ASSERTION, DENIAL AND NON-CLASSICAL THEORIES

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JULY 31, 2009
version 0.99

Abstract: In this paper I urge friends of truth-value gaps and truth-value gluts – proponents of paracomplete and paraconsistent logics – to consider theories not merely as sets of sentences, but as *pairs* of sets of sentences, or what I call 'bitheories,' which keep track not only of what *holds* according to the theory, but also what *fails* to hold according to the theory. I explain the connection between bitheories, sequents, and the speech acts of assertion and denial. I illustrate the usefulness of bitheories by showing how they make available a technique for characterising different theories while abstracting away from logical vocabulary such as connectives or quantifiers, thereby making theoretical commitments independent of the choice of this or that particular non-classical logic.

Examples discussed include theories of numbers, classes and truth. In the latter two cases, the bitheoretical perspective brings to light some heretofore unconsidered puzzles for friends of naïve theories of classes and truth.

1 ASSERTION, DENIAL & SEQUENTS

Friends of truth-value GAPS and truth-value GLUTS both must distinguish the *assertion of a negation* (asserting $\ulcorner\neg p\urcorner$) and *denial* (denying $\ulcorner p\urcorner$). If you take there to be a truth-value *glut* at $\ulcorner p\urcorner$ the appropriate claim to make (when asked) is to assert $\ulcorner\neg p\urcorner$ without thereby denying $\ulcorner p\urcorner$. If you take there to be a truth-value *gap* at $\ulcorner p\urcorner$ the appropriate claim to make (when asked) is to deny $\ulcorner p\urcorner$ without thereby asserting $\ulcorner\neg p\urcorner$.

*Comments from readers on this paper are very welcome. Please check the webpage http://consequently.org/writing/adnct for the latest version of the paper, to post comments and to read comments left by others. ¶ Thanks to the Logic Seminar at the University of Melbourne and the audience at WCP4 (including Jc Beall, Patrick Girard and Jerry Seligman), and to Michael De, for comments on this material, and to an anonymous referee, who helpfully pressed on certain points, and asked me to write up the little proof (in footnote 22) that there are at most two truth values. ¶ This research is supported by the Australian Research Council, through grant DP0343388, and Max Richter's *24 Postcards in Full Colour*.

This is why a taking ⌜p⌝ to be in a truth-value gap is not the same attitude as *ignorance* or *agnosticism* concerning ⌜p⌝. If I am ignorant of ⌜p⌝, I assert neither ⌜p⌝ nor ⌜¬p⌝ and neither do I deny them. I am open to the possibilities. Taking ⌜p⌝ to be in truth-value gap involves *denying* it, together with denying its negation. Similarly, this is why a taking ⌜p⌝ to suffer from a truth-value glut is not the same attitude as being *confused* concerning ⌜p⌝. I might mistakenly believe both ⌜p⌝ and ⌜¬p⌝,[1] but in *that* case I take my assertion (when asked) of ⌜¬p⌝ to rule out assertion of ⌜p⌝, and I take my assertion of ⌜p⌝ to rule out assertion of ⌜¬p⌝. I am, alas, in two minds concerning ⌜p⌝. Someone who takes ⌜p⌝ to be genuinely both true and false is not in this state. To take ⌜p⌝ to be both true and false is to be prepared to assert ⌜¬p⌝ without thereby denying ⌜p⌝.

I think this is important, because logical consequence has something to say not only about assertion but also about denial and the connection between assertion and denial [16]. To take an argument to be valid does not mean that when one asserts the premises one should also assert the conclusion (that way lies madness, or at least, making *many* assertions). No, to take an argument to be valid involves (at least as a part) the commitment to take the assertion of the premises to stand against the denial of the conclusion. In general, we can think of logical consequence as governing *positions* involving statements asserted and those denied. Logical validity governs positions in the following way:

- If $A \vdash B$, then the *position* consisting of asserting ⌜A⌝ and denying ⌜B⌝ *clashes*.

If ⌜B⌝ deductively follows from ⌜A⌝, and I assert ⌜A⌝ and deny ⌜B⌝, I have made a mistake. This generalises in the case of more than one assertion and more than one denial.

- If $\Gamma \vdash \Delta$, a position in which we assert each member of Γ and deny each member of Δ *clashes*.

What can we say about this relation of logical consequence, between collections of premises and conclusions, governing positions involving assertions and denials? At the very least we can say that the following rule (*Id*) holds, meaning that a position is a clash if the same thing is both asserted and denied.

$$\Gamma, A \vdash A, \Delta \quad (Id)$$

Furthermore, if there is no clash in asserting every member of Γ and denying every member of Δ, we can see that together with asserting each member of Γ and denying each member of Δ either there is no clash in asserting ⌜A⌝ or there is no clash in denying ⌜A⌝. In other words, if asserting ⌜A⌝ is ruled out by means of the rules of the game alone, then since ⌜A⌝ is unassertible, its denial is implicit in the assertion of every member of Γ and the denial of every member

[1] A nice example of confusion is David Lewis' discussion of the orientation of Nassau Street in "Logic for Equivocators" [6].

of Δ, so its explicit denial involves no clash. Contraposing this, if there is a clash in denying $\ulcorner A \urcorner$ (together with asserting every member of Γ and denying every member of Δ) and there is also a clash in asserting $\ulcorner A \urcorner$ (together with asserting every member of every member of Γ and denying every member of Δ), then there is a clash in asserting every member of Γ and denying every member of Δ alone. In other words, we have the rule (*Cut*).

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ (Cut)$$

We should note three things concerning the structural features of consequence understood in this way. First, Γ and Δ here are *sets* of statements. While the use of more discriminating collections (multisets, lists, etc.) can be very useful from a proof-theoretic point of view, as long as there is no normative difference between a position in which something has been asserted *twice* and where it has been asserted merely *once*, this seems to be a distinction that makes no difference. Second, it is important to notice that implicit in the rule (*Id*) of identity, is the rule of *weakening*. If a position has a clash, this is not alleviated by the addition of *more* assertions or denials. It could well be alleviated by a *retraction* of something formerly asserted or denied, but a retraction is not the same thing as a denial or an assertion. Retracting a claim means moving to a position in which that claim is taken 'off the table' as an assertion (or as a denial), which need not involve any further assertion or denial (of that claim, its negation, or anything else). If I discover that the claim that p has untoward consequences, I can retract an assertion of $\ulcorner p \urcorner$ without being commital to its truth or falsity. Third, the vocabulary of sequents here is, so far, *independent* of the logical vocabulary used in the statements that are themselves asserted and denied. We have only sketched some structural features which quite plausibly govern the practice of making asertions and denials.[2]

Now, let's consider logical vocabulary, and in particular the operator of negation. Gentzen's own sequent rules for negation are simple:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

They tell us that asserting $\ulcorner \neg A \urcorner$ has the same status as denying $\ulcorner A \urcorner$. If there is a clash in denying $\ulcorner A \urcorner$ (in the context of asserting every member of Γ and denying every member of Δ), there is a clash in asserting $\ulcorner A \urcorner$ too (in that context). Similarly, denying $\ulcorner \neg A \urcorner$ has the same status as asserting $\ulcorner A \urcorner$.

[2] I have argued for them in some detail elsewhere [16]. I do not take these considerations to be *conclusive*, but on the other hand, I have not seen any rival account of the norms of assertion and denial that is in any way a plausible alternative to this picture. I urge defenders of non-classical logic who take the assumptions I have made to be mistaken to develop an alternative account, explaining when the rules (*Id*) or (*Cut*) might fail, when governing assertion and denial, and what should be put in their place.

Clearly, given what we have already said about friends of gaps and gluts, not everyone will find these rules acceptable. Depending on our attitudes to truth-value gaps and gluts, we may find some rules acceptable and others not. $(\neg L)$ corresponds to *Ex Contradictione Quodlibet*[3] and $(\neg R)$ corresponds to the Law of the Excluded Middle.[4] The four different possibilities seem to be these:

1. *No gaps, no gluts*: both $(\neg L)$ and $(\neg R)$ are acceptable.

2. *Gaps, no gluts*: $(\neg L)$ is acceptable, but $(\neg R)$ is not. We can have $\nvdash \neg A, A$.

3. *No gaps, gluts*: $(\neg L)$ is not acceptable, but $(\neg R)$ is acceptable. We can have $A, \neg A \nvdash$.

4. *Gaps, gluts*: both $(\neg L)$ and $(\neg R)$ are not acceptable We have both $\nvdash \neg A, A$ and $A, \neg A \nvdash$.

In other words, the interpretation of sequents in terms of assertion and denial gives us a way to characterise different treatments of negation. While there is something to be said taking $(\neg L)$ and $(\neg R)$ as a *definition* of a concept of negation, this will be a definition that friends of gaps or gluts take to fail to characterise the true concept of negation.

Does that mean there is a *false* concept of negation defined by $(\neg L)$ and $(\neg R)$, or does it mean that these rules don't define a concept at all? These are subtle matters for the friend of gaps or gluts. Graham Priest, for one, takes the rules $(\neg L)$ and $(\neg R)$ to not define a concept at all. Priest takes there to be *no* concept satisfying those two rules [11].

The fact that $(\neg L)$ is not acceptable for friends of gaps, and $(\neg R)$ is not acceptable for friends of gluts does not mean that friends of gaps or gluts must reject any rules that together *look* like $(\neg L)$ and $(\neg R)$. For example, rules of this general shape are satisfied by negation in linear logic [4, 15], without allowing a derivation of either the law of the excluded middle or the law of non-contradiction. You can prove a sequent of the form $\vdash p, \neg p$, but this does not mean that $\vdash p \vee \neg p$, only that $\vdash p + \neg p$, where '+' is an *intensional* disjunction, and where $p + p$ does not entail p, and where p does not entail $p + q$ (both *weakening* and *contraction* fail for this disjunction). The disagreement is not over the general shape of the rules, but over the rules $(\neg L)$ and $(\neg R)$ where we have interpreted the structure (the comma, the turnstile) in just the way we have here, governing assertion and denial in just this matter. We must be clear on how we are interpreting our vocabulary, *especially* when using non-classical logics, in which common assumptions are questioned or rejected.

Just as there are rules governing negation, there are rules governing other connectives. Different logics, classical and non-classical, have different rules for the connectives such as conjunction, disjunction, the conditional (or many

[3]Since $p \vdash p, q$, we have by $(\neg L)$, $p, \neg p \vdash q$.

[4]Since $p \vdash p$, we have by $(\neg R)$, $\vdash p, \neg p$, and by the disjunction rule, $\vdash p \vee \sim p$.

conditionals) and quantifiers. In what follows, these rules will be unimportant, as the topics we are considering can be characterised without reference to those connectives.

Let's consider the position of the dialetheist, or in fact, any proponent of a paraconsistent logic. We will presume (*Id*) and (*Cut*) in what follows. This is not to say that they cannot be resisted by a dialetheist: of course they can. However, to resist them is to open up the question: what is to be held in their place? It seems that maintaining (*Id*) and (*Cut*) are *no* bar to holding a paraconsistent logic, nor even being a dialetheist, who holds that some contradictions are true. We can very well make sense of this position, agree that (*Id*) and (*Cut*) are valid, and simply reject ($\neg L$) as a rule satisfied by a genuine negation operator. That is an understandable position for the dialetheist. Were the dialetheist to go *further*, to reject either of (*Id*) or (*Cut*), they would need to answer further questions. Chief among them is this: how *are* we to constrain assertion and denial? If not (*Id*) and (*Cut*), then what? If we have a valid argument from premises to conclusion, how does this constrain assertion and denial? Do we not take there to be a mistake in asserting the premises of a valid argument and denying the conclusion? Something must be said here, and it is a challenge for the dialetheist who wishes to reject (*Id*) or (*Cut*) to sail between the Scylla of agreeing with (*Id*) and (*Cut*) and the Charybdis of rejecting so much that logic has no evaluatve force.

2 THEORIES, COTHEORIES & BITHEORIES

With that background on assertion and denial, granting the role of (*Id*) and (*Cut*) constraining assertion and denial, but allowing different accounts of negation and the other logical connectives to vary from logic to logic, it is time to consider the notion of a *theory*. For among many different logics, such as classical logic, constructive logics, logics with truth-value gaps and—especially for our discussion here, logics with truth-value gluts—the notion of a *theory* makes sense, and has a prominent role. Given any of these choices of logic, and given the context of the formalisation of mathematical concepts or the presentation of other 'theories', intuitively understood, it is common to treat a formal *theory* as a collection of statements. Perhaps when presented it is characterised as the consequences of a number of basic axioms. Perhaps instead it is characterised as the application of a number of basic rules. Perhaps, thirdly, it is characterised as the collection of statements true in some class of models. However they are characterised, the result is the following condition: a THEORY is a collection T of sentences closed under logical consequence. That is,

$$T \text{ is a THEORY iff } (\forall A)(T \vdash A \;\Rightarrow\; A \in T)$$

If some statement is a consequence of the theory, it is also a part of the theory. So, if you *endorse* the theory, commitment to this theory means that you are making a mistake if you deny any statement in the theory. The consequences of T are *undeniable*, granted commitment to T.[5] What does the theory tell us we *should* deny, or contrapositively, what we *shouldn't* assert? As far as the theory goes, if assertion of a negation does not bring denial along with it (as it doesn't, for friends of gluts), the commitment to the theory itself need carry no consequences concerning what is not to be asserted (or what should be denied). The fact that a theory tells us $\ulcorner\neg A\urcorner$ does not give us guidance on the matter of ruling $\ulcorner A\urcorner$ out, at least if we have countenanced gluts.[6]

What is there to do? It seems that we must not only keep track of what is to be accepted, on the terms of a theory, but we should also keep track of what is to be *rejected*. Dual to a theory is the notion of a COTHEORY, a collection U of sentences closed 'over' logical consequence. That is,

$$U \text{ is a COTHEORY iff } (\forall A)(A \vdash U \;\Rightarrow\; A \in U)$$

That is, if some statement has the cotheory as a consequence, it is also a part of the cotheory. So, if you *reject* the cotheory, this rejection means that you are making a mistake if you assert any statement in the cotheory.[7] The statements which have U as a consequence are *unacceptable*, granted commitment to deny U.

A cotheory is the natural dual partner to a theory. However, we don't want to restrict our attention to treating either a theory or a cotheory in isolation, or merely in tandem. Perhaps given the assertion of some members of T and the denial of some other members of U, some other statements are unassertible, or are undeniable. In each case, these statements belong in U or in T respectively. In other words, what we *really* need is a BITHEORY, consisting of both a theory and also a cotheory.

$$\langle T, U\rangle \text{ is a BITHEORY } \textit{iff } (\forall A)(T \vdash A, U \Rightarrow A \in T \textit{ and } T, A \vdash U \Rightarrow A \in U)$$

In other words, $\langle T, U\rangle$ gives us direction both on what is to be asserted (T) and what is to be denied (U). And if, in this context, $\ulcorner A\urcorner$ is undeniable, it also belongs in T, and if $\ulcorner A\urcorner$ is unassertible, it belongs in U.

If we were merely to consider logics in which $(\neg L)$ and $(\neg R)$ held in their generality, we would not need to consider either cotheories or bitheories. In that

[5] Since T is closed under consequence, that is not saying much of corse. We could strengthen things by noting that the statements of the theory are undeniable, given commitment to the *axioms* of the theory, if the axioms X form a set from which all members of T follow.

[6] It might be thought that commitment to something like $\ulcorner A \to \bot\urcorner$ would do it, where $\ulcorner\bot\urcorner$ is to be rejected always. Perhaps that will express a feature appropriate for denial, but now the trouble is that it is too strong. In non-classical logics used for the paradoxes, $\ulcorner A \vee (A \to \bot)\urcorner$ is rejected (and it must be, lest the liar paradox arise for the 'negation' of implying $\ulcorner\bot\urcorner$), so $\ulcorner A\urcorner$ and $\ulcorner A \to \bot\urcorner$ are to be rejected. But this means we must have some way of rejecting $\ulcorner A\urcorner$ which does not involve accepting $\ulcorner A \to \bot\urcorner$. So, $\ulcorner A \to \bot\urcorner$ may express *one* kind of rejection, but it is not enough to express the entirety of the notion.

[7] Note, to reject the cotheory is not to reject the conjunction of its members, since the cotheory marks what is to be rejected. To reject it is to reject each member, just as to accept a theory is to accept each member of the theory.

case $\ulcorner T, A \vdash U \urcorner$ is equivalent to $\ulcorner T \vdash \neg A, U \urcorner$, so membership in U is decided by membership (of the negation) in T. In cases where we do not have a negation connective with such an intimate connection with assertion and denial, this trick will not always work, and hence the need to explicitly consider both components of a bitheory.

Does this distinction actually matter in practice? I think it does. In the rest of this paper I'll look at three non-classical theories as bitheories: they are theories of Numbers, Classes, and Truth. We will see that attention to considerations of assertion and denial – considering these theories as *bi*theories – will provide a range of insights obscured when we consider presentation in the guise of theories alone.

3 NUMBERS, CLASSES & TRUTH

We will start with a simple case. Numbers: theories of arithmetic

3.1 NUMBERS

Axiomatic presentations of theories of arithmetic typically involve many connectives: axioms take the form of conditional statements such as $x' = y' \to x = y$, and so on. It is noticeable, however, that the details of the logic of the conditional in question often does not matter very much [9, 17, 18]. Now that we have the machinery of sequents, and their interpretation in terms of assertion and denial, it turns out that we can strip the extra logical vocabulary away from the core of the presentation of arithmetic. The language for our statements of arithmetic will involve the following items:

$$= \qquad 0 \qquad ' \qquad + \qquad \times$$

Identity is a two-place predicate, $\ulcorner 0 \urcorner$ is a constant, successor is a one-place function, and addition and multiplication are binary two-place functions. For addition and multiplication, the salient requirements in the theory are simple recursive *equations*. We endorse the following:

$$\vdash x + 0 = x \qquad \vdash x + y' = (x + y)'$$

$$\vdash x \times 0 = 0 \qquad \vdash x \times y' = x \times y + x$$

The use of free variables indicates at least the commitment to each *instance* for any choice of terms to fill in for $\ulcorner x \urcorner$ and for $\ulcorner y \urcorner$, but also commitment to each further instance whenever we extend our language to contain more terms of the same type. If you wish to consider quantificational statements, then the logic of the universal quantifier should dictate that not only do we endorse $\ulcorner x + 0 = x \urcorner$ but its generalisation $\ulcorner (\forall x)(x + 0 = x) \urcorner$.[8]

[8] In sequent presentations of logic, that would be a direct consequence of $\ulcorner x + 0 = x \urcorner$ by ($\forall R$), since x occurs nowhere else in the sequent: it is *arbitrary*.

These recursive equations are items to be asserted. They say nothing about what is to be denied.[9] More interesting are the rules governing identity and the successor function. These involve denial:

$$x' = y' \vdash x = y \qquad 0 = x' \vdash$$

The first of the rules is an axiomatic sequent: indicating that successor is a one-to-one function. It pairs an assertion and a denial, dictating that it would be a clash to assert $\ulcorner x' = y' \urcorner$ but to deny $\ulcorner x = y \urcorner$. Given a position in which we have asserted $\ulcorner x' = y' \urcorner$, the only option for $\ulcorner x = y \urcorner$ is to assert it, as it is undeniable. This seems quite plausible: to take $\ulcorner x' = y' \urcorner$ to hold but to reject $\ulcorner x = y \urcorner$ seems to involve a mistaken conception of numbers or of the successor function. Conversely, if we reject $\ulcorner x = y \urcorner$ then the only option for $\ulcorner x' = y' \urcorner$ is to reject it, too.

Similarly, the claim that 0 is not a successor, often formalised as an *axiom*, of the form $\ulcorner 0 \neq x' \urcorner$, is better formulated as a denial. While it is interesting to observe that there are models of arithmetic that get arithmetical truths correct while also including a claim of the form $\ulcorner 0 = n' \urcorner$ for some n, while also committing us to $\ulcorner 0 \neq x' \urcorner$ in general,[10] there is no doubt that these models get something *wrong*. They endorse something that is to be rejected, by the lights of the concepts of arithmetic. They may endorse everything that arithmetic tells us is to be endorsed, but that is not enough to be a model of the *bi*theory of arithmetic, and only a bi-theoretical perspective is enough to draw out this fact.

From the rules so far, we may derive simple statements, such as these:

$$\vdash 0 = 0 \qquad \vdash 0'' + 0'' = 0'' \times 0''$$

$$0 = 0' \vdash \qquad 0' \times 0'' = 0''' \vdash$$

using the recursive equations and (*Cut*). However, it is harder to prove things in *generality*. For this, we need principles of induction. It seems harder to do away with logical vocabulary when it comes to induction, for an induction axiom is typically formulated with a thicket of connectives and quantifiers:

$$(\phi(0) \land (\forall x)(\phi(x) \to \phi(x'))) \to (\forall x)\phi(x)$$

Here, the logic truly makes a difference.[11] However, it seems like the logic of the choice of this or that conditional used in the formulation of an induction *axiom* should not make a difference. Induction is a least number principle. It tells us

[9] To accept that $\vdash x + 0 = x$ is to take $\ulcorner x + 0 = x \urcorner$ to be undeniable, so it tells us about what is *not* to be denied. For positive advice on what is to be denied, however, we need to look elsewhere.

[10] These are the so-called 'mod' models of arithmetic, over the integers modulo n [7, 8].

[11] In the absence of contraction for the conditional or in the absence of weakening, it makes a difference as to whether the induction scheme is formulated as above, or as $\phi(0) \to ((\forall x)(\phi(x) \to \phi(x'))) \to (\forall x)\phi(x))$ or as $(\forall x)(\phi(x) \to \phi(x')) \to (\phi(0) \to (\forall x)\phi(x))$ or as a myriad of other formulations, each subtly different.

that when a property fails to hold of all numbers and it holds of 0, there is a number for which it holds where for the *next* number it fails. Contrapositively, it tells us that when a property holds of some numbers, and it doesn't hold of 0, there is a number for which it fails, but where it holds at the *next* number. In other words, we have the following two principles of *ascent* and *descent*.

$$\frac{\Gamma \vdash \phi(0), \Delta \quad \Gamma, \phi(x) \vdash \phi(x'), \Delta}{\Gamma \vdash \phi(x), \Delta}\ (\textit{Ascent})$$

$$\frac{\Gamma, \phi(x') \vdash \phi(x), \Delta \quad \Gamma, \phi(0) \vdash \Delta}{\Gamma, \phi(x) \vdash \Delta}\ (\textit{Descent})$$

Reading these principles from bottom-to-top, *ascent* tells us that if we have denied $\ulcorner\phi(x)\urcorner$ we should either be prepared to deny $\ulcorner\phi(0)\urcorner$, or we should be prepared to (for some term x) assert $\ulcorner\phi(x)\urcorner$ and deny $\ulcorner\phi(x')\urcorner$. Or from top-to-bottom, if we have claimed $\ulcorner\phi(0)\urcorner$ and if $\ulcorner\phi(x)\urcorner$ brings with it $\ulcorner\phi(x')\urcorner$, then we have claimed $\ulcorner\phi(x)\urcorner$ in general, ascending the tower of numbers.

The *descent* prinicple is the dual. If we have asserted $\ulcorner\phi(x)\urcorner$ we should either be prepared to assert $\ulcorner\phi(0)\urcorner$, or we should be prepared to deny $\ulcorner\phi(x)\urcorner$ while asserting $\ulcorner\phi(x')\urcorner$ (for some term x). Or from top-to-bottom, if we have denied $\ulcorner\phi(0)\urcorner$ and if $\ulcorner\phi(x')\urcorner$ brings with it $\ulcorner\phi(x)\urcorner$, then we had better deny $\ulcorner\phi(x)\urcorner$ in general, lest we be able to descend the tower of numbers to 0, from wherever we started.

Clearly, no conditional in the object-language is required for this formulation of induction principles, but it is just as clear that we have not managed to rid the induction conditions of all conditionality entirely—the turnstile of consequence expresses a conditional connection: there is no escaping that. However, we have been able to formulate induction in such a way that patterns of assertion and denial of statements—themselves not containing conditionals—are enough for us to judge whether induction has been violated or not. This is an advance, for now we can formulate bitheories of arithmetic in which the induction principle is present, yet where we need make no choice over what kind of object-language conditional is present in the theory. We can get some way with arithmetic without having to make that choice at all.

Induction here has split into two rules because in the absence of a negation satisfying both $(\neg L)$ an $(\neg R)$, we have no way, in general, to get from one prinicple to the other, yet it seems that both are equally appropriate rules governing arithmetic. Anyone prepared to endorse the premises of either rule, without endorsing the appropraite conclusion would seem to thereby have a non-standard understanding of the concept of *number*, so they seem appropriate to countenance as axiomatic principles. They are essentially bi-theoretic, governing both assertion and denial. Better still, they apply in the absence of other connectives, so we can examine a great deal of the theory of arithmetic without deciding between a logic for gaps or gluts.

3.2 CLASSES

The aim of this paper is to introduce a new line of inquiry, not to pursue any of those directions to any length. So, instead of exploring arithmetic further, let's now consider class theories. Non-classical logics, of gaps or of gluts, are often proposed as the right means for a 'solution' to the paradoxes confronting Frege's general conception of classes. Frege's axiom (V), the general principle of comprehension for classes, has this form

$$a \in \{x : \phi(x)\} \textit{ if and only if } \phi(a) \quad \text{(V)}$$

Membership in the class $\{x : \phi(x)\}$ is found by way of the membership condition. An object a is in that class if and only if the defining condition $\ulcorner\phi(a)\urcorner$ holds. Again, considered as a single *axiom* it features a biconditional, so the question must be raised: which conditional, what logic? Non-classical logics for axiom (V) differ in their choice at this point [1, 2, 3, 12, 14].[12] Independent of concerns over conditionality, there is a central core to commitment to axiom (V): $\ulcorner\phi(a)\urcorner$ and $\ulcorner a \in \{x : \phi(x)\}\urcorner$ stand and fall together. The assertion of $\ulcorner\phi(a)\urcorner$ has the same upshot as the assertion of $\ulcorner a \in \{x : \phi(x)\}\urcorner$; a denial of $\ulcorner\phi(a)\urcorner$ has the same upshot as a denial of $\ulcorner a \in \{x : \phi(x)\}\urcorner$. Anyone prepared to *assert* $\ulcorner\phi(a)\urcorner$ but to *deny* $\ulcorner a \in \{x : \phi(x)\}\urcorner$ rejects condition (V). Similarly, anyone prepared to *deny* $\ulcorner\phi(a)\urcorner$ but to *assert* $\ulcorner a \in \{x : \phi(x)\}\urcorner$ also rejects condition (V). We have the following two introduction rules for membership in classes:

$$\frac{\Gamma, \phi(a) \vdash \Delta}{\Gamma, a \in \{x : \phi(x)\} \vdash \Delta}\,(\in L) \qquad \frac{\Gamma \vdash \phi(a), \Delta}{\Gamma \vdash a \in \{x : \phi(x)\}, \Delta}\,(\in R)$$

Now, what makes $\ulcorner\{x : \phi(x)\}\urcorner$ an expression denoting a *class* and not a *property* is the commitment to extensionality, the commitment that

Classes with *the same members* are *the same*.

For this, we require some means to express the binary relation of identity. It is traditional, again, to express this as an axiom involving quantification and conditionals (actually, a conditional and a *bi*conditional) something like this:

$$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \to x = y)$$

[12] This is not an idle worry: Curry's paradox wreaks havoc with axiom (V), so in the presence of a conditional, the inference from $\ulcorner p \to (p \to q)\urcorner$ to $\ulcorner p \to q\urcorner$ is to be rejected [10]. But then, $\ulcorner p \to (p \to q)\urcorner$ and $\ulcorner p \to q\urcorner$ express different conditional connections between $\ulcorner p\urcorner$ and $\ulcorner q\urcorner$. For the first, *two* instances of *modus ponens* are required to get from $\ulcorner p\urcorner$ to $\ulcorner q\urcorner$, for the second, one suffices. Which of these conditional notions is to be used in the statement of axiom (V)? This is a genuinely hard problem. Suppose I write $\ulcorner p \to (p \to q)\urcorner$ as $\ulcorner p \to_2 q\urcorner$, and replace my theory expressed in terms of $\ulcorner\to\urcorner$ with one expressed in terms of $\ulcorner\to_2\urcorner$. *Modus ponens* holds for $\ulcorner\to_2\urcorner$ as much as it does for $\ulcorner\to\urcorner$. What changes? Which is the *real* conditional? In the absence of a wider semantic story, the difference is vacuous. Yet for the proponent of a non-classical logic, the difference is important, for if there was no difference, the theory is trivial. So, a wider semantic story of some kind must be told.

and again, there are many debates concerning the appropriate formulation of this condition.[13] Again, we can avoid such baroque discussions by expressing the commitment to extensionality as, at root, commitment to an inference rule in which conditionality is eliminated altogether from the object language.[14]

$$\frac{\Gamma, x \in a \vdash x \in b, \Delta \qquad \Gamma, x \in b \vdash x \in a, \Delta}{\Gamma \vdash a = b, \Delta}\ (Ext_{\in})$$

The rule $(Ext_{\in})$ tells us that if we are prepared to deny $\ulcorner a = b \urcorner$, then we must be prepared either to assert $\ulcorner x \in a \urcorner$ and deny $\ulcorner x \in b \urcorner$, or *vice versa*.[15] I cannot see how anyone prepared to reject any of $(\in L)$, $(\in R)$ or $(Ext_{\in})$ truly accepts an extensional theory of classes satisfying law (V) in its intended meaning.[16] This is problematic, since we shall see that, independently of any fancy footwork concerning the logic of the propositional connectives, $(\in L)$, $(\in R)$ and $(Ext_{\in})$ are already very strong. To explain the untoward consequences of these rules, we need to explain how to understand the logic of identity in this context. I take it that the appropriate notion of identity is one in which the following three rules are satisfied.

$$\frac{\Gamma, \phi(a) \vdash \Delta}{\Gamma, a = b, \phi(b) \vdash \Delta}\ (=L_l) \qquad \frac{\Gamma \vdash \phi(a), \Delta}{\Gamma, a = b \vdash \phi(b), \Delta}\ (=L_r)$$

$$\frac{\Gamma, Xa \vdash Xb, \Delta \quad \Gamma, Xb \vdash Xa, \Delta}{\Gamma \vdash a = b, \Delta}\ (=R)$$

Identity is, at its heart, a second-order notion.[17] If I assert $\ulcorner a = b \urcorner$ and $\ulcorner \phi(b) \urcorner$, then I am thereby committed to $\ulcorner \phi(a) \urcorner$. After all, if I were to assert $\ulcorner \phi(a) \urcorner$

[13]Not only are there debates concerning contraction: there are also debates over *relevance*. Should the main conditional be taken to express a *relevant* connection, or should we weaken the condition to involve a prophylactic $\ulcorner t \urcorner$?

$$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \wedge t \to x = y)$$

Furthermore, does the identity $\ulcorner x = y \urcorner$ *entail* $\ulcorner (\forall z)(x \in z \leftrightarrow y \in z) \urcorner$ or is the connection here not relevance preserving? Options abound.

[14]In the rule $(Ext_{\in})$ we have the side condition that x is absent from Γ and Δ.

[15]The terms $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ denote classes, nothing else here. There is no implicit commitment to the effect that different *numbers*, *electrons* or *tables* must have different $\ulcorner \in \urcorner$-members.

[16]I have myself explored theories and models in which a kind of 'naïve comprehension' holds but in which $(\in L)$, $(\in R)$ fail. The simple LP-models of naïve comprehension [14] validate 'extensionality' in the weak form

$$a \in \{x : \phi(x)\} \equiv \phi(a)$$

where $\ulcorner \equiv \urcorner$ is a material conditional. Here, models do not truly validate $(\in L)$ and $(\in R)$, for a class B in which everything both *is* and *isn't* a member validates that material biconditional, doing the job for $\{x : \phi(x)\}$ for *any* predicate $\ulcorner \phi \urcorner$. A material biconditional with one side 'both' true and false is, at least, true. In models in which B does the job of the empty set $\{x : \bot\}$, we have $\ulcorner a \in B \equiv \bot \urcorner$ materially true, but we are prepared to *assert* $\ulcorner a \in \{x : \bot\} \urcorner$ but at the same time *deny* $\bot$. Here $(\in L)$ fails. The case for B standing in for the universal set is dual.

[17]This is clearly articulated by Stephen Read in his paper 'Identity and Harmony' [13].

and deny $\ulcorner\phi(b)\urcorner$, what more evidence do I need to the effect that $a \neq b$?[18] Similarly, if I assert $\ulcorner a = b\urcorner$ and I *deny* $\ulcorner\phi(b)\urcorner$: I am thereby committed to denying $\ulcorner\phi(a)\urcorner$. This motivates the left identity rules. For the right identity rule ($=R$), if I *deny* $\ulcorner a = b\urcorner$ I must be prepared to countenance something (perhaps not in the vocabulary I already have: it may be a schematic 'property' not expressible in my own vocabulary) holding of a but not of b, or *vice versa*. The second-order nature of the identity rule ($=R$) may seem worrying. In what follows we need not worry at all. For our purposes we need only appeal to ($=L_1$), and for that rule we need only the case where $\ulcorner\phi(x)\urcorner$ is $\ulcorner t \in x\urcorner$. Nothing more is required.

Now there is a puzzle. Friends of Frege's Law (V) have long worried about Russell's paradox, involving class $\mathfrak{R}$ defined as $\{x : x \notin x\}$. For us, this is not the main concern, for nothing *we* have said involves negation, at least in the object-language.[19] Russell's paradox, if it is a paradox at all, is meant as a problem for naïve theories of classes, and as we have seen, we can express these as bitheories governed by the rules ($\in L$), ($\in R$), ($Ext_\in$), and the rules of identity. Can we express the core idea of Russell's paradox in the absence of negation or other propositional connectives in the object-language used to define conditions on classes? It turns out that we can. Using an idea from a paper of Hinnion and Libert [5], we can express the paradox using class abstraction, membership and identity alone: using only the concepts we have used in the rules ($\in L$), ($\in R$), ($Ext_\in$) and nothing else. Therefore, we avoid all of the argument concerning the design of the logical vocabulary governing the predicates $\ulcorner\phi\urcorner$. We cut across all discussion of truth-value gaps or truth-value gluts, contraction, intensional connectives, or anything else. Hinnion and Libert give the following definition [5, p. 831],[20] which I will call the *Hinnion* class:

$$\mathfrak{H} =_{df} \{x : \{y : x \in x\} = \{y : p\}\}$$

Notice, the vocabulary is what is given in the statements of comprehension and extensionality. There is no negation, conditionality, quantifiers, in the definition. It turns out that using only the rules ($\in L$), ($\in R$), ($Ext_\in$), ($=L_1$), (Cut) and (Id) we can derive $\ulcorner p\urcorner$. The derivation is agnostic concerning gaps, gluts and any detail other than these rules.

[18] Yes, we must be careful of the nature of the context $\ulcorner\phi(\cdot)\urcorner$ and the terms $\ulcorner a\urcorner$ and $\ulcorner b\urcorner$. Here there will be no such opaque contexts or non-rigid designators.

[19] Yes, we have kept track of assertion and denial. We have not committed ourself to any particular theory of negation, or even the claim that our language has a single concept of negation. Just as we may be able to express a range of conditional notions, why not a range of negative notions? To think that there is one Russell set is to think that there is one negation.

[20] Actually, they use $\ulcorner\bot\urcorner$, not an abitrary $\ulcorner p\urcorner$ used here. Nothing hangs on this, except the formulation here is slightly more general, designed to apply even in the case where we have no special statement taken to entail all others.

Here is the first part of the derivation. Call it δ_1.

$$\dfrac{\dfrac{\mathfrak{H}\in\mathfrak{H}\vdash\mathfrak{H}\in\mathfrak{H}}{\mathfrak{H}\in\mathfrak{H}\vdash x\in\{y:\mathfrak{H}\in\mathfrak{H}\}}(\in R) \qquad \dfrac{\dfrac{p\vdash p}{x\in\{y:p\}\vdash p}(\in L)}{x\in\{y:\mathfrak{H}\in\mathfrak{H}\},\{y:\mathfrak{H}\in\mathfrak{H}\}=\{y:p\}\vdash p}(=L_1)}{\dfrac{\mathfrak{H}\in\mathfrak{H},\{y:\mathfrak{H}\in\mathfrak{H}\}=\{y:p\}\vdash p}{\mathfrak{H}\in\mathfrak{H}\vdash p}(\in L)}(Cut)$$

Now consider the next part. Call it δ_2.

$$\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\\ \mathfrak{H}\in\mathfrak{H}\vdash p\end{matrix}}{x\in\{y:\mathfrak{H}\in\mathfrak{H}\}\vdash x\in\{y:p\},p}(\in L) \qquad \dfrac{p\vdash x\in\{y:\mathfrak{H}\in\mathfrak{H}\},p}{x\in\{y:p\}\vdash x\in\{y:\mathfrak{H}\in\mathfrak{H}\},p}(\in L)}{\dfrac{\vdash\{y:\mathfrak{H}\in\mathfrak{H}\}=\{y:p\},p}{\vdash\mathfrak{H}\in\mathfrak{H},p}(\in R)}(Ext_{\in})$$

Finally, we paste the two pieces together, to conclude $\ulcorner p\urcorner$.

$$\dfrac{\begin{matrix}\delta_2\\ \vdots\\ \vdash\mathfrak{H}\in\mathfrak{H},p\end{matrix} \qquad \begin{matrix}\delta_1\\ \vdots\\ \mathfrak{H}\in\mathfrak{H}\vdash p\end{matrix}}{\vdash p}(Cut)$$

Given that $\ulcorner p\urcorner$ is to be denied (for some $\ulcorner p\urcorner$ or other), *everyone* has to reject one of the rules ($\in L$), ($\in R$), ($Ext_{\in}$), ($=L_1$), (*Cut*) and (*Id*). At some stage the derivation of $\ulcorner p\urcorner$ is to break down, but where? Orthodoxy tells us that the rules to reject (at least where $\ulcorner\in\urcorner$ expresses class membership) are ($\in L$) or ($\in R$), and the underlying asumption that every predicate determines a set: to reject Law (V). For defenders of Law (V), however, some other move must be rejected. For defenders of Law (V) concerning *classes*, the pickings seem extremely thin: either defend Law (V) despite rejecting ($\in L$) or ($\in R$) — in the face of criticism that to reject ($\in L$) or ($\in R$) is to reject what we meant by Law (V) in the first place — or reject ($Ext_{\in}$) in the face that this was what we meant by *extensionality* in the first place — or finally, find fault in ($=L$), (*Cut*) or (*Id*).

What option can the defender of Law (V) take? The bitheoretical perspective seems to constrain the options for non-classical theories of classes much more stark. Evading this paradox will, at least, help clarify what is at stake in taking a non-classical position on classes in defence of Law (V).

3.3 TRUTH

In the last section, I will see to what extent these results apply to theories of truth defending Tarski's T-scheme in the face of paradoxes like the liar. The structure is similar to Russell's paradox, but at face value there seems to be nothing playing the role of extensionality. Tarski's T-schema is often presented as a biconditional

$$T\{\!\{A\}\!\} \textit{ if and only if } A$$

where $\ulcorner\{\!\{\cdot\}\!\}\urcorner$ is some quotation device, and there is some biconditional connecting $\ulcorner T\{\!\{A\}\!\}\urcorner$ with $\ulcorner A\urcorner$. But as is probably obvious by now, we will not take that route. Instead, we will notice that anyone preparted to assert $\ulcorner T\{\!\{A\}\!\}\urcorner$ and deny $\ulcorner A\urcorner$ or to deny $\ulcorner T\{\!\{A\}\!\}\urcorner$ and assert $\ulcorner A\urcorner$ is rejecting the equivalence. We have the following two rules, governing the bitheory of truth, governing expressions of the form $\ulcorner T\{\!\{A\}\!\}\urcorner$ in positions of assertion and of denial respectively.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, T\{\!\{A\}\!\} \vdash \Delta}\,(TL) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash T\{\!\{A\}\!\}, \Delta}\,(TR)$$

To understand the significance of these rules, we need to ask ourselvs this: what kind of object is $\{\!\{A\}\!\}$? In particular, when is $\{\!\{A\}\!\}$ equal to $\{\!\{B\}\!\}$? One option is to think of $\{\!\{A\}\!\}$ as the *truth value* of $\ulcorner A\urcorner$. If that were the case we would have *extensionality* for the term $\ulcorner\{\!\{A\}\!\}\urcorner$.

$$\frac{\Gamma, T\{\!\{A\}\!\} \vdash T\{\!\{B\}\!\}, \Delta \qquad \Gamma, T\{\!\{B\}\!\} \vdash T\{\!\{A\}\!\}, \Delta}{\Gamma \vdash \{\!\{A\}\!\} = \{\!\{B\}\!\}, \Delta}\,(Ext_{T\{\!\{\ \}\!\}})$$

Anyone prepared to deny $\ulcorner\{\!\{A\}\!\} = \{\!\{B\}\!\}\urcorner$ must either be prepared to assert $\ulcorner T\{\!\{A\}\!\}\urcorner$ and deny $\ulcorner T\{\!\{B\}\!\}\urcorner$ or vice versa. Given extensionality for $\ulcorner\{\!\{\ \}\!\}\urcorner$ in this form, we have the analogue of the Hinnion–Libert paradox. We let the term $\ulcorner\mathfrak{L}\urcorner$ name the truth value of the expression $\ulcorner\{\!\{T\mathfrak{L}\}\!\} = \{\!\{p\}\!\}\urcorner$.[21] So we have the following definition

$$\mathfrak{L} =_{\mathrm{df}} \{\!\{\{\!\{T\mathfrak{L}\}\!\} = \{\!\{p\}\!\}\}\!\}$$

With this in place, we can form the followin derivation. First, δ_3:

$$\dfrac{\dfrac{\dfrac{T\mathfrak{L} \vdash T\mathfrak{L}}{T\mathfrak{L} \vdash T\{\!\{T\mathfrak{L}\}\!\}}\,(TR) \qquad \dfrac{\dfrac{p \vdash p}{T\{\!\{p\}\!\} \vdash p}\,(TL)}{T\{\!\{T\mathfrak{L}\}\!\}, \{\!\{T\mathfrak{L}\}\!\} = \{\!\{p\}\!\} \vdash p}\,(=L_1)}{T\mathfrak{L}, \{\!\{T\mathfrak{L}\}\!\} = \{\!\{p\}\!\} \vdash p}\,(Cut)}{T\mathfrak{L} \vdash p}\,(TL)$$

[21] Diagonalisation, demonstratives, or other devices give you 'self-reference' enough for this.

Then using δ_3, we form δ_4:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\delta_3\\ \vdots\\ T\mathfrak{L} \vdash p\end{array}}{T\{\!|T\mathfrak{L}|\!\} \vdash p}\,(TL)}{T\{\!|T\mathfrak{L}|\!\} \vdash T\{\!|p|\!\}, p}\,\textit{Weakening} \qquad \dfrac{p \vdash T\{\!|T\mathfrak{L}|\!\}, p}{T\{\!|p|\!\} \vdash T\{\!|T\mathfrak{L}|\!\}, p}\,(TL)}{\dfrac{\vdash \{\!|T\mathfrak{L}|\!\} = \{\!|p|\!\}, p}{\vdash T\mathfrak{L}, p}\,(TR)}\,(\mathit{Ext}_{T\{\!|\ |\!\}})$$

Then δ_3 and δ_4 give us:

$$\dfrac{\begin{array}{c}\delta_4\\ \vdots\\ \vdash T\mathfrak{L}, p\end{array} \qquad \begin{array}{c}\delta_3\\ \vdots\\ T\mathfrak{L} \vdash p\end{array}}{\vdash p}\,(\mathit{Cut})$$

It follows that *everyone* who rejects some proposition $\ulcorner p \urcorner$ has to reject one of (TL), (TR), $(\mathit{Ext}_{T\{\!|\ |\!\}})$, $(=L_\iota)$, (*Cut*) and (*Id*). Here the problem does not seem to be so stark, as the commitment to truth *values* in the form required by $(\mathit{Ext}_{T\{\!|\ |\!\}})$ seems rather strong for the defender of a non-classical logic with gaps or gluts.[22]

However, the problem does not go away. Instead of focussing on *truth values* perhaps we should consider *propositions*. We can replace the appeal to $(\mathit{Ext}_{T\{\!|\ |\!\}})$ by appeal to identity of *co-entailing* propositions. Think of $\llbracket A \rrbracket$ as the *proposition* to the effect *that* A. Here the criterion of intensional identity is that denying $\ulcorner \llbracket A \rrbracket = \llbracket B \rrbracket \urcorner$ involves either asserting $\ulcorner T\llbracket A \rrbracket \urcorner$ and denying $\ulcorner T\llbracket B \rrbracket \urcorner$ or

[22]How many truth values are there? Using $(\mathit{Ext}_{T\{\!|\ |\!\}})$ it seems there are only *two*, since we can derive $\vdash \{\!|A|\!\} = \{\!|B|\!\}, \{\!|B|\!\} = \{\!|C|\!\}, \{\!|C|\!\} = \{\!|A|\!\}$. Using the form of $(\mathit{Ext}_{T\{\!|\ |\!\}})$ with weakening built in:

$$\dfrac{\Gamma, T\{\!|A|\!\} \vdash T\{\!|B|\!\}, \Delta \quad \Gamma', T\{\!|B|\!\} \vdash T\{\!|A|\!\}, \Delta'}{\Gamma, \Gamma' \vdash \{\!|A|\!\} = \{\!|B|\!\}, \Delta, \Delta'}\,[\mathit{Ext}]$$

simply to make the proofs *narrow* enough to fit on the page, we have

$$\dfrac{\dfrac{T\{\!|B|\!\}, T\{\!|C|\!\} \vdash T\{\!|C|\!\}, T\{\!|A|\!\} \quad T\{\!|A|\!\} \vdash T\{\!|C|\!\}, T\{\!|A|\!\}}{T\{\!|B|\!\} \vdash T\{\!|C|\!\}, T\{\!|A|\!\}, \{\!|C|\!\} = \{\!|A|\!\}}\,[\mathit{Ext}] \qquad T\{\!|B|\!\}, T\{\!|C|\!\} \vdash T\{\!|B|\!\}}{T\{\!|B|\!\} \vdash T\{\!|A|\!\}, \{\!|B|\!\} = \{\!|C|\!\}, \{\!|C|\!\} = \{\!|A|\!\}}\,[\mathit{Ext}]$$

and similarly, we can prove $T\{\!|A|\!\} \vdash T\{\!|B|\!\}, \{\!|B|\!\} = \{\!|C|\!\}, \{\!|C|\!\} = \{\!|A|\!\}$, which together give us

$$\dfrac{T\{\!|A|\!\} \vdash T\{\!|B|\!\}, \{\!|B|\!\} = \{\!|C|\!\}, \{\!|C|\!\} = \{\!|A|\!\} \qquad T\{\!|B|\!\} \vdash T\{\!|A|\!\}, \{\!|B|\!\} = \{\!|C|\!\}, \{\!|C|\!\} = \{\!|A|\!\}}{\vdash \{\!|A|\!\} = \{\!|B|\!\}, \{\!|B|\!\} = \{\!|C|\!\}, \{\!|C|\!\} = \{\!|A|\!\}}\,[\mathit{Ext}]$$

In other words, of any three truth values, two are equal. To prove that there are *at least two* truth values, more must be done. I suggest finding sentences $\top$ and $\bot$ such that $\{\!|\top|\!\} = \{\!|\bot|\!\} \vdash$

vice versa, now no longer keeping other assumptions as side-conditions. Reading the rule from top-to-bottom, it means merely that if ⌜A⌝ *entails* ⌜B⌝ and ⌜B⌝ *entails* ⌜A⌝ then the propositions $[\![A]\!]$ and $[\![B]\!]$ are identical.

$$\dfrac{T[\![A]\!] \vdash T[\![B]\!] \qquad T[\![B]\!] \vdash T[\![A]\!]}{\vdash [\![A]\!] = [\![B]\!]}\ (\mathit{Int}_{T[\![\ ]\!]})$$

This is, as with $(\mathit{Ext}_{T\{\!|\ |\!\}})$, a substantial commitment. However, it is a commitment to the heart of the model-theory of many non-classical logics. In these models if the entailment from ⌜A⌝ to ⌜B⌝ fails, there is some *point* (world, situation, whatever) where ⌜A⌝ holds and ⌜B⌝ does not. Construing a proposition as a set of points (perhaps satisfying some kind of closure or coherence condition), if ⌜A⌝ and ⌜B⌝ are co-entailing, they hold at the same points, and so correspond to the same propositions, construed as sets of points. So the condition $(\mathit{Int}_{T[\![\ ]\!]})$ is not a foreign idea.

However, $(\mathit{Int}_{T[\![\ ]\!]})$ causes nearly as much trouble as $(\mathit{Ext}_{T\{\!|\ |\!\}})$. If we have a statement $\bot$, from which anything follows (which is always unassertable), we can replace $(\mathit{Ext}_{T\{\!|\ |\!\}})$ by $(\mathit{Int}_{T[\![\ ]\!]})$ in our problematic derivation. Using fixed-points, define our problematic proposition term ⌜$\mathfrak{P}$⌝ by setting

$$\mathfrak{P} =_{\mathrm{df}} [\![[\![T\mathfrak{P}]\!] = [\![\bot]\!]]\!]$$

Then we have δ_5:

$$\dfrac{\dfrac{\dfrac{\dfrac{T\mathfrak{P} \vdash T\mathfrak{P}}{T\mathfrak{P} \vdash T[\![T\mathfrak{P}]\!]}\ (TR) \qquad \dfrac{\dfrac{\dfrac{}{\bot \vdash}\ (\bot L)}{T[\![\bot]\!] \vdash}\ (TL)}{T[\![T\mathfrak{P}]\!], [\![T\mathfrak{P}]\!] = [\![\bot]\!] \vdash}\ (=L_\iota)}{T\mathfrak{P}, [\![T\mathfrak{P}]\!] = [\![\bot]\!] \vdash}\ (\mathit{Cut})}{T\mathfrak{P} \vdash}\ (TL)}{}$$

Using δ_5 we can construct δ_6:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\delta_5\\ \vdots\\ T\mathfrak{P} \vdash\end{matrix}}{T\mathfrak{P} \vdash \bot}\ (\bot R)}{T[\![T\mathfrak{P}]\!] \vdash \bot}\ (TL)}{T[\![T\mathfrak{P}]\!] \vdash T[\![\bot]\!]}\ (TR) \qquad \dfrac{\bot \vdash T[\![T\mathfrak{P}]\!]}{T[\![\bot]\!] \vdash T[\![T\mathfrak{P}]\!]}\ (TL)}{\vdash [\![T\mathfrak{P}]\!] = [\![\bot]\!]}\ (\mathit{Int}_{T[\![\ ]\!]})}{\vdash T\mathfrak{P}}\ (TR)$$

Together, δ_5 and δ_6 give us

$$\dfrac{\begin{matrix}\delta_6 \\ \vdots \\ \vdash \mathsf{T}\mathfrak{P}\end{matrix} \qquad \begin{matrix}\delta_5 \\ \vdots \\ \mathsf{T}\mathfrak{P} \vdash\end{matrix}}{\vdash}\ (Cut)$$

which is a problematic conclusion, since $\Gamma \vdash \Delta$ follows for every Γ, Δ. This tells us that there is a clash in every position.

It follows that *everyone* has to reject one of ($\mathsf{T}L$), ($\mathsf{T}R$), ($Int_{\mathsf{T}[\![\ ]\!]}$), ($=L_1$), ($\bot L$), ($\bot R$), (*Cut*) and (*Id*). If, in particular, the defender of ($\mathsf{T}L$) and ($\mathsf{T}R$) wishes to reject ($Int_{\mathsf{T}[\![\ ]\!]}$), the onus is on her or him to give an account of the semantics of the non-classical logic in use in such a way as to not allow for a definition of propositions which motivates ($Int_{\mathsf{T}[\![\ ]\!]}$). Given the widespread use of world-like semantics, this seems to be a significant challenge.

CONCLUSION

Here is the moral of the story so far: bitheories (and sequent rules) give us a way to specify natural conditions on concepts, such as

$$numbers \quad \in \quad \{\ :\ \} \quad \mathsf{T} \quad \{\!|\ |\!\} \quad [\![\ ]\!]$$

in a way that abstracts away from debates over this or that logic. The results here apply to logics with gaps, with gluts, with any number of different connectives. Attending instead to the way entailment constrains assertion and denial allows us to avoid stepping in to those difficult debates, to uncover common structure underlying many different theories in many different logics.

Let me end with some homework for everyone interested in these issues.

1. *For everyone*: Use *bitheories*. The bitheoretical formulation of theories of numbers, classes and truth has proved to be clarifying. You do not need to be a partisan in favour of non-classical logics to be interested in a formulation of arithmetic which allows for the arithmetic rules to be independent from the connectives and quantifiers. In this way, we have a natural account of *positive* arithmetic (arithmetic without negation), of the shared core between classical Peano arithmetic and intuitionist Heyting arithmetic, and many other connections may be explored.

2. *For friends of gaps or gluts*: Articulate and defend your commitments connecting consequence, negation, assertion and denial. I have attempted to sketch what I take to be those connections. Perhaps the story told here is wrong. Regardless, it is certainly incomplete. Friends of gaps and gluts should not merely present *theories* of things they take to be true. Given a gap or a glut at the boundary between truth and falsity, the presentation

of a theory is more complicated, and the connections between logical consequence, assertion and denial — and the role of the concept (or concepts?) of negation must be articulated. The role of (*Cut*) and (*Id*) sketched here (and defended elsewhere [16]) are crucial in everything we have done. If there was some way to live without (*Cut*) or (*Id*), that would open up more space for strong non-classical theories of classes and truth. But what can we leave in their place? What is the connection between assertion and denial, consequence and negation if not the one sketched here? What story can be told?

3. Finally, *for friends of strong theories such as* ($\in L/R$) *or* ($\mathsf{T}L/R$): articulate and defend your response to these paradoxes. In particular, a friend of Law (V) for classes, or the T-scheme for truth must explain which of (*Id*), (*Cut*), ($=L_1$) and (*Ext* & *Int*) are to be rejected. In particular, the defender of these theories needs to isolate a point in each derivation where it breaks down: a rule cannot fail merely because it does not satisfy this or that strong constraint on validity, but rather, in the cases in question we must find a spot in the derivation where we are prepared to grant the premises of a rule but reject the conclusion.

   Answering this challenge will involve work. (In particular, it will involve giving an answer to Homework Task 2.) No matter how this challenge is met, an answer will help us understand non-classical theories of classes and truth much more.

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