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In my first lecture, I introduced INFERENTIALIST SEMANTICS, giving an account of how you can understand propositional logical vocabulary as *definable* using invertible rules of inference. The defining rules, together with the fundamental structural rules governing proof *as such* are the only transitions in our gap-free proofs. Since these defining rules conservatively extend those background structural rules, they are safe to adopt.

$$\frac{\mathcal{C}, A \succ B}{\mathcal{C} \succ A \to B}\ {\to}Df$$

In the second lecture, I showed how this account generalises to quantifiers and to modal operators. For the quantifiers, our language must involve a category of singular terms, satisfying constraints on substitution of one term for another.

$$\frac{\mathcal{C} \succ A}{\mathcal{C} \succ \forall x A}\ \forall Df\ \ (x \text{ is not free in } \mathcal{C}.)$$

The defining rules for the modal operators involve expanding our background structural account of proof to encompass properly *modal* supposition. To modally suppose A is to add A to the conversational context in a *new zone*. To say take it that A is necessary is to take it that A not only holds, but *would* hold had things gone otherwise. Tagging zones with labels, there is no clash between asserting A at a and denying it at another zone b, but there *is* a clash between asserting □A at a and denying A at b.

$$\frac{\mathcal{C} \succ A \cdot i}{\mathcal{C} \succ \Box A \cdot j}\ \forall Df\ \ (i \text{ is not present in } \mathcal{C}.)$$

Combining defining rules for the first-order quantifiers and modal operators gives an inferentialist semantics for first-order quantified modal logic, where the inference rules governing the logical connectives describe what we *do* to use the concepts in making assertions, denials, inferences, and suppositions, i.e., to use them in our talk and thought. ¶ I showed how familiar models for first-order modal logics arise as the limit of a process of settling issues in the *positions* that feature in inference, and that model-theoretic 'semantics' for modal logic can be defended on inferentialist lines.

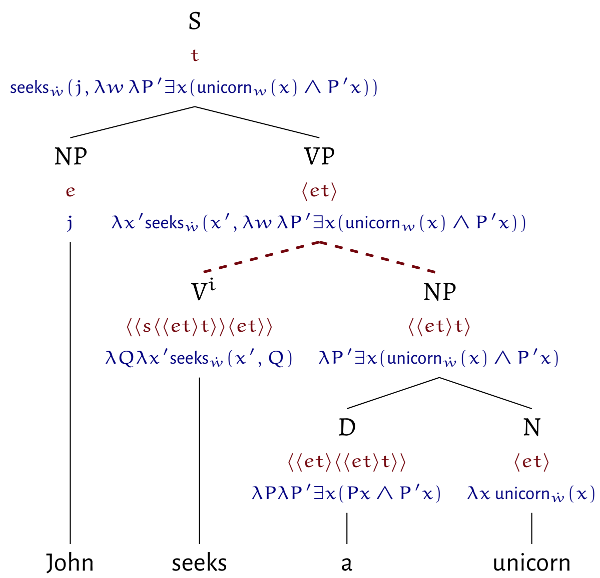
This modal logic contains a varying domain inner *existence-entailing* quantifiers and constant domain outer *possibilist* quantifiers, given the discipline governing variables discussed in the previous lecture.

3.1 THERE IS MORE TO LANGUAGE THAN LOGIC

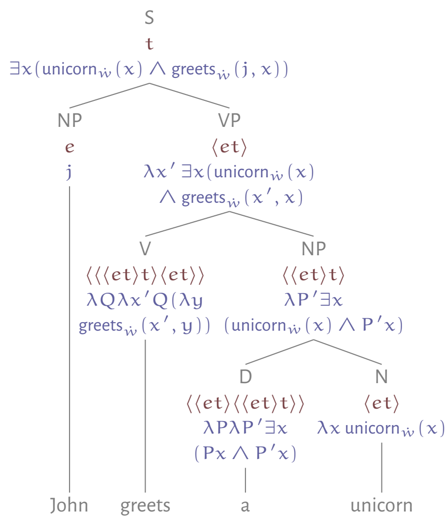
Semantics is an incredibly rich and diverse field, and I cannot do justice to it in a single lecture [6]. I will focus on the *truth-conditional* semantics, as pioneered by Richard Montague [10], David Lewis [5] and Barbara Partee [7, 9], in the 1970s. Modal model theory is central to this enterprise. Semantic values are supplied for natural language lexical items, in the vocabulary of an *intensional type theory*.

Barbara Partee (1940–)

I will consider the two-sorted type theory Ty_2 of Daniel Gallin [3]. It is an extension of a two-sorted first-order logic, with one sort e for *entities* and another sort s for *states*. These basic sorts are also the basic *types*. *Sentences* in the language have type t. ¶ For any type α and β there is a *functional* type, ⟨αβ⟩ of functions from items of type α to items of type β. ¶ Here is a Ty_2 derivation of a semantic value for the expression 'John seeks a unicorn':



Here is a Ty_2 derivation for 'John greets a unicorn'. Notice that for it to be true that John greets a unicorn, there must *be* a unicorn that John greets.



Elizabeth Coppock and Lucas Champollion's textbook is a particularly comprehensive and accessible introduction to intensional type theory in general, and Ty_2 in particular [2].

Linguists have noticed that their commitments concerning the entities and worlds of their semantic theories are not quite the same as the commitments of metaphysicians. Here is Barbara Partee, writing in the late 1980s, during the heyday of the discussion of the metaphysics of possible worlds.

One motivation for this line of inquiry this challenge from Timothy Williamson. Speaking of Robert Brandom, he says his "inferentialism has remained at an even more programmatic stage than Dummett's, lacking an equivalent of Dummett's connection with technical developments in proof theory by Dag Prawitz and others. As a result, inferentialism has been far less fruitful than referentialism for linguistics. In that crude sense, referentialism beats inferentialism by pragmatic standards" [11, p. 34]. While I do not so much care about what perspective 'beats' another, Williamson does prompt the question of how inferentialism bears on model-theoretic semantics and its 'referentialist' commitments.

> A non-absolutist picture seems to fit linguistic semantics better than an absolutist one, where by absolutist I mean the position that there is one single maximal set (or class, if it's too big to be a set) of possible worlds. If a philosopher could find arguments that in the best metaphysical theory there is indeed a maximal set, I suspect that would for the linguist be further confirmation that his enterprise is not metaphysics, and I would doubt that such a maximal set would ever figure in a natural language semantics. As various people have noted, *possible worlds* are really not so different in this respect from *entities*: every model-theoretic semantic theory I'm familiar with takes entities to be among the primitives—but puzzles about the identity conditions of individuals and about whether there is a maximal set of all of them are just as problematic, and it is just as questionable whether semantic theory has to depend on settling such questions. [. . .] it is the structure provided by the possible worlds theory that does the work, not the choice of particular possible worlds, if the latter makes sense at all. [8, p. 118]

How should we understand the semanticists commitment to a domain of possible worlds and to a domain of entities in their semantic theories? My own orientation when answering this question is owed to my second inspiration here, Nuel Belnap. He writes:

Nuel Belnap (1930–2024)

> … in the tolerant spirit of Carnap, we believe that one is likely to want a *variety* of complementary (noncompeting) pre-semantic analyses—and most especially, a variety of pre-semantic treatments of one and the same 'language.' One does not have to 'believe in alternative logics' to repudiate the sort of absolutism that comes not from logic itself, but from narrow-gauge metaphysics or epistemology … although Carnap's beneficent influence is legendary, it seems worth repeating the lesson: There can and should be multiple useful, productive, insightful and pertinent analyses of the *same* target. Pre-semantics therefore emphasizes the usefulness of thinking in terms of a *variety* of pre-semantic systems. [1, p. 1]

3.2 TAKING A GOD'S EYE VIEW

First, recall that *positions* are structured into distinct zones. One zone distinguished as the 'actual' zone, the other zones record claims that are endorsed as *possible*. A position might look like *this*:

$$[_{@}\, p, q, \cancel{r} \mid p, s, \cancel{r} \mid r, \cancel{p}, \cancel{q}]$$

We could *label* the alternate zones:

$$[_{@}\, p, q, \cancel{r} \mid_{a} p, s, \cancel{r} \mid_{b} r, \cancel{p}, \cancel{q}]$$

Add to our conceptual arsenal the capacity to use these labels in further judgements. It makes sense to say that *in zone* b r *is granted*, or for short '$r : b$'. Introduce it to our vocabulary with this defining rule:

$$\frac{\mathcal{C} \succ A \cdot i}{\mathcal{C} \succ A : i \cdot j}\ {:}\,Df$$

This is conservatively extending and uniquely defining. ¶ It looks like we have singular terms referring to *possible worlds*. Such a reading is *permissible*, it not *required*.

» «

With zone tags *in formulas*, it is one small step to allow *quantification* into zone index position. With the obvious defining rules for universal and existential quantification, $\forall w(A : w)$ is equivalent to $\Box A$, and $\exists w(A : w)$ is equivalent to $\Diamond A$. ¶ We have moved from a zone-internal vocabulary to a zone-neutral *external* vocabulary. This shift incurs no greater ontological commitment than was made previously, but the resulting two-sorted first-order language brings us closer to Ty_2.

3.3 GOING UP THE LADDER

$$\frac{\mathcal{C} \succ A^{x}_{b} \cdot i}{\mathcal{C} \succ (\lambda x A)b \cdot i}\ \lambda_{\langle et\rangle}Df \qquad \frac{\mathcal{C} \succ A^{P}_{B} \cdot i}{\mathcal{C} \succ (\lambda P\, A)B \cdot i}\ \lambda_{\langle \alpha t\rangle}Df$$

Predicate abstraction is governed by a straightforward defining rule $\lambda_{\langle et\rangle}Df$. ¶ What goes for predicate abstraction can go for other types, too. In general, given any type α, we can define the abstraction operator $\langle \alpha t\rangle$ generalising the previous defining rule to $\lambda_{\langle \alpha t\rangle}Df$. where A has type t, and P and B have type α.

Gallin defines the general λ reduction rule with a type-general *identity* relation. A more general rule is:

$$\frac{\mathcal{C} \succ D\, A^{P}_{B} \cdot i}{\mathcal{C} \succ D\, (\lambda P\, A)B \cdot i}\ \lambda_{\langle \alpha\beta\rangle}Df$$

for A type β, D type $\langle \beta t\rangle$, P and B type α.

The *power* of abstraction is unleashed, when you combine it with *quantification* at all type levels. ¶ Given variables at each type level, it is natural to *generalise* into those variable positions:

$$\frac{\mathcal{C} \succ A \cdot i}{\mathcal{C} \succ \forall P\, A \cdot i}\ \forall_\alpha Df \qquad \frac{\mathcal{C}, A \succ C \cdot i}{\mathcal{C}, \exists P\, A \succ C \cdot i}\ \exists_\alpha Df$$

Here, the variable P has type α, and is required to not occur free in the context $\mathcal{C}$ (or the tag i). ¶ These rules will have their desired effect only in the presence of an underlying principle of SUBSTITUTION, according to which the variable P is inferentially general among terms of type α. ¶ As before, the addition of quantifiers at every type level is uniquely defining, and conservatively extending. However, the conservative extension result comes with significant caveats. ¶ First, the richer structure available when quantifying into higher types (even in the predicate type $\langle et\rangle$) means that quantification is *impredicative*. This means that our usual argument to conservative extension is blocked. ¶ The appropriate *left* rule for the universal quantifier is $\forall L$, where B is any item of type α. Since this type might involve terms of great logical complexity, it might even contain the formula $\forall P\, A$ as a *subformula*, and hence, there is absolutely no guarantee that the left and right rules only add material and never destroy it, read from top to bottom. ¶ The rules are impredicative. It is no surprise that the higher type quantifier rules behave differently to the first-order quantifiers.

The 'usual argument' goes like this: take the invertible defining rules, and provide equivalent left-right rules which satisfy the subformula property (anything above the inference line is also present below the line). Then show that the only remaining rule in your calculus (the *Cut* rule) may be eliminated, and hence, any derivable sequent can be derived without *Cut*, and so, has a properly *analytic* derivation using only the vocabulary in the endsequent.

$$\frac{\mathcal{C}, A^P_B \succ C \cdot i}{\mathcal{C}, \forall P\, A \succ C \cdot i}\ \forall_\alpha L$$

Nonetheless, as a family, the λ and quantifier rules at each type *are* conservatively extending over the original two-sorted first-order modal logic. It is not too hard to see why. ¶ Any model of the two-sorted first-order modal logic has two domains, a zone domain D_s, and an entity domain D_e (which has a distinguished subset D_e^w for each zone w, of the objects that are taken, in zone w, to exist). For uniformity, we will take there to be a special domain $D_t = \{0, 1\}$ of truth values. A domain $D_{\langle\alpha\beta\rangle}$ for type $\langle\alpha\beta\rangle$ is a set of functions from D_α to D_β. ¶ *Which* functions? The easiest choice, in one sense, is to say *all* such functions.

The choice of 'all' such functions is *easy*, but it does involve some serious mathematical and logical baggage. If our entity domain D_e is infinite, the domains $D_{\langle et\rangle}$, $D_{\langle\langle et\rangle t\rangle}$, $D_{\langle\langle\langle et\rangle t\rangle t\rangle}, \ldots$ climb up the hierarchy of infinite cardinals, and the logic of such 'standard' models is beholden to the commitments of the underlying set theory in which it is formulated. Nothing like this dependence obtains for first-order theories.

Although the 'standard models' of Ty_2 are forbiddingly mathematically rich, and implicated in issues in set theory that are far away from concerns in natural language semantics, these models are merely models, used to show that the higher type vocabulary conservatively extends our first-order commitments. There is good evidence that the inferences that actually do the *work*, in formal linguistics, are given in the defining rules for abstraction and the quantifiers [4]. The inferentially defined system suffices.

3.4 WHAT THIS MEANS MEANING MIGHT MEAN

Ty_2 is a structuring tool for the *theorists* to describe and explain compositional patterns that are implicit in how we *use* our vocabulary in our natural languages.

There is no requirement that the everyday language user, competent in the use of the indefinite article 'a', has to understand that is a term of type $\langle\langle et\rangle\langle\langle et\rangle t\rangle\rangle$ or that it means $\lambda P \lambda P' \exists x(Px \land P'x)$.

The basic materials in the models for Ty_2 have the same interpretation that the inferentialist gave for models of a first-order modal language. ¶ The *functional* domains of higher types provide a space in which the linguist can describe the capacities and commitments of language users. An item of type $\langle\alpha t\rangle$ corresponds to a *distinction* between items of type α. ¶ Given any type α, we have a corresponding *intension* of type $\langle s\alpha\rangle$. Something of this type a choice of something of type α, in each different zone.

There is much more to be said, but we can see how these types have cognitive and communicative significance. We can *grasp* the meaning of some part of speech (or the corresponding concept), to a lesser or greater extent, as we are able to work with at concept more narrowly or broadly. ¶ The inferentialist pre-semantics points to how an intensional type theory like Ty_2 provides a useful structuring vocabulary for those capacities. ¶ Truth-conditional semantics is *vindicated* by its inferentialist pre-semantics.

Let me return full circle. In the first lecture, I defined $\rightarrow$ like this:

I have made the zone tag explicit, as this rule applies in each zone.

$$\frac{\mathcal{C}, A \succ B \cdot i}{\mathcal{C} \succ A \rightarrow B \cdot i}\ {\rightarrow}Df$$

When viewed from Ty_2, the material conditional is a lexical item of type $\langle t\langle tt\rangle\rangle$, as it conjoins two sentences. Items of type $\langle t\langle tt\rangle\rangle$ are interpreted in models of Ty_2 as binary

It can also be lifted to operate on intensions, taking type $\langle\langle st\rangle\langle\langle st\rangle\langle st\rangle\rangle\rangle$.

truth functions. ¶ The one binary truth function can be presented in many different ways. The lexical entry for material conditional '$\rightarrow$' is constrained only by the traditional boolean valuation condition. To grasp a concept with this interpretation, it suffices to affirm $A \rightarrow B$ when A is denied or B is affirmed, and to deny $A \rightarrow B$ when A is affirmed and B is denied. This is, of course, altogether too strong a constraint, because we might affirm A and deny B and the issue of whether $A \rightarrow B$ holds simply does not arise for us. Second, even when the issue does arise, we might find it hard to interpret, due to its complexity. ¶ Different rules can deliver the same pattern of verdicts, when applied correctly, and each would have the same *intension*, as the interpretation is kept fixed under subjunctive suppositions.

$v(A \rightarrow B) = 0$ iff $v(A) = 1$ and $v(B) = 0$.

QUICK: is $((p \rightarrow q) \rightarrow p) \rightarrow p$ true?

Extension and intension do not exhaust meaning. $A \rightarrow B$ and $\neg A \vee B$ agree on extension and on intension. They are given different *definitions*. The rules introducing a concept that is given by definition are *basic* for that concept. ¶ Truth-conditional semantic theories, insofar as they represent meaning by extension and intension, focus on the *result* of interpretation, drawing our attention away from the *process*. ¶ Supplementing your truth-conditional semantics with an inferentialist *pre*-semantics means you can keep your representational theory while gaining new insight into its foundations.

If I ask my logic students to show that $A \rightarrow B$ and $\neg A \vee B$ are equivalent, there is some work that I am asking them to *do*. I am not asking them to show that $A \rightarrow B$ and $A \rightarrow B$ are equivalent, which is a much simpler task.

WHERE TO, FROM HERE?

First, this is one effort at giving a Belnap-inspired alternative pre-semantic analysis of a familiar semantic system. I would like to see more!

Proof theoretical approaches are a natural home for *hyperintensional* distinctions. Even though A and B might be logically equivalent, a proof of A is not necessarily a proof of B. Truth-conditional semantics tends to flatten out distinctions between logically equivalent statements, while they might have different *semantic*, *epistemic*, and *metaphysical* significance. ¶ An inferentialist pre-semantics gives scope for modelling natural and motivated hyperintensional distinctions.

If the proof of the equivalence of A and B is complex and difficult to find, then of course there will be proofs of A that are not also proofs of B.

This investigation was motivated by the prevailing contours of *logic*, as a foundational discipline with its own insights. I have made use of both its proof theoretic and model theoretic techniques, because doing so gives us more to work with, as philosophers, and exploring their connections with other foundational issues—in this case, in semantics—gives us new insights which, in turn, means we return to those logical techniques with greater understanding.

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