BRADWARDINE HYPERSEQUENTS

Greg Restall*

Philosophy Department,
The University of Melbourne,
Parkville 3010, Australia

restall@unimelb.edu.au

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Abstract: According to Stephen Read, Thomas Bradwardine's theory of truth provides an independently motivated solution to the paradoxes of truth, such as the liar [7, 9]. In a series of papers, I have discussed modal models for Read's reconstruction of Bradwardine's theory [12, 13]. In this paper, provide a hypersequent calculus for this theory, and I show that the cut rule is admissible in the hypersequent calculus.

It gives me great pleasure to honour Professor Stephen Read, whose work has been a profound influence on my own. His work on relevant logic [4, 5], on proof theoretical harmony [6], on the logic of identity [8] and on Thomas Bradwardine's theory of truth [7, 9, 10] have been a rich source of insight, of stimulation and of provocation. In an attempt to both honour Stephen, and hopefully to give him some pleasure, I am going to attempt to cook up something original using some of the many and varied ingredients he has provided us. In this paper, I will mix and match ideas and techniques from Stephen's papers on proof theory, on Bradwardine's theory of truth, and on identity to offer a harmonious sequent system for a theory of truth inspired by Stephen Read's recovery of the work of Thomas Bradwardine.

1 BACKGROUND

I have written before on models for Bradwardine's theory of truth as recovered by Stephen Read [12, 13]. The crucial syntactic innovation of the formal theory of truth is the fact that truth is a defined notion and not a primitive one. The primitive notion in the theory is that expressed by the 'connecticate' ":", or "says that". The connecticate is half *connect*ive and half pred*icate*. Like a *predicate*, it takes a singular term (here on the left), and like a *connective* it takes a

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the sentence (here, on the right). In words, "$t : p$" says that t says that p. The connecticate expresses *signification*, the idea that certain objects (sentences, utterances, inscriptions, whatever) signify some things, and fail to signify others, and we express this in our formal language using signification. If s is a sentence that says that $2+2=4$ we can say

$$s : 2+2=4$$

but if that very sentence does *not* say that $2 \times 2 = 4$ we can say

$$\neg(s : 2 \times 2 = 4)$$

The parentheses are there to help with disambiguation, but they are not strictly necessary. Our convention is that ':' binds as strongly as grammatically possible, so '$t : p \supset q$' is a conditional whose antecedent is '$t : p$', while in '$t : 2+2=4$' the colon cannot bind anything smaller than '$2+2=4$' since this is the smallest available sentence commencing after the colon.

Now, when is an object true? Presumably if it says something, and if whatever it says is the case. To express this, we will make use of propositional quantification. The intuitive idea of truth was straightforward:

$$Tx =_{df} (\exists p)x : p \wedge (\forall p)(x : p \supset p)$$

For x to be true, there must be something that x says, and of anything x says, it is the case. For falsity we require the opposite. x is false if and only if there is something x says that is not the case.

$$Fx =_{df} (\exists p)(x : p \wedge \neg p)$$

Given just this raw definition of truth and of falsity, we can already learn something about the paradoxes. If λ is a liar sentence, then it says of itself that it is not true. This can be specified precisely. λ is a liar if and only if we have

$$\lambda : \neg T\lambda$$

Given the *definition* of the truth predicate we can reason as follows: Suppose $T\lambda$. Then we would have $(\forall p)(\lambda : p \supset p)$, and since $\neg\lambda : \neg T\lambda$ we would have $\neg T\lambda$. So, it follows that $\neg T\lambda$, and furthermore, that $F\lambda$ (since there is something that λ does say). So, if λ is a liar sentence, then indeed it is not true.

Now, we do not necessarily land in self-contradictary reasoning, because it does not follow that $T\lambda$ (that λ is also true), unlike the situation with Tarski's theory of truth. For all we can conclude in this case is that *something* that λ says (namely $\neg T\lambda$) holds, not that *everything* that λ says holds, which is what is required for truth. So, it follows that *something* that λ says fails to hold.[1]

[1] What is this further something that λ says? It is hard to say something general about this. Read argues (reconstructing an argument of Bradwardine) that of anything that says of itself that it is false also says of itself that it is true [9, page 311]. I have shown elsewhere that this conclusion does not follow from the premises we have so far accepted [13]. There are models of Bradwardine's theory in which some liars say of themselves that they are true, and others do not.

This is enough to specify liar sentences and to say a little about their properties. However, it is not enough to give an account of all of the distinctive behaviour of signification. What can we say about the signification relation? Here are three possible principles concerning signification. (1) if $t : p$ and $t : q$, does it follow that $t : p \wedge q$? Or (2) if $t : p$ and if $p \vdash q$ does it follow that $t : q$? Finally, (3) $t : p$ and $p \supset q$ holds, does it follow that $t : q$? In an earlier paper [12], I provided a modal model theory for the signification relation, according to which the principles (1) and (2) generally hold, so signification is closed under conjunction, and under entailment, but for which (3) does not hold: we may have $t : p$ and $p \supset q$ may contingently hold without $t : q$ holding. For example, if something thing (say t) says that Socrates is mortal, then the mere contingent truth the material conditional 'If Socrates is mortal then the 2012 Olympics was held in London' is not enough to ensure that t also says that the 2012 Olympics was held in London. Signification, on these models, is closed under *entailment* but not under the material conditional. Things say all those things *entailed* by what they say, but not necessarily those things contingently *implied* by what they say.

2 MODAL MODELS

In "Modal Models for Bradwardine's Theory of Truth" I introduced modals in which '$x : p$' is read as an indexed necessity operator '$\Box_x p$.' We will do the same here, with a number of simplifying assumptions. In that paper, a model consists of a frame $\langle W, O, D, \{R_d : d \in D\}\rangle$, featuring of a class of worlds W, a domain O of objects, and a subset D of O of those objects which are *declarative* (those objects which signify), and for each declarative object, a binary accessibility relation R_d on W, together with an evaluation relation $\Vdash$ relating worlds (and assignments of values to the variables) to formulas. For our presentation in this paper, we make two simplifying assumptions: (1) that *all* objects signify, so $D = O$, and (2) we will assume that signification is modally fixed: that is, for each $w, v, v' \in W$, $vR_d w$ iff $v'R_d w$. In other words, *what* an object signifies does not vary from world to world. So, the accessibility relation R_d for an object d can be defined by way of a class of worlds W_d, where we set $wR_d v$ if and only if $v \in W_d$. (We can think of W_d as the set of worlds which are as d describes.)

Along with those two simplifying assumptions, we will make one *liberalising* assumption, to the effect that the domain of quantification for the second order quantifiers (of which the propositional quantifiers are a special case) is a *subclass* of the standard (full) domain. We will in particular focus on Henkin models, where the domain of quantification is closed under the usual logical operations definable in the language, in the standard manner [16].

Summing up, we have the following definition of a frame, of a model and truth in a model.

DEFINITION [SIMPLE BRADWARDINE FRAMES] A structure $\langle W, D_1, D_2, \{R_d : d \in D_1\}\rangle$ is a *simple Bradwardine frame* when it is made up of

» A non-empty set W of *worlds*

» A non-empty set D_1 of *objects*

» A non-empty set D_2 consisting of sets D_2^n of $n+1$-tuples from $D_1 \times \cdots \times D_1 \times W$. D_2^n is the range of quantification of n-place predicates (which vary from world to world), so D_2^0, a set of sets of worlds is the range of quantification of the propositional quantifiers.

» A relation $R_d \subseteq W \times W$ for each $d \in D_1$, such that for each $w, v, v' \in W$, wR_dv iff wR_dv'.

The language we will interpret on simple Bradwardine frames is straightforward. We have already seen the connecticate "$:$". In addition to this, we will use full second order quantification with n-ary predicates X^n (for $n = 0, 1, 2, \ldots$) rather than restrict outselves to the 0-ary case of propositional quantification, which is all that is strictly necessary for formulating the theory of truth. The reason is simple: unary predicate quantification will play a role in the final section on the logic of identity, and the proof theory works smoothly with arbitrary second order quantification, so there is no problem in including it.

Furthermore, it simplifies our presentation to use explicit λ-abstraction to create complex n-ary predicates. Given a formula A and the variables $x_1, \ldots, x_n$ the n-place λ-abstract $(\lambda x_1 \cdots x_n\, A)$ is a complex n-place predicate, so given n terms $t_1, \ldots, t_n$, $(\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n$ is a formula. The truth conditions for this formula are exactly the same as that of the formula $A|^{x_1\cdots x_n}_{t_1\cdots t_n}$ found by simultaneously substituting the terms $t_1, \ldots, t_n$ into the instances of the variables $x_1, \ldots, x_n$ that are free in A.

With all of that, given a Simple Bradwardine Frame, we can interpret sentences in our formal language on the frame in the usual manner:

DEFINITION [SIMPLE BRADWARDINE MODELS] An evaluation on a simple Bradwardine frame is provided by giving an extension to every relation in the language, and a denotation for every name. The *variables* in the language will be interpreted with the aid of an assignment α of values to variables. The value α assigns to an *objectual* variable such as x is an object $[\![x]\!]_\alpha$ in D_1. The value that α assigns to a predicate variable X^n is a *set* $[\![X^n]\!]_\alpha$ in D_2^n of $n+1$-tuples (of n domain elements from D_1 and one world). The projection of $[\![X^n]\!]_\alpha$ on the world w is thus a set of n-tuples from D_1, the extension of the n-place predicate at the world w.

Given such an assignment of values to atomic expressions, we can assign values to complex expressions, relative to the choice of a world and the choice of an assignment of values to variables.

» $\mathfrak{M}, \alpha, w \Vdash Rt_1 \cdots t_n$ iff the n-tuple $\langle [\![t_1]\!]_\alpha, \ldots, [\![t_n]\!]_\alpha \rangle$ is in the projection of the extension $[\![R]\!]$ at world w.

» $\mathfrak{M}, \alpha, w \Vdash p$ iff w is in $[\![p]\!]_\alpha$.

» $\mathfrak{M}, \alpha, w \Vdash \neg A$ iff $\mathfrak{M}, \alpha, w \nVdash A$.

» $\mathfrak{M}, \alpha, w \Vdash A \wedge B$ iff $\mathfrak{M}, \alpha, w \Vdash A$ and $\mathfrak{M}, \alpha, w \Vdash A$.

» $\mathfrak{M}, \alpha, w \Vdash (\forall x)A$ iff $\mathfrak{M}, \alpha', w \Vdash A$ for every x-variant α' of α.

» $\mathfrak{M}, \alpha, w \Vdash (\forall X^n)A$ iff $\mathfrak{M}, \alpha', w \Vdash A$ for every X^n-variant α' of α.

» $\mathfrak{M}, \alpha, w \Vdash (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n$ iff $\mathfrak{M}, \alpha, w \Vdash A|^{x_1 \cdots x_n}_{t_1 \cdots t_n}$, where $A|^{x_1 \cdots x_n}_{t_1 \cdots t_n}$ is the result of simultaneously substituting the free instances of $x_1, \ldots, x_n$ in A by $t_1, \ldots, t_n$, respectively.

» $\mathfrak{M}, \alpha, w \Vdash \Box A$ iff $\mathfrak{M}, \alpha, v \Vdash A$ for every world v.

These clauses are completely standard. They model constant domain quantified S5 with a universal accessibility relation, and with second order quantification ranging over a possibly restricted second order domain. The innovation is in the treatment of 'says that'.

» $\mathfrak{M}, \alpha, w \Vdash t : A$ iff for each v where $wR_{[\![t]\!]_\alpha}v$, we have $\mathfrak{M}, \alpha, v \Vdash A$.

In other words, 't :' functions as a normal modal operator, using the accessibility relation $R_{[\![t]\!]}$.

This completes our definition of simple Bradwardine models and the recursive satisfaction relation. In the rest of this paper I will provide a sound and complete sequent system for the logic of this class of models.

3 HYPERSEQUENTS

Modal reasoning makes use of not only flat assertion and denial, but also assertion and denial under *suppositions*. In the modal case, we can suppose that things obtain in some way other than they have *actually* obtained. If a sequent

$$A \vdash B$$

tells us that it would be a mistake to assert A and deny B, then the *hypersequent*

$$A \vdash B \mid C \vdash D$$

tells us that it would be a mistake to (in one context) assert A and deny B and (in another context) assert C and deny D. So, while

$$A \vdash A$$

is an axiomatic sequent—there is a clash between asserting A and denying A. The corresponding hypersequent

$$A \vdash \;\mid\; \vdash A$$

involves no such clash. There is no clash involved in asserting A in one part of a discourse and (under a different supposition) denying A. After all, the whole point of modal reasoning is to consider alternate possibilities.

There would be nothing to be gained in this larger hypersequent structure unless there were some way to bridge the gap between zones of a hypersequent. This is the point of the modal operators. This is a sequent that *does* involve a clash.

$$\Box A \vdash \;\mid\; \vdash A$$

Asserting that A is *necessary* and (in some alternate zone) denying A does involve a clash, since the job a of a claim to necessity is to range over alternate possibilities. I develop the modal proof theory for S5 and more complex modal logics in this manner [2, 11, 14].

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This sort of reasoning helps with signification too. Given some object t that signifies, we can suppose that things are *as* t *describes*. This motivates taking the following sequent to involve a clash:

$$t{:}A \vdash \;\mid\; \vdash_t A$$

where now, the *tagged* sequent $\vdash_t A$ represents a t-zone—denying A in a context governed by the term t, which is taking things to be as t describes. Asserting that $t{:}A$ (that t says that A) and denying A in a t-zone therefore involves a clash. This is the kind of structure we will employ in our reasoning: a hypersequent in which the zones may be tagged with terms.

DEFINITION [BRADWARDINE HYPERSEQUENT] A non-empty multiset of sequents of formulas

$$\Gamma_1 \vdash_{t_1} \Delta_1 \mid \cdots \mid \Gamma_m \vdash_{t_m} \Delta_m \mid \Gamma_{m+1} \vdash \Delta_{m+1} \mid \cdots \mid \Gamma_n \vdash \Delta_n$$

in which m sequents are tagged with terms, and $n - m$ sequents are untagged is said to be a *Bradwardine sequent*. In this Bradwardine sequent we must have $n + m \geqslant 1$ but either n or m can be 0.

It will very soon get quite tedious to keep singling out both those component sequents which are tagged and those which are not. So, from now, we will use the convention that when tagging a sequent $\Gamma \vdash_t \Delta$, the t is either a term or is *empty*, so the sequent in this case is untagged. In the case that t is empty, then the signification formula $t{:}A$ is the necessitation $\Box A$.

Bradwardine sequents can be interpreted in a simple Bradwardine model.

DEFINITION [HYPERSEQUENTS IN MODELS] The model $\mathfrak{M}$ is a *counterexample* to the hypersequent $\Gamma_1 \vdash_{t_1} \Delta_1 \mid \cdots \mid \Gamma_n \vdash_{t_n} \Delta_n$ if and only if there are worlds $w_1, \ldots, w_n$ where for each i, $w1R_{[\![t_i]\!]}w_i$ (and if t_i is absent, then $R_{[\![t_i]\!]}$ is the universal relation) and where each element of Γ_i is true at w_i and each element of Δ_i is false at w_i. In other words, for every tagged sequent $\Gamma_i \vdash_{t_i} \Delta_i$ we want an world w_i, accessible using the $R[\![t_i]\!]$ relation, where each member of Γ_i holds and each member of Δ_i fails. If we can do that for *each* sequent in the hypersequent, we have a counterexample.

If a hypersequent has no counterexample, then we say that the hypersequent is *valid*.

» «

Now we will use hypersequents to give an account of the rules of derivation appropriate for our theory of signification. In these statements of the rules, instead of writing out a long hypersequent, we will often write '$\Gamma \vdash_t \Delta \mid \mathcal{H}$' to indicate a hypersequent in which $\Gamma \vdash_t \Delta$ is one component sequent. Then the hypersequent '$\Gamma, A \vdash_t \Delta \mid \mathcal{H}$,' for example, is to be found by adding the formula A into the anteedent of the component squent $\Gamma \vdash_t \Delta$ of the original hypersequent.

So, we start with the structural rules *Identity* and *Cut*, and *Contraction*, which govern the logical structure of formulas as such, without singling out the behaviour of any particular logical constants.

$$\Gamma, A \vdash_t A, \Delta \mid \mathcal{H} \ (Id) \qquad \frac{\Gamma \vdash_t A, \Delta \mid \mathcal{H} \qquad \Gamma, A \vdash_t \Delta \mid \mathcal{H}}{\Gamma \vdash_t \Delta \mid \mathcal{H}} \ (Cut)$$

$$\frac{\Gamma, A, A \vdash_t \Delta \mid \mathcal{H}}{\Gamma, A \vdash_t \Delta \mid \mathcal{H}} \ (WL) \qquad \frac{\Gamma \vdash_t A, A, \Delta \mid \mathcal{H}}{\Gamma \vdash_t A, \Delta \mid \mathcal{H}} \ (WR)$$

The rules for propositional logical connectives behave in the expected manner, locally to a component in a sequent. Here are the rules for conjunction and negation

$$\frac{\Gamma, A, B \vdash_t \Delta \mid \mathcal{H}}{\Gamma, A \wedge B \vdash_t \Delta \mid \mathcal{H}} \ (\wedge L) \qquad \frac{\Gamma \vdash_t A, \Delta \mid \mathcal{H} \qquad \Gamma \vdash_t B, \Delta \mid \mathcal{H}}{\Gamma \vdash_t A \wedge B, \Delta \mid \mathcal{H}} \ (\wedge R)$$

$$\frac{\Gamma \vdash_t A, \Delta \mid \mathcal{H}}{\Gamma, \neg A \vdash_t \Delta \mid \mathcal{H}} \ (\neg L) \qquad \frac{\Gamma, A \vdash_t \Delta \mid \mathcal{H}}{\Gamma \vdash_t \neg A, \Delta \mid \mathcal{H}} \ (\neg R)$$

Other connectives can be defined in terms of conjunction and negation, and they have rules of the structure one would expect. The first-order quantifier rules are also as expected

$$\frac{\Gamma, A|^x_s \vdash_t \Delta \mid \mathcal{H}}{\Gamma, (\forall x)A \vdash_t \Delta \mid \mathcal{H}} \ (\forall L) \qquad \frac{\Gamma \vdash_t A, \Delta \mid \mathcal{H}}{\Gamma \vdash_t (\forall x)A, \Delta \mid \mathcal{H}} \ (\forall R)$$

with the usual side condition that the variable x is not free in the premise hypersequent at any place other than A (so, we have derived A on the basis of no assumptions concerning x, so the derivation is purely general: we have the werewithal to derive $(\forall x)A$).

The second order quantifier rules are similar, with the only complication being the syntax, according to which X^n is an n-ary predicate variable, and P^n is an n-place predicate—either a primitive predicate or a λ abstraction.

$$\frac{\Gamma, A|^{X^n}_{P^n} \vdash_t \Delta \mid \mathcal{H}}{\Gamma, (\forall X^n)A \vdash_t \Delta \mid \mathcal{H}}\ (\forall^n_2 L) \qquad \frac{\Gamma \vdash_t A, \Delta \mid \mathcal{H}}{\Gamma \vdash_t (\forall X^n)A, \Delta \mid \mathcal{H}}\ (\forall^n_2 R)$$

Exactly the same variable condition holds on the premise of $\forall^n_2 R$: the variable X^n occurs nowhere free in the premise of the rule other than in the consequent formula A.

The second order quantifier rules make use of complex predicates, and for that we need to have rules for λ abstraction. These are straightforward:

$$\frac{\Gamma, A|^{x_1 \cdots x_n}_{t_1 \cdots t_n} \vdash_t \Delta \mid \mathcal{H}}{\Gamma, (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n \vdash_t \Delta \mid \mathcal{H}}\ (\lambda L) \qquad \frac{\Gamma \vdash_t A|^{x_1 \cdots x_n}_{t_1 \cdots t_n}, \Delta \mid \mathcal{H}}{\Gamma \vdash_t (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n, \Delta \mid \mathcal{H}}\ (\lambda L)$$

Finally, we need rules for signification and necessitation.

$$\frac{\Gamma \vdash_{t'} \Delta \mid A, \Gamma' \vdash_t \Delta' \mid \mathcal{H}}{\Gamma, t : A \vdash_{t'} \Delta \mid \Gamma' \vdash_t \Delta' \mid \mathcal{H}}\ (:L) \qquad \frac{\Gamma \vdash_{t'} \Delta \mid\ \vdash_t A \mid \mathcal{H}}{\Gamma \vdash_{t'} t : A, \Delta \mid \mathcal{H}}\ (:R)$$

$$\frac{\Gamma \vdash_{t'} \Delta \mid A, \Gamma' \vdash_t \Delta' \mid \mathcal{H}}{\Gamma, \Box A \vdash_{t'} \Delta \mid \Gamma' \vdash_t \Delta' \mid \mathcal{H}}\ (\Box L) \qquad \frac{\Gamma \vdash_{t'} \Delta \mid\ \vdash A \mid \mathcal{H}}{\Gamma \vdash_{t'} \Box A, \Delta \mid \mathcal{H}}\ (\Box R)$$

This completes the presentation of the hypersequent system. Here are some example derivations. The first shows that signification is closed under *modus ponens*. (We use the obvious defined rules for the material conditional.)

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash \mid A \vdash_t A \quad \vdash \mid B \vdash_t B}{\vdash \mid A \supset B, A \vdash_t B}\ (\supset L)}{t : A \supset B \vdash \mid A \vdash_t B}\ (:L)}{t : A \supset B, t : A \vdash \mid \vdash_t B}\ (:L)}{t : A \supset B, t : A \vdash t : B}\ (:R)$$

The next derivation demonstrates the modal flavour of signification in our proof theory and in our models:

$$\cfrac{\cfrac{\cfrac{\vdash \mid \vdash_s \mid p \vdash_t p}{t : p \vdash \mid \vdash_s \mid \vdash_t p}\ (:L)}{t : p \vdash \mid \vdash_s t : p}\ (:R)}{t : p \vdash s : t : p}\ (:R)$$

This is a consequence of the modal inertness of signification: signification facts are true in *every* circumstance, and so, are signified by *anything*. For a richer and more realistic theory, we would need not only to label zones with terms, but keep track of an accessbility relation [2], so finer distinctions can be made. For here, however, this is enough to show some of the distinctive features of this labelled modal system.

For our next feature of the hypersequent system, we will first prove a simple lemma.

LEMMA *The rule*

$$\frac{\Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash_t \Delta \mid \mathcal{H}}$$

is height preserving admissible. This means that if there is a derivation of the hypersequent $\Gamma \vdash \Delta \mid \mathcal{H}$ *then there is a derivation of* $\Gamma \vdash_t \Delta \mid \mathcal{H}$ *of the same height.*

Proof: Take the derivation of $\Gamma \vdash \Delta \mid \mathcal{H}$ and trace upwards from the sequent $\Gamma \vdash \Delta$ to keep track of all of the ancestor sequents, and label each with t. The result is still a derivation. The only rule in which a sequent *must* be unlabelled is $\Box R$ and in this case it is a *premise* hypersequent and not a *conclusion* hypersequent, and this sequent is not the ancestor of any component sequent in the conclusion, so any unlabelled sequents in a conclusion may be freely labelled (and the process continue) without disrupting any rules in place. ∎

LEMMA *The rule*

$$\frac{A \vdash B}{t{:}A \vdash t{:}B}$$

is also admissible.

Proof: Given a derivation of $A \vdash B$, transform it into a derivation of $A \vdash_t B$ as in the previous lemma. From here, we reason as follows:

$$\cfrac{\cfrac{A \vdash_t B}{t{:}A \vdash \mid \vdash_t B}\,(:L)}{t{:}A \vdash t{:}B}\,(:R)$$

and we have our result. ∎

As a final example at this stage, let us look at the resoning we have already seen, to the effect that if $\lambda{:}\neg T\lambda$ then $\neg T\lambda$. What is perhaps a little surprising is that the crucial reasoning for about this liar sentence uses absolutely *nothing* of the logic of signification. It uses only the definition of T and the rules governing the logical connectives and quantifiers. Recall that $T\lambda$ is shorthand for $(\exists p)x:p \wedge (\forall p)(x:p \supset p)$.

$$\dfrac{\dfrac{\lambda:\neg T\lambda \vdash \lambda:\neg T\lambda \qquad \dfrac{T\lambda \vdash T\lambda}{T\lambda, \neg T\lambda \vdash}(\neg L)}{\dfrac{\lambda:\neg T\lambda, T\lambda, \lambda:\neg T\lambda \supset \neg T\lambda \vdash}{\dfrac{\lambda:\neg T\lambda, T\lambda, (\forall p)(\lambda:p \supset p) \vdash}{\lambda:\neg T\lambda, T\lambda \vdash}(\wedge L)}(\forall L)}(\supset L)}{\lambda:\neg T\lambda \vdash \neg T\lambda}(\neg R)$$

So, the fact an object that signifies its own untruth is, in fact, untrue, depends solely on the logical vocabulary, and the definition of truth in terms of that vocabulary. Nothing need be assumed about the logical structure of signification in this reasoning. We do not use any rules for ':'.

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The results of the rest of the paper follow the general structure of those in "A Cut-Free Sequent System for Two Dimensional Modal Logic" [14]. Since the results are standard, I will sketch them here. Full proofs of this kind can be found that paper.

We have already seen what it is for a hypersequent to have a counterexample in a model, and we have defined the *valid* hypersequents as those that hold in every model.

DEFINITION [SOUNDNESS] Every derivable hypersequent holds in every simple Bradwardine model. That is, derivable hypersequents are *valid*.

Proof: A simple induction on the length of the derivation. Axioms are all valid, and if the premises of a rule are valid, so is the conclusion. As a result, all derivable hypersequents are valid. ∎

So, the sequent calculus does not overreach the class of Bradwardine models. So far, so good. The converse is harder to show. Significantly harder.

4 COMPLETENESS AND THE ADMISSIBILITY OF CUT

To show completeness, we will show for any underivable hypersequent, we can find a model in which it fails. For this, we will show more, that if we have a hypersequent which cannot be derived in the absence of cut, then we have a model which forms a counterexample. We will show this using the technique of "A Cut-Free Sequent System for Two Dimensional Modal Logic" [14]. We will embed any underivable hypersequent into a directed family of hypersequents, satisfying certain closure conditions. First, we need the notion of *extension* for hypersequents.

DEFINITION [EXTENSION OF HYPERSEQUENTS] $\mathcal{H}'$ extends $\mathcal{H}$ iff there is some map f from the formula instances in $\mathcal{H}$ to instances of the same formulae in $\mathcal{H}'$ which preserves all hypersequent structure. This means the following two things: *first*, for each formula occurrence A in $\mathcal{H}$ its corresponding occurrence $f(A)$ in $\mathcal{H}'$ shares its *shape* (it is an instance of the same formula), its *position in a sequent* (in the left or the right of the sequent), its *sequent label*, if it started with one (so if A is in a t-sequent in $\mathcal{H}$, so is $f(A)$ in $\mathcal{H}'$); and *second*, if A an B are in the same sequent in $\mathcal{H}$ so are $f(A)$ an $f(B)$ in $\mathcal{H}'$.

So, for example, the hypersequent

$$A \wedge B \vdash_t C \mid \vdash_s t{:}A \mid \neg D \vdash$$

is extended by

$$A \wedge B , A \vdash_t C \mid A \wedge B, C \vdash B \mid \neg D , C \vdash_s t{:}A$$

by the mapping marked here. The relation of extension is reflexive and transitive (but not antisymmetric). It is a preorder but not a partial order.

DEFINITION [DIRECTED SETS OF HYPERSEQUENTS] A set $\mathfrak{D}$ of hypersequents is DIRECTED if and only if it is (1) *closed under extension*: whenever $\mathcal{H}$ is in $\mathfrak{D}$, and $\mathcal{H}$ extends $\mathcal{H}'$ then $\mathcal{H}'$ is in $\mathfrak{D}$ too and (2) *contains upper bounds* if $\mathcal{H}$ and $\mathcal{H}'$ are in $\mathfrak{D}$ there is some hypersequent in $\mathfrak{D}$ extending both $\mathcal{H}$ and $\mathcal{H}'$.

LEMMA [MODELS DETERMINE DIRECTED SETS] *The set of all hypersequents failing in some given model is directed.*

Proof: That the set of hypersequents failing in model is directed is straightforward. If $\mathcal{H}$ fails in a model, then so does any hypersequent $\mathcal{H}$ extends. If $\mathcal{H}$ and $\mathcal{H}'$ fail in some model, then the disjoint union of the two hypersequents extends both and also fails in that model. ∎

A directed set $\mathfrak{D}$ of hypersequents will determine a frame in the following way.

DEFINITION [THE FRAME OF A DIRECTED SET] Given a directed set $\mathfrak{D}$ of hypersequents, a component sequent in a hypersequent in $\mathfrak{D}$ determines a directed set of sequents: those to which this sequent is *extended* and each sequent that also extends to those sequents. This directed set is a *world* in the frame. A world is $R_{[\![t]\!]}$ accessible from another world if that first world is tagged by the term t.

DEFINITION [TRUTH AND FALSITY] Given a world w in a frame of a directed set $\mathfrak{D}$ we will say that a formula A is true in w if it appears in the left of a sequent in w (once it appears in the left of a sequent in w, it appears in the left of all extending sequents), and it is false in w if it appears in the right of a sequent in w.

DEFINITION [DOWNWARD CLOSURE] A directed family of hypersequents is said to be CLOSED DOWNWARDS if and only if the following closure conditions are satisfied.

Negation Closure: If $\neg A$ is true at a world, then A is false at that world. If $\neg A$ is false at a world, then A is true at that world. Given an underivable hypersequent $\mathcal{H}$ featuring a negation $\neg A$ as true (resp. false) at some world, it may be extended into an underivable hypersequent where A is false (resp. true) at that world, because we have the following derivations, which show that if that wasn't the case, $\mathcal{H}$ would be derivable.

$$\dfrac{\dfrac{\Gamma, \neg A \vdash_t A, \Delta \mid \mathcal{H}}{\Gamma, \neg A, \neg A \vdash_t \Delta \mid \mathcal{H}}\,(\neg L)}{\Gamma, \neg A \vdash_t \Delta \mid \mathcal{H}}\,(WL) \qquad \dfrac{\dfrac{\Gamma, A \vdash_t \neg A, \Delta \mid \mathcal{H}}{\Gamma \vdash_t \neg A, \neg A, \Delta \mid \mathcal{H}}\,(\neg R)}{\Gamma \vdash_t \neg A, \Delta \mid \mathcal{H}}\,(WR)$$

Conjunction Closure: If $A \wedge B$ is true at a world, then A and B are true at that world. If $A \wedge B$ is false at a world, then either A or B is false at that world. Given an underivable hypersequent $\mathcal{H}$ featuring $A \wedge B$ as true (resp. false) at some world, it may be extended into an underivable hypersequent where A and B are true (resp. either A is false or B is false) at that world, because we have the following derivations, which show that if that wasn't the case, $\mathcal{H}$ would be derivable.

$$\dfrac{\dfrac{\Gamma, A, B, A \wedge B \vdash_t \Delta \mid \mathcal{H}}{\Gamma, A \wedge B, A \wedge B \vdash_t \Delta \mid \mathcal{H}}\,(\wedge L)}{\Gamma, A \wedge B \vdash_t \Delta \mid \mathcal{H}}\,(WL) \qquad \dfrac{\dfrac{\Gamma \vdash_t A, A \wedge B, \Delta \mid \mathcal{H} \quad \Gamma \vdash_t B, A \wedge B, \Delta \mid \mathcal{H}}{\Gamma \vdash_t A \wedge B, A \wedge B, \Delta \mid \mathcal{H}}\,(\wedge R)}{\Gamma \vdash_t A \wedge B, \Delta \mid \mathcal{H}}\,(WR)$$

Necessity Closure: If $\Box A$ is true at a world, then A is true at each alternative to A. If $\Box A$ is false at a world, then A is false at some alternative to A. Given an underivable hypersequent $\mathcal{H}$ featuring $\Box A$ as true at some world, and featuring some alternative to that world, $\mathcal{H}$ may be extended into an underivable hypersequent where A is true at that alternative; and if $\Box A$ is false at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where A is false at some subjunctive alternative to that world, because of the following derivations:

$$\dfrac{\dfrac{\Gamma, \Box A \vdash_t \Delta \mid \Gamma', A \vdash_{t'} \Delta' \mid \mathcal{H}}{\Gamma, \Box A, \Box A \vdash_t \Delta \mid \Gamma' \vdash_{t'} \Delta' \mid \mathcal{H}}\,(\Box L)}{\Gamma, \Box A \vdash_t \Delta \mid \Gamma' \vdash_{t'} \Delta' \mid \mathcal{H}}\,(WL) \qquad \dfrac{\dfrac{\vdash A \mid \Gamma \vdash_t \Box A, \Delta \mid \mathcal{H}}{\Gamma \vdash_t \Box A, \Box A, \Delta \mid \mathcal{H}}\,(\Box R)}{\Gamma \vdash_t \Box A, \Delta \mid \mathcal{H}}\,(WR)$$

Significantion Closure: If $t{:}A$ is true at a world, then A is true at each t-zone. If $\Box A$ is false at a world, then A is false at some t-zone. Given an underivable hypersequent $\mathcal{H}$ featuring $\Box A$ as true at some world, and featuring some t-zone, $\mathcal{H}$ may be extended into an underivable hypersequent where A is true at that t-zone; and if $\Box A$ is false at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where A is false at some t-zone, because of the following derivations:

$$\dfrac{\dfrac{\Gamma, t\!:\!A \vdash_{t'} \Delta \mid \Gamma', A \vdash_t \Delta' \mid \mathcal{H}}{\Gamma, t\!:\!A, t\!:\!A \vdash_{t'} \Delta \mid \Gamma' \vdash_t \Delta' \mid \mathcal{H}}\,(:L)}{\Gamma, t\!:\!A \vdash_{t'} \Delta \mid \Gamma' \vdash_t \Delta' \mid \mathcal{H}}\,(WL) \qquad \dfrac{\dfrac{\vdash_t A \mid \Gamma \vdash_{t'} t\!:\!A, \Delta \mid \mathcal{H}}{\Gamma \vdash_{t'} t\!:\!A, t\!:\!A, \Delta \mid \mathcal{H}}\,(:R)}{\Gamma \vdash_{t'} t\!:\!A, \Delta \mid \mathcal{H}}\,(WR)$$

λ *Closure*: If $(\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n$ is true at a world, then so is $A|^{x_1 \cdots x_n}_{t_1 \cdots t_n}$. If $(\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n$ is false at a world, then so is $A|^{x_1 \cdots x_n}_{t_1 \cdots t_n}$. Given an underivable hypersequent $\mathcal{H}$ featuring $(\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n$ as true at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where $A|^{x_1 \cdots x_n}_{t_1 \cdots t_n}$ is true at that world; and if $(\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n$ as false at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where $A|^{x_1 \cdots x_n}_{t_1 \cdots t_n}$ is false at that world, because of the following derivations:

$$\dfrac{\dfrac{\Gamma, (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n, A|^{x_1 \cdots x_n}_{t_1 \cdots t_n} \vdash_t \Delta \mid \mathcal{H}}{\Gamma, (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n, (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n \vdash_t \Delta \mid \mathcal{H}}\,(\lambda L)}{\Gamma, (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n \vdash_t \Delta \mid \mathcal{H}}\,(WL)$$

$$\dfrac{\dfrac{\Gamma \vdash_t (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n, A|^{x_1 \cdots x_n}_{t_1 \cdots t_n}, \Delta \mid \mathcal{H}}{\Gamma \vdash_t (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n, (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n, \Delta \mid \mathcal{H}}\,(\lambda R)}{\Gamma \vdash_t (\lambda x_1 \cdots x_n\, A)t_1 \cdots t_n, \Delta \mid \mathcal{H}}\,(WR)$$

$(\forall x)$ *Closure*: If $(\forall x)A$ is true at a world then so is $A|^x_s$ for any term s in the vocabulary. If $(\forall x)A$ is false at a world then so is $A|^x_s$ for some term s in the vocabulary. Given an underivable hypersequent $\mathcal{H}$ featuring $(\forall x)A$ as true at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where $A|^x_s$ is true at that world; and if $(\forall x)A$ as false at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where $A|^x_s$ is false at that world, for a term s new to the hypersequent, because of the following derivations:

$$\dfrac{\dfrac{\Gamma, (\forall x)A, A|^x_s \vdash_t \Delta \mid \mathcal{H}}{\Gamma, (\forall x)A, (\forall x)A \vdash_t \Delta \mid \mathcal{H}}\,(\forall L)}{\Gamma, (\forall x)A \vdash_t \Delta \mid \mathcal{H}}\,(WL) \qquad \dfrac{\dfrac{\Gamma \vdash_t (\forall x)A, A|^x_y \Delta \mid \mathcal{H}}{\Gamma \vdash_t (\forall x)A, (\forall x)A, \Delta \mid \mathcal{H}}\,(\forall R)}{\Gamma \vdash_t (\forall x)A, \Delta \mid \mathcal{H}}\,(WR)$$

where in the second derivation we choose a variable y fresh to the hypersequent, so the result is indeed an instance of the rule $\forall R$. $(\forall X^n)$ *Closure*: If $(\forall X^n)A$ is true at a world then so is $A|^{X^n}_{P^n}$ for any n-place predicate P^n in the vocabulary. If $(\forall X^n)A$ is false at a world then so is $A|^{X^n}_{P^n}$ for some n-place predicate P^n in the vocabulary. Given an underivable hypersequent $\mathcal{H}$ featuring $(\forall X^n)A$ as true at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where $A|^{X^n}_{P^n}$ is true at that world; and if $(\forall X^n)A$ as false at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where $A|^{X^n}_{P^n}$ is false at that world, for predicate

P^n new to the hypersequent, because of the following derivations:

$$\dfrac{\dfrac{\Gamma, (\forall X^n)A, A|^{X^n}_{P^n} \vdash_t \Delta \mid \mathcal{H}}{\Gamma, (\forall X^n)A, (\forall X^n)A \vdash_t \Delta \mid \mathcal{H}}\,(\forall^n_2 L)}{\Gamma, (\forall X^n)A \vdash_t \Delta \mid \mathcal{H}}\,(WL) \qquad \dfrac{\dfrac{\Gamma \vdash_t (\forall X^n)A, A|^{X^n}_{Y^n}\Delta \mid \mathcal{H}}{\Gamma \vdash_t (\forall X^n)A, (\forall X^n)A, \Delta \mid \mathcal{H}}\,(\forall^n_2 R)}{\Gamma \vdash_t (\forall X^n)A, \Delta \mid \mathcal{H}}\,(WR)$$

where in the second derivation we choose a variable Y^n fresh to the hypersequent, so the result is indeed an instance of the rule $\forall^n_2 R$.

So, if we start with an underivable sequent (even a sequent that cannot be derived without the use of *Cut*), we may close under these conditions to get a downard closed, directed family $\mathfrak{D}$ of hypersequents.

LEMMA [DOWNWARD CLOSED DIRECTED FAMILIES] *For any hypersequent $\mathcal{H}$ that cannot be derived without cut, there is a directed family $\mathfrak{D}$ of hypersequents also underivable without cut, satisfying the downward closure conditions.*

Now we have the raw materials to prove completeness.

THEOREM [COMPLETENESS] *Any hypersequent which has no Cut-free derivation has a counterexample in some model.*

Proof [*sketch*]: Take an underivable hypersequent. By the previous lemma, there is a downward closed directed family $\mathfrak{D}$ containing our starting hypersequent. Consider the frame of $\mathfrak{D}$. The worlds are the worlds of the family. The first order domain is the class of terms (including variables), the n-place second-order domain is the class of $n+1$-tuples of n terms and one world $\langle t_1, \ldots, t_n, w\rangle$ such that there is some predicate (complex or simple) S where $S|^{x_1,\ldots,x_n}_{t_1,\ldots,t_n}$ is true at w. Then take the extension of the atomic predicate P at world w to be the set of n-tuples $\langle t_1, \ldots, t_n\rangle$ where $Pt_1, \ldots, t_n$ is true at w. This is the model. This model will, in general, make *more* statements true or false at worlds than the downward closed directed family $\mathfrak{D}$, because $\mathfrak{D}$ need not be complete at each world, while a model is. This means that more predicates may be definable in the *model* than in are given in $\mathfrak{D}$. However, a simple ordinal inductive construction due to Prawitz [3] fills out the second order domain at stage $\alpha+1$, adding extensions of predicates which are definable in the model defined at stage α. The construction here is completely standard, and at the limit stage, no new predicates are added to the domain and the limit is a Henkin model. ∎

COROLLARY [CUT IS ADMISSIBLE] *If a hypersequent is derivable with* Cut, *it is derivable without* Cut *too.*

Proof: We prove the contrapositive. If $\mathcal{H}$ is not derivable without *Cut*, then by the completeness theorem, it has a counterexample in some model. By the soundness theorem, it follows that this sequent is not derivable using *Cut*. So, contraposing, if $\mathcal{H}$ is derivable with *Cut*, it is also derivable without. ∎

It follows that this proof system is remarkably well behaved. The rules are in a kind of harmony, in that we do not need the rule of Cut to deliver the transitivity of the consequence relation. The rules by themselves do this well enough as it is. The left rules and the right rules are in balance.

5 THE LOGIC OF IDENTITY

We'll end with a short discussion of another of Stephen Read's interests: the logic of identity [8]. I haven't included an identity predicate in the language as it stands, but this is no great loss. Given the power of second order quantification, it is possible to *define* the identity relation on the first-order domain in the usual manner.

$$t = t' =_{df} (\forall X^1)(X^1 t \equiv X^1 t')$$

Given this definition, anything *true of* t is *true of* t', and vice versa. This allows a substitution of t by t' in any sentence. Given a sentence $A|^x_t$ in which t occurs somewhere, we can see that A is equivalent to $(\lambda x\, A)t$, where $(\lambda x\, A)$ is a complex one-place predicate. So, $(\lambda x\, A)t \equiv (\lambda x\, A)t'$ is an instance of the universally quantified sentence $(\forall X^1)(X^1 t \equiv X^1 t')$. So, $A|^x_t \equiv A|^x_{t'}$ follows from $t = t'$, for arbitrary sentences A, and so, arbitrary substitution of one by another is possible.

However, in our proof theory, we want not only to substitute one term for another in a *sentence*, but also in an arbitrary *hypersequent*. In other words, we there is reason to hope that the following sequent to should derivable:

$$a = b \vdash_t \ \mid Ra \vdash_{t'} Rb$$

as it is a consequence of $Ra \vdash_{t'} Ra$, where we substitute b for the second a in that sequence, at the cost of adding $a = b$ in *some* zone. In general, one can define identity using the following kind of rule:

$$\frac{\mathcal{H}|^x_a}{a = b \vdash_t \ \mid \mathcal{H}|^x_b}\ (=L_1) \qquad \frac{\mathcal{H}|^x_b}{a = b \vdash_t \ \mid \mathcal{H}|^x_a}\ (=L_2)$$

which allows for the substitution of a for b (and vice versa) in an arbitrary hypersequent. (This rule is a generalisation of Barwise's sequent rule for identity [1].[2]) It turns out that if the language is expressive enough—as it is here—we can encode an arbitrary hypersequent as an individual *sentence*, so the second order conception of identity will be strong enough for us to justify these hypersequent substitution rules as derived rules in the calculus. As an example, to give us a derivation of $\vdash_t a = b \mid Ra \vdash_{t'} Rb$, notice that we can reason as

[2] Thanks to Jeremy Seligman for pointing me to this formulation of the rules or identity [15].

follows:

$$
\dfrac{
\dfrac{\dfrac{\vdash_t \;\mid Ra \vdash_{t'} Ra \mid Ra \vdash_{t'} Rb}{\vdash_t \;\mid\; \vdash_{t'} Ra \supset Ra \mid Ra \vdash_{t'} Rb}\,(\supset L)}{\vdash_t t' : (Ra \supset Ra) \mid Ra \vdash_{t'} Rb}\,(:R)
\qquad
\dfrac{\dfrac{\vdash_t \;\mid Rb, Ra \vdash_{t'} Rb \quad \vdash_t \;\mid Ra \vdash_{t'} Ra, Rb}{\vdash_t \;\mid Ra \supset Rb, Ra \vdash'_t Rb}\,(\supset L)}{t' : (Ra \supset Rb) \vdash_t \;\mid Ra \vdash_{t'} Rb}\,(:L)
}{
\dfrac{t' : (Ra \supset Ra) \supset t' : (Ra \supset Rb) \vdash_t \;\mid Ra \vdash_{t'} Rb}{\dfrac{t' : (Ra \supset Ra) \equiv t' : (Ra \supset Rb) \vdash_t \;\mid Ra \vdash_{t'} Rb}{a = b \vdash_t \;\mid Ra \vdash_{t'} Rb}\,(=df)}\,(\wedge L)
}\,(\supset L)
$$

From $a = b$ in the t zone, we can allow a substitution of an a by a b in the t' zone, by encoding this transition under "$t' : .$" If $a = b$ then $t' : (Ra \supset Ra)$ sufices for $t' : (Ra \supset Rb)$, and (as the right branch in this derivation shows) $t' : (Ra \supset Rb)$ is enough to ensure that we can transition from Ra to Rb in the t' zone.

LEMMA *In general, for any hypersequent $\mathcal{H}$ there is a formula H such that there are two cut-free derivations, the first* (Pack) *from the premise hypersequent $\mathcal{H}$ to the conclusion $\vdash_t H$, and the second* (Unpack)*, from axioms, to the conclusion $H \vdash_t \;\mid \mathcal{H}$.*

Proof: For the hypersequent $\Gamma_1 \vdash_{t_1} \Delta_1 \mid \cdots \mid \Gamma_n \vdash_{t_n} \Delta_n$ the relevant formula is

$$\bigvee_{i=1}^{n} t_i : (\bigwedge \Gamma_i \supset \bigvee \Delta_i)$$

(where if 't_i' is absent then '$\Box$' takes the place of '$t_i :$', and as usual vacuous conjunctions and disjunctions are replaced by tautologies and conjunctions respectively). A simple induction on the construction of the formula is enough to construct the derivations *Pack* and *Unpack*. Here is a fully general case where $n = 2$, and each Γ_i and Δ_i are small: we choose the hypersequent $A, B \vdash_{t_1} C \mid D \vdash_{t_2} E, F$. Here is the '*Pack*' derivation, encoding this hypersequent as a single formula.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, B \vdash_{t_1} C \mid D \vdash_{t_2} E, F}{A \wedge B \vdash_{t_1} C \mid D \vdash_{t_2} E, F}\,(\wedge L)}{A \wedge B \vdash_{t_1} C \mid D \vdash_{t_2} E \vee F}\,(\vee R)}{\vdash_{t_1} A \wedge B \supset C \mid D \vdash_{t_2} E \vee F}\,(\supset R)}{\vdash_{t_1} A \wedge B \supset C \mid\; \vdash_{t_2} D \supset E \vee F}\,(\supset R)}{\vdash_t t_1 : (A \wedge B \supset C) \mid\; \vdash_{t_2} D \supset E \vee F}\,(:L)}{\vdash_t t_1 : (A \wedge B \supset C), t_2 : (D \supset E \vee F)}\,(:L)}{\vdash_t t_1 : (A \wedge B \supset C) \vee t_2 : (D \supset E \vee F)}\,(\vee R)
$$

And here is the *Unpack* derivation:

$$\dfrac{\dfrac{\dfrac{C \vdash_{t_1} C \qquad \dfrac{A \vdash_{t_1} A \quad B \vdash_{t_1} B}{A, B \vdash_{t_1} A \land B}\,(\land R)}{A \land B \supset C, A, B \vdash_{t_1} C}\,(\supset L)}{t_1 : (A \land B \supset C) \vdash_t \ \mid A, B \vdash_{t_1} C}\,(:L) \qquad \dfrac{\dfrac{D \vdash_{t_2} D \qquad \dfrac{E \vdash_{t_2} E \quad F \vdash_{t_2} F}{E \lor F \vdash_{t_2} E, F}\,(\lor L)}{D, D \supset E \lor F \vdash_{t_2} E, F}\,(\supset L)}{t_2 : (D \supset E \lor F) \vdash_t \ \mid D \vdash_{t_2} E, F}\,(:L)}{t_1 : (A \land B \supset C) \lor t_2 : (D \supset E \lor F) \vdash_t \ \mid A, B \vdash_{t_1} C \mid D \vdash_{t_2} E, F}\,(\lor L)$$

■

The fact that we can *pack* and *unpack* a hypersequent $\mathcal{H}$ into a single formula H means that we can get the full effect of substitution into a hypersequent by means of the second order identity rule.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{H}|^x_a\\ \vdots\\ \textit{Pack}\\ \vdots\\ \vdash_t Ha\end{array} \qquad \begin{array}{c}\textit{Unpack}\\ \vdots\\ Hb \vdash_t \ \mid \mathcal{H}|^x_b\end{array}}{Ha \supset Hb \vdash_t \ \mid \mathcal{H}|^x_b}\,(\supset L)}{Ha \equiv Hb \vdash_t \ \mid \mathcal{H}|^x_b}\,(\land L)}{a = b \vdash_t \ \mid \mathcal{H}|^x_b}\,(=df)$$

since the formula Ha packing the hypersequent $\mathcal{H}|^x_a$ is just a substitution variant of the formula Hb packing $\mathcal{H}|^x_b$. The result means that any substitution of one hypersequent by another can be encoded by a substitution at the level of individual formulas. It follows that the expressive power of the vocabulary of hypersequents does not outstrip that of the object language vocabulary.

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