$$\frac{x \cdot i, \mathcal{C} \succ C}{\mathcal{C} \succ C}\ \Diamond \mathit{Var}$$

(Here, the zone i must not be used elsewhere in $\mathcal{C}$ and C.) ❡ With this in place, the quantifier rules come in two forms: one is existentially committing in the current zone, and the other ranges across all zones in scope.

$$\frac{\mathcal{C}, x \cdot j \succ A \cdot j}{\overline{\mathcal{C} \succ \forall x A \cdot j}}\ \forall Df \qquad \frac{\mathcal{C}, x \cdot j, A \cdot j \succ C}{\overline{\mathcal{C}, \exists x A \cdot j \succ C}}\ \exists Df \qquad \frac{\mathcal{C} \succ A \cdot j}{\overline{\mathcal{C} \succ \forall^{\Diamond} x A \cdot j}}\ \forall^{\Diamond} Df \qquad \frac{\mathcal{C}, A \cdot j \succ C}{\overline{\mathcal{C}, \exists^{\Diamond} x A \cdot j \succ C}}\ \exists^{\Diamond} Df$$

In these rules we have the proviso that the variable x bound by the quantifier does not occur free in among the assumptions $\mathcal{C}$ or the conclusion C.

With the existentially committing rules for ∀ and ∃, the Barcan Formula fails, without the principle to the effect that something existing *there* also exists *here*. We start …

$$\cfrac{\cfrac{x \cdot a \qquad \forall x \Box Fx \cdot a}{\Box Fx \cdot a}\ \forall E}{Fx \cdot b}\ \Box E$$

… but stop there. We cannot introduce the universal quantifier unless we can derive Fx at b under the supposition that x's value is present *at* b. Unless we can derive $x \cdot a$ from $x \cdot b$, we are stuck. ❡ One way to verify that we *cannot* derive $\Box\forall x Fx$ from $\forall x \Box Fx$, using these rules, is to systematically reason about what can be derived. ❡ Instead, we will consider what this perspective can tell us about *models*, and use this to give counterexamples to invalid reasoning.

2.4 PUSHING THINGS TO THE LIMIT

In Lecture 1, we saw how *limit* positions—the available positions that decide every issue in a given language—provide another way to think of traditional two-valued models. ❡ Given a purely propositional language $\mathcal{L}$, with no quantifiers or modal operators, these positions settle all $\mathcal{L}$. ❡ Quantifiers and modal operators complicate this picture.

Start with *quantifiers*: Consider a tiny language $\mathcal{L}^F_\exists$ with one unary predicate F, no connectives, a family of variables and singular terms, and the existential quantifier, governed by the defining rule ∃*Df*. ❡ There is no contradiction in affirming ∃xFx while denying Ft for every term t in the language (including variables). A position denying every formula of the form Ft and affirming ∃xFx is available, and can be extended into a partition of all of $\mathcal{L}^F_\exists$. ❡ There is an issue left unsettled by this position: we have affirmed ∃xFx, but have no insight into *which* item has feature F.

This is a fact concerning the inference rules for $\mathcal{L}^F_\exists$. There is no way to derive a contradiction from ∃xFx and any number of denials of the form $\not\vdash t$.

Let's call a position ∃-WITNESSED if whenever it includes ∃xA, it also includes A^x_y for some variable y. (We say, *of some item* y that it has the relevant feature.)

The defining rule for ∃ ensures that any available finite position including ∃xA, is extended by an available position that contains A^x_y for some fresh variable y. ❡ (Why is this? Suppose that our finite position contains ∃xA. Choose a variable y not already present in the position. Given any proof leading from premises in that position together with A^y_x to a contradiction, since y is absent from the position, we can use ∃*Df* to derive the contradiction from ∃xA alone. Since the original position is available, the extended position is, too.) ❡ This ensures that any available (small) position may be extended to a limit position that is also ∃-*witnessed*. ❡ What goes for ∃ goes also for ∀. I leave those details for you to consider.

The finiteness condition ensures that we can choose a variable that is, as yet, unused by the position. This can be weakened in natural ways, but those details are not necessary for our purposes.

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There the modal operators raise parallel issues. Given a position in which I affirm ◇A (in zone i), this raises the question: *How* could A have been the case? To appropriately *settle* the issue, our position also involve a zone in which A is affirmed. That will

If I say it's possible that $2 + 2 = 5$, you're within your rights to ask me *how*, and if you suppose that $2 + 2 = 5$ and reduce that (counterfactual) supposition to absurdity, using agreed premises and principles, you've ruled out my claim that it's possible.

show how this issue can be answered. (Perhaps at the cost of raising further issues, of course!) ¶ Call positions which contain, for each $\Diamond A \cdot i$, some $A \cdot j$, $\Diamond$-WITNESSED. ¶ And just as with the $\exists$ rule, that this constraint can be met arises out of the defining rule $\Diamond$*Df*. Any available position containing $\Diamond A \cdot i$ may be extended to contain $A \cdot j$ for a fresh zone j. ¶ (What goes for $\Diamond$ goes also for $\Box$. I leave those details for you to consider.)

Witnessed limit positions in a language $\mathcal{L}$ settle every $\mathcal{L}$-issue in this extended sense.

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Unsurprisingly, witnessed limit positions describe Kripke models. ¶ Each *zone* describes what is true at a possible world. The terms ruled *in* at a zone are the inner domain of objects *existing* at that world, while the remaining terms that are ruled in at *some* world are the outer domain of *possibilia*. ¶ The usual truth conditions for a varying domain quantified S5 (with actuality) are satisfied, given the defining rules for the quantifiers, connectives and modal operators, and the witnessing conditions governing the construction of our position.

For the technical details, see my draft "Modal Logic and Contingent Existence (Generality and Existence 2)" [8].

Here is a sketch of how such positions are constructed. Start here, attempting to refute a Barcan formula, affirming $\Diamond\exists xFx$ and denying $\exists x\Diamond Fx$.

$$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx}]$$

For the witnessing condition for $\Diamond$ we must add a zone affirming $\exists xFx$. We have:

$$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx} \mid \exists xFx]$$

We must add a witness for $\exists xFx$, which is also present at that zone.

$$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx} \mid x, \exists xFx, Fx]$$

Since Fx is affirmed in the second zone, $\Diamond Fx$ is undeniable in each zone (given $\Diamond$*Df*), so as we fill out the position, $\Diamond Fx$ must be settled affirmatively in each zone.

$$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx}, \Diamond Fx \mid x, \exists xFx, Fx, \Diamond Fx]$$

Now, since $\exists x\Diamond Fx$ is *denied* but $\Diamond Fx$ is affirmed in the first zone, the object x must fail to be present there.

$$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx}, \Diamond Fx, \cancel{x} \mid x, \exists xFx, Fx, \Diamond Fx]$$

If the predicate F expresses an existence entailing property (as all predicates do in Feferman's logic) then Fx must also be rejected in our starting position:

$$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx}, \Diamond Fx, \cancel{Fx}, \cancel{x} \mid x, \exists xFx, Fx, \Diamond Fx] \qquad (\mathfrak{P})$$

This position describes a tiny model, settling every issue concerning the single object x and the predicate F. It provides a counterexample to the Barcan formula.

» «

This argument provides an solution to Prior's modal semantics puzzle. Possible worlds models manage to model modal vocabulary when that vocabulary is governed by the defining rules, and given that inferentialist semantics, witnessed limit positions describe these models.

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ISSUE 1: What about the *ontological* significance of possibilia and of possible worlds?

Witnessed limit positions are *abstracta*. They are *constructions* arising out of our linguistic practice. However, saying that these models are *abstracta*, and therefore, that there are no interesting issues in the ontological commitment undertaken when adopting those models is not addressing the important issues involved in modal semantics.

The *vital* issue concerns what it is to *take* such a position. To *endorse* a position is to take it that what is in the zone marked @ holds, while what is in the other zones is *possible*. So, according to $(\mathfrak{P})$, $\Diamond\exists xFx$ is true, and it's jointly possible (1) that there exists some item, x, and (2) it is F. From this it follows in *that* scenario that $\exists xFx$ and $\Diamond Fx$. Also, according to $(\mathfrak{P})$, the item featuring in that other possibility is actually existent.

This is the kind of ideological and ontological commitment involved in taking these models to describe *how things are*. They systematically spell out the commitments made when using modal and quantificational vocabulary. ¶ What more do you want from your models than that?

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ISSUE 2: The models constructed by constructing witnessed limit positions are a *scale models*. They need not be one-to-one models of the universe. ¶ Suppose $\mathcal{L}$ is countably infinite. You can construct a witnessed limit position with only countably many items in the domain, no matter what your starting position is like. ¶ You could start with a standard mathematical theory, like ZFC, according to which there are uncountable sets. ¶ Regardless, the limit position constructed will be countably infinite, from the outside. We have countable models of ZFC. This is familiar to mathematicians.

Other models are possible, of course. But they aren't made by *this* construction.

Does this mean that $\forall$ doesn't mean *all*, that one cannot quantify over *absolutely everything*? No. A model, as a set-theoretic object, is a *model*, of the language, it is not the language as it is *actually used*. The *reality* is what is described by the vocabulary in use. ¶ You are genuinely *saying* that everything has feature F when you say $\forall x Fx$, even if the *models* representing that statement are small and do not exhaust everything. ¶ As conceptual capacities expand, and our language is extended, our original commitment to $\forall x Fx$ will, if it is retained, be about the things we talk about *then*, too. ¶ As we saw before, the semantics of the quantifiers are not substitutional in the flat sense, and there is no bar to '$\forall$' meaning *all*, even if every *model* of this vocabulary is small.

Similarly, if models are *classes*. That merely punts the issue up to another level. If a model is a *thing*, it is extremely difficult (if not inconsistent) for a one-to-one scale model of the universe to contain itself.

Similarly, there is no requirement here that an adequate $\mathcal{L}$-model of a modal language must exhaust modal space. An adequate model for an extended language $\mathcal{L}'$ may have more worlds. (Zones may be $\mathcal{L}$-indistinguishable without being $\mathcal{L}'$-indistinguishable, in the sense that a complete $\mathcal{L}$-zone might be expanded with $\mathcal{L}'$ vocabulary coherently in two different ways.) Again, that is unproblematic. The models of a modal language give a tractable models of what is possible and what is necessary, describing 'worlds' only as far as necessary for the purposes of that spelling out commitments expressible in that language. ¶ So, there is no problem in using a model of modal vocabulary, without having to address thorny questions settling how many objects or worlds there in fact are. A model can be totally adequate about the statements expressible in that language without that model settling once-and-for-all difficult questions how many objects or worlds there are. *That* is not what models are for.

FOR NEXT TIME

We will take this perspective on models for modal vocabulary, and see how this can give a new angle on the value of truth-conditional semantics for natural languages.

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