MODELS FOR LIARS IN BRADWARDINE'S THEORY OF TRUTH

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Abstract: Stephen Read's work on Bradwardine's theory of truth is some of the most exciting work on truth and insolubilia in recent years [4, 5]. In this paper, I give models for Read's formulation of Bradwardine's theory of truth, and I examine the behaviour of liar sentences in those models. I conclude by examining Bradwardine's argument to the effect that if something signifies itself to be untrue then it signifies itself to be true as well. We will see that there are models in which this conclusion fails. This should help us elucidate the hidden assumptions required to underpin Bradwardine's argument, and to make explicit the content of Bradwardine's theory of truth.

As has been made clear in many of the papers in this volume, the crucial feature in Bradwardine's theory of truth is the notion of signification. Expressed by a 'connecticate', which I shall write with the simple infix colon ":", whenever t is a singular term and p is a sentence

$$t : p$$

is another sentence, to be read 't signifies that p', or simply 't says that p.' Bradwardine uses signification to define predicates of truth and falsehood: t is false if and only if it signifies something that is not the case, and it is true if and only if it signifies something, and everything it signifies is the case. Truth and

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falsity are defined notions, where the definitions utilise signification and what we now call propositional quantification.[1] Ft is $(\exists p)(t : p \,\&\, \neg p)$ — t is false if and only if it says something that is not the case. (Notice the syntax. The colon for "says that" binds more tightly than the conjunction, so "$t : p \,\&\, \neg p$" is the conjunction of "$t : p$" and "$\neg p$".) Similarly, Tt is $(\exists p)(t : p) \,\&\, (\forall p)(t : p \to p)$, where '&' expresses some kind of conjunction and '$\to$' expresses some notion of implication. (For smooth exposition, I will introduce yet one more definition: Dt is $(\exists p)(t : p)$. This says that t is *declarative*: it says something. So, Tt is $Dt \,\&\, (\forall p)(t : p \to p)$.)

The distinctive feature of Bradwardine's approach is *not* merely this definition of truth. It is what I will call *Bradwardine's axiom*:

DEFINITION [BRADWARDINE'S AXIOM] Every proposition signifies or means contingently or necessarily everything which follows from it contingently or necessarily [4].

We may render the condition in the following way:

$$\text{If } t : p \text{ then if (if } p \text{ then } q\text{) then } t : q$$

Rearranging the conditionals, we might have another formulation

$$\text{If (if } p \text{ then } q\text{) then (if } t : p \text{ then } t : q\text{).}$$

The crucial issue in understanding Bradwardine's axiom is what form of conditional expression might be used in formulating it. What conditionals feature?

1 CLASSICAL COLLAPSE

If all of these conditionals are material, then we have

$$(p \supset q) \supset (x : p \supset x : q)$$

which we might call the *material Bradwardine Axiom*. The material axiom collapses almost all distinctions concerning signification.

FACT [BRADWARDINE'S COLLAPSE] *Under the material Bradwardine axiom*

» *the non-declarative objects* (*the* x *such that* $\neg Dx$) *say* nothing

» *the false declarative objects say* everything, *and*

» *the true declarative objects say* all and only what is the case.

[1] The vocabulary may cause some confusion. What Bradwardine calls a 'propositio' is the denotation of the singular term 't' in $t : p$. The *propositio* signifies. Propositional quantification, in modern terminology, is quantification into *sentence* position. I will attempt to avoid talk of 'propositions' in any substantial sense which might ask us to choose between modern and medieval terminology here.

In other words, what an object signifies is completely determined by whether or not it is declarative, and, if declarative, whether it is true or false.

Proof: The proof turns on the behaviour of the material conditional. We have

$$q \supset (x : p \supset x : q)$$

since q entails $p \supset q$. It follows from this that if q is the case, then if x says anything (if x is declarative) then x says q. All declarative objects say everything that is the case. On the other hand, we have

$$\neg p \supset (x : p \supset x : q)$$

In other words, if there is some p that x says, that is not the case, then x says that q too. But q is arbitrary. In other words, if x is false, then x says everything.

All that remains are the non-declarative objects, but by definition, these are those that say nothing at all. It follows that in the presence of the material Bradwardine's axiom, the falsehoods say everything, the truths say all and only what is the case, and the non-declaratives say nothing. What is said collapses into this tripartite division. ■

Notice that the principles grounding this collapse are that q entails $p \supset q$ and that $\sim q$ entails $p \supset q$. Both of these principles are valid in *intuitionistic* logic as well, so an intuitionistic understanding of Bradwardine's axiom fares little better that the classical one. An intuitionist cannot conclude (on the basis of logic alone), that t is either true or false: the argument for this relies on the intuitionistically invalid law of the excluded middle. However, even intuitionistically, given the material Bradwardine axiom, all truths say the same thing (namely, everything that is the case), and all falsehoods say the same thing (namely everything). The extra 'wriggle room' provided by the failure of the law of the excluded middle may provide more discrimination in some things may say, but this collapse of signification among the truths and the falsehoods is nonetheless crippling for Bradwardine's programme.

So, we must move further afield in logical space to find an appropriate conditional to express the connection required in Bradwardine's axiom. It seems to me that there are two major options on the table for retaining Bradwardine's axiom without collapse. The first is to move to a kind of *relevant* implication. This is Read's preferred option, and it has the virtue of explicitly allowing for what Bradwardine draws to our attention: both necessary and contingent consequences [3]. Relevant implication may be very robustly *contingent* without being material or extensional. A crucial principle in strong relevant logics is the principle of assertion:

$$p \to ((p \to q) \to q)$$

which cannot hold if the conditional has any *modal* force. The fact that p is the case does not mean that in *other* circumstances where $p \to q$ is the case, then

q is the case, if those other circumstances need not be ones where p is true.[2] In relevant logics such as Anderson and Belnap's R, the principle of assertion holds and the conditional expressed is contingent.

The other option is to ignore contingent consequences to concentrate on necessary ones: we require merely that if p *strictly* implies q, then $t : p$ strictly implies $t : q$. In this paper, I will consider models which encompass both choices. We shall look at models in which

$$p \to q \text{ entails } t : p \to t : q$$

where '$\to$' expresses some kind of non-truth-functional conditional, whether relevant or strict. These are *intensional* conditionals, and so, modelling them will require intensional models.

2 INTENSIONAL MODELS

The models of this section will be structures in which we can interpret sentences in the language in which Bradwardine's theory is expressed. That is, sentences in a language containing the connecticate ':', the conditional '$\to$,' quantification over objects '$(\forall x)$', and quantification into sentence position '$(\forall p)$'. (We will later consider conjunction and negation, but the current suite of items of the language will suffice for this section.) The models we will consider will allow us to interpret sentences in the language

DEFINITION [FRAMES] A *frame* is a structure $\langle P, R, O, \{D_a : a \in P\}, \{S_d : d \in D\}\rangle$ with

» a set P of *points*

» a ternary relation R on P, interpreting the conditional

» a non-empty set O of *objects*

» subsets D_p of O of objects *declarative* at point p

» a binary relation S_d on P for each $d \in D_p$[3]

The points are points of evaluation in a frame: sentences are evaluated as holding or not relative to a point in the frame. In modal models, these points are consistent and complete evaluations of the language (they model "possible worlds" if you like to think of it that way). In models for relevant logics, the points also evaluate formulas in the same way, but the requirement of consistency and completeness (with respect to negation) is not imposed. Formulas with free variables are also evaluated at points, but this must take into account not only the

[2] So, this rules out *temporal* understandings of the conditional, in which $p \to q$ holds if *whenever* p holds, q holds. We may have p true now. It does not follow that whenever $p \to q$ holds, q holds, since later, p may fail to hold.

[3] We will only evaluate S_d from point a if d is in D_a, so we can add the restriction that $aS_d b$ only if $d \in D_a$, but we need not do so, as it makes no difference to do without it.

point at which evaluation occurs, but also the *value* of the variable. The value of a variable in *term* position will be a member of the domain O of objects, since this is the possible semantic value for a term. The value of a variable in *sentence* position will be a set of points, since this is the possible semantic value for a sentence. (If two sentences hold at exactly the same set of points, they are indistinguishable as far as a model is concerned.) So, an assignment α of values to the variables in our language will assign to each object variable an element of O and each propositional variable a subset of P.

Now we have enough structure to define *evaluations* on our frames.

DEFINITION [MODELS] Given a language with a number of atomic predicates, a *model* $\mathfrak{M}$ on a frame $\langle P, R, O, \{D_a : a \in P\}, \{S_d : d \in D\}\rangle$ is determined by the interpretation of the non-logical vocabulary.

» For an n-place predicate, $[\![F]\!] : P \to \mathcal{P}(O^n)$. The extension of an n-place predicate is a function returning a set of n-tuples of objects for each point in the domain.
» For a constant term t, $[\![t]\!] \in O$. The extension of a term t is a choice of an object in O. We extend the notion of the denotation of a term to be relative to an assignment of variables $[\![t]\!]_\alpha$ to include the interpretation of variables: $[\![x]\!]_\alpha$ is the value that α assigns to the variable x. For uniformity, we write "$[\![t]\!]_\alpha$" for all terms t, even for constant terms where the extension does not depend on the assignment α.

Given this information, a model $\mathfrak{M}$ defines a relation $\Vdash$ of satisfaction, evaluating formulas relative to assignments α and points $a \in P$:

» $\mathfrak{M}, \alpha, a \Vdash p$ iff the point a is in the value that α assigns p.
» $\mathfrak{M}, \alpha, a \Vdash Ft_1 \cdots t_n$ iff the n-tuple $\langle [\![t_1]\!]_\alpha, \ldots, [\![t_n]\!]_\alpha\rangle$ is in the extension $[\![F]\!]_a$.
» $\mathfrak{M}, \alpha, a \Vdash A \wedge B$ iff $\mathfrak{M}, \alpha, a \Vdash A$ and $\mathfrak{M}, \alpha, a \Vdash B$.
» $\mathfrak{M}, \alpha, a \Vdash \neg A$ iff $\mathfrak{M}, \alpha, a \nVdash A$.[4]
» $\mathfrak{M}, \alpha, a \Vdash A \to B$ iff whenever $Rabc$, if $\mathfrak{M}, \alpha, b \Vdash A$ then $\mathfrak{M}, \alpha, c \Vdash B$.
» $\mathfrak{M}, \alpha, a \Vdash (\forall x)A$ iff $\mathfrak{M}, \alpha', a \Vdash A$ for every x-variant α' of α.
» $\mathfrak{M}, \alpha, a \Vdash (\forall p)A$ iff $\mathfrak{M}, \alpha', a \Vdash A$ for every p-variant α' of α.
» $\mathfrak{M}, \alpha, a \Vdash t : A$ iff $[\![t]\!]_\alpha \in D_a$ and for each a' where $aS_{[\![t]\!]_{\alpha,a}}a'$, we have $\mathfrak{M}, \alpha, a' \Vdash A$.

The novelty in this definition is the last clause, evaluating signification. A claim of the form $t : A$ holds at point a if and only if (1) the denotation of the term t is an object that is declarative at point a, and (2) in every point accessible

[4]This clause treats negation as boolean. In a wider range of models for relevant logics, we would rather do without boolean negation in favour for a negation that goes further in respecting relevance considerations. Here, however, boolean negation will suffice.

from a, by way of the binary relation $S_{[\![t]\!]_\alpha}$ (the relation determined by the denotation of t) the formula A holds. This makes $t : A$ act as a normal modal operator when the denotation of the term t is declarative. I have discussed the choice of this interpretation elsewhere [7]. The choice of this interpretation is motivated by two factors. First, it allows us to simply construct models in which we can investigate what *does not* follow from the Bradwardine axiom. Given a model in which the Bradwardine axiom holds, if something else does not hold, then it is not a consequence (relative to the background logic of the model, at least) of the Bradwardine axiom. Second, the novel consequences of this particular interpretation are not, in themselves, overly problematic. For example, one consequence of this interpretation is as follows. If we interpret $A \wedge B$ as holding at a point if and only if A and B both hold at that point,[5] then it follows that $t : A$ and $t : B$ entails $t : A \wedge B$. If $t : A$ and $t : B$ both hold at a point a, then $[\![t]\!]_\alpha$ is declarative at a, and A and B both hold at all of the points $S_{[\![t]\!]_\alpha}$-accessible from a. It follows that $A \wedge B$ holds at all of these points, and hence that $t : A \wedge B$ holds at a.

So, the closure of signification under conjunction is a consequence of how we have interpreted it in these models. I do not know whether or not Bradwardine explicitly or implicitly assumes this condition, but it does not seem unduly implausible. So, models in which it holds do not (on this account at least) look problematic.

The reader with a little experience of models of modal logic may think that I have skewed these models in favour of relevant logics by using a three-place relation to interpret the conditional, instead of a two-place relation. It is true that the generality of a three-place relation is used to model relevant logics such as R (in which logical truths such as $p \to p$ need not hold at every point, since we want to find counterexamples to the validity of the argument from q to $p \to p$). However, they may be used to interpret strict conditionals from modal logic. For example, if we set $Rabc$ to hold if and only if $b = c$, then the resulting conditional "$\to$" interpreted by R is the strict implication of the logic S5, where necessity is interpreted as truth at every point.[6]

Now, we have enough information to interpret the language, and to define entailment as preservation of holding at points in our models. These models make satisfaction closed under *entailment* (if all A points are B points, then all points at which $t : A$ holds are points at which $t : B$ holds), but the Bradwardine axiom does not necessarily hold. It is not necessarily the case that $A \to B$ entails $t : A \to t : B$. For this, we need to impose one condition connecting the relation R and the relations S_d.

DEFINITION [BRADWARDINE FRAMES] A frame is a *Bradwardine frame* if and only if the following conditions hold between R and S_d:

[5] This is the case in standard models for modal logics, and also in ternary relational models for relevant logics, so it is not a particularly controversial assumption

[6] These are the models discussed in my earlier paper [7].

- For all points a, b, c, if $Rabc$ and $d \in D_b$ then $d \in D_c$ too.
- For all points a, b, c, c', if $Rabc$, $d \in D_b$ and $cS_d c'$, then there is some point b' where $bS_d b'$ and $Rab'c'$.

This suffices to ensure that if $A \to B$ holds at a then so does $t : A \to t : B$, so $A \to B$ entails $t : A \to t : B$ in our models, as required.

FACT [BRADWARDINE'S AXIOM IN BRADWARDINE FRAMES] *In any model on a Bradwardine frame, if* $A \to B$ *holds at* a, *then so does* $t : A \to t : B$.

Proof: Suppose that $A \to B$ holds at a in our model. To show that $a \Vdash t : A \to t : B$ (we suppress mention of α and $\mathfrak{M}$ since these do not vary in this proof) we consider b and c where $Rabc$, and $b \Vdash t : A$. We wish to show that $c \Vdash t : B$ too. Since $b \Vdash t : A$ we have $[\![t]\!]_\alpha \in D_b$. By the first condition on Bradwardine frames, we have $[\![t]\!]_\alpha \in D_c$ too.[7] Now that $[\![t]\!]_\alpha \in D_c$, we can ask the second part of the question concerning $t : B$ at c. Suppose that $cS_{[\![t]\!]_\alpha}c'$. Does $c' \Vdash B$? The second condition on Bradwardine frames tells us that since $Rabc$ and $cS_{[\![t]\!]_\alpha}c'$, we have some b' where $bS_{[\![t]\!]_\alpha}b'$ and $Rab'c'$. Since $b \Vdash t : A$ and $bS_{[\![t]\!]_\alpha}b'$, we have $b' \Vdash A$ and since $Rabc$ and $a \Vdash A \to B$ we have $c' \Vdash B$ as desired. This concludes the proof.[8] ■

Many frames are Bradwardine. In fact, all frames for strict conditionals are Bradwardine. A conditional is strict if the ternary relation modelling it is essentially a binary relation. We have $A \to B$ at a if and only if all of the points accessible from a are such that if A holds there, so does B. As ternary relational frames, $Rabc$ only when $b = c$.

FACT [STRICT IMPLICATION FRAMES ARE BRADWARDINE] *If* $Rabc$ *only if* $b = c$, *then the frame satisfies the Bradwardine condition.*

Proof: Immediate consequence of the definition of the condition. ■

If R is a genuinely ternary relation, the Bradwardine condition has some bite. Not every frame on a ternary relation is Bradwardine.

EXAMPLE Let P be the set of positive natural numbers $\{1, 2, 3, \ldots\}$, and define $Rlmn$ if and only if n divides both l and m evenly.[9]

[7] This reasoning would have failed had we allowed the denotation of t to vary from point to point, or we would have had to impose a more complex condition connecting R and the declarative objects.

[8] This result is straightforwardly extended to a correspondence result. Suppose that a frame is not a Bradwardine frame. Then it is not difficult to construct a model such that there is a point a at which $p \to q$ holds but $x : p \to x : q$ does not.

[9] This is a model for the positive fragment of the relevant logic R.

If we have some object d that is declarative at 4 but not 2 then the first part of the Bradwardine frame condition fails, since $R6\,4\,2$ (2 divides 6 and 4) but d is declarative at 4 but not 2.

If we have an object d that is declarative everywhere, and a relation S_d such that and $nS_d m$ if and only if $n \neq m$, then we have, for example, $R2\,2\,2$ and $2S_d 3$, but there is *no* number m such that $R2\,m\,3$ since 3 does not divide 2.

So, many frames do not satisfy the Bradwardine condition. However, it is not too difficult[10] to construct frames for the relevant logic R which satisfy the Bradwardine condition. These frames will occupy us for the next sections.

3 liars and bradwardine's argument

Bradwardine's most interesting contribution to the discussion of the Liar paradox is the argument to the effect that if something says of itself it is false, then it also says of itself that it is true. The argument, as discussed by Read, is as follows:

> ... suppose $s : Fs$, that is, suppose some proposition, s, says of itself that it is false, and suppose that it is false. By [the definition of F], it follows that something s says fails to obtain: $(\exists p)(t : p \,\&\, \neg p)$, if not that s is false then something else s says, call it q. Then if it's not q that fails to hold, it must be Fs that fails to hold, i.e., $Fs \Rightarrow (q \to \neg Fs)$ $(*)$, indeed, by Residuation and Bivalence, $(Fs \,\&\, q) \Rightarrow Ts$. But $s : Fs$, and $s : q$, so by [the Bradwardine condition], $s : Ts$. Thus any proposition which says of itself that is not true (or false), also says of itself that it is true. [5, page 311][11]

This argument essentially uses a notion of conjunction, expressed by "&," and a notion of entailment, expressed by "⇒," together with negation. The argument uses a number of principles: the definition of the falsity predicate F, and a strong version of the Bradwardine axiom — we infer from $s : Fs$ and $s : q$, with $(Fs \,\&\, q) \Rightarrow Ts$ to $s : Ts$. This is not only the closure of signification under entailment, but also the kind of conjunction expressed by "&." This principle seems properly stronger than Bradwardine's axiom as I have stated it, but (given a reasonable interpretation of "&"), it seems not unreasonable.

The other significant step in the argument is the inference to $(*)$: the conclusion that Fs entails $q \to \neg Fs$, for the particular choice of q — the "something else" said by s. I will show that this step does not follow from the Bradwardine condition, by constructing models in which it fails.

[10] Not too difficult with the aide of a computer, at least.

[11] I have harmonised the notation with that used in this paper, and marked a step with '(∗)' for later reference.

EXAMPLE [A MODAL MODEL] Let P be the two points a and b. Let $Rabc$ if and only if $b = c$, so the logic of implication is the strict implication of the modal logic S5. Let O, D_a and D_b contain the object l. We set $xS_l y$ if and only if $x \neq y$. That is, $aS_l b$ and $bS_l a$, but neither $aS_l a$ nor $bS_l b$. Let λ be a term whose denotation is the object l. We will show that $\lambda : F\lambda$ holds at a and at b, but that $\lambda : T\lambda$ fails at both a and at b. This model provides a counterexample to Bradwardine's argument in the background logic S5.

First, let p hold at b but not a. Then $\lambda : p$ holds at a, since p is true at all of the points S_l accessible from a (namely, b). However, $\neg p$ also holds at a, so we have $(\lambda : p) \wedge \neg p$, so $(\exists p)(\lambda : p \wedge \neg p)$ holds at a. In other words, $F\lambda$ holds at a. By symmetric reasoning, $F\lambda$ holds at b too, since here, $(\lambda : \neg p) \wedge \neg\neg p$ holds.

Therefore, $F\lambda$ holds both at a and at b. Therefore, since l is a declarative object at both a and b, $\lambda : F\lambda$ holds at a and b too, since $F\lambda$ holds at every point accessible from a, and at every point accessible from b. So, the object l is a liar: it signifies of itself that it is false.

Does it signify of itself that it is *true*? No. Since $F\lambda$ holds at a and b, $T\lambda$ fails at both a and b. Therefore, $\lambda : T\lambda$ fails at a (since at the S_l accessible point, b, $T\lambda$ fails), and it fails at b (since at the S_l accessible point, a, $T\lambda$ still fails).

So, in this model the Bradwardine axiom holds, we have an object that signifies itself to be false (in each point of the model), it *is* false (in each point of the model), yet it does not signify itself to be true. This is a counterexample to Bradwardine's argument. Step $(*)$ in the argument fails, since the other principles used hold in our model. What this means we will consider soon. Before that, however, we will show that counterexamples may be constructed in *relevant* models as well.

EXAMPLE [A RELEVANT MODEL] This is more difficult construction, since models for relevant logics are more complicated. I will not go through the details of models for the relevant logic R here. Appropriate texts to read are numerous and widely available [1, 2, 3, 6]. Instead, I will sketch some simple models of a logic stronger than R, the *Boolean* relevant logic KR. These models are simple, and they do not force us to answer difficult questions concerning the interaction between signification, Truth, Falsity and an intensional (non-Boolean) notion of negation. A relation R is a KR relation if and only if

» There is some point 0 such that $R0ab$ if and only if $a = b$.

» $Raaa$ for every point a.

» $Rabc$ iff $Rbac$ iff $Racb$ for all points a, b, c.

» If $R(ab)cd$ then $Ra(bc)d$. That is, if there is some e where $Rabe$ and $Recd$ then there is some f where $Rbcf$ and $Rafd$.

The conditions are motivated as follows: the first, $R0ab$ if and only if $a = b$ tells us that there is a point at which the conditionals that hold are those that are valid on the model. In particular, at 0, $A \to A$ holds. (It may fail at other points:

if $Rabc$ where $b \neq c$, and A holds at b but not c then $A \to A$ fails at a.) For the second, $Raaa$ tells us that if A and $A \to B$ hold at a then B holds at a too. In other words, $A, A \to B$ entails B. For the third, the first component ($Rabc$ iff $Rbac$) tells us that if A holds at a then $(A \to B) \to B$ holds at a too. The second component ($Rabc$ iff $Racb$) tells us that if $A \to B$ holds at a, then $\neg B \to \neg A$ holds at a too, if we interpret "$\neg$" in the usual Boolean manner: $\neg p$ holds at a point iff p fails at that point. The final and most complicated condition tells us that if $A \to B$ holds at a, then $(C \to A) \to (C \to B)$ holds at a too. These models interpret the whole of the classical relevant logic KR.

Now we can construct our particular KR model in which Bradwardine's conclusion fails. The domain of points of this model is the set $\{0, 1, 2\}$. We interpret the ternary relation R in the following way:

R	0	1	2
0	0	1	2
1	1	012	12
2	2	12	012

where $Rabc$ holds if and only if the number c is found in the a-row and b-column of the table. In other words, we have $R110$, $R111$ and $R112$ (this is the "012" in the middle of the table), but $R121$ and $R122$ but not $R120$ (since we have "12" in the 1-row and 2-column), and so on. This is a model for KR. Verifying this (especially the last condition) is a non-trivial matter.

Now, take λ to be a term in our language, and let its denotation be an object l that is declarative at every point. Let the relation S_l be defined by setting $0S_l1$, $1S_l2$ and $2S_l1$. We can think of S_l as a function, where $0^{s_l} = 1$, $1^{s_l} = 2$ and $2^{s_l} = 1$. When we model "λ :" we can then say that $\lambda : A$ holds at a if and only if A holds at a^{s_l}.

The first thing to verify is that the Bradwardine condition holds in this frame. In this case, we need to verify that if $Rabc$ then $Rab^{s_l}c^{s_l}$. This condition is satisfied in our model, and it is not tedious to check. If $a = 0$, then $R0bc$ iff $b = c$ and then, $R0b^{s_l}c^{s_l}$ too, since $b^{s_l} = c^{s_l}$. When $a \neq 0$ if $Rabc$, we have $Rab^{s_l}c^{s_l}$, since $a \neq 0$ and $b^{s_l} \neq 0$ (s_l sends each point to 1 or 2 but not 0) and in this case, $Rab^{s_l}1$ holds and $Rab^{s_l}2$ holds. We know that c^{s_l} is either 1 or 2, and hence $Rab^{s_l}c^{s_l}$. So, the condition holds. It is a Bradwardine frame.

Now $T\lambda$ is false at every point, and $F\lambda$ is true at every point. Choose a point b. We choose the extension of the proposition p so that it is true at every point other than b. Then at b, $\lambda : p$ is true, since p holds at b^{s_l}. Yet at b, $\neg p$ is true (negation is Boolean). So, at b we have $(\exists p)(\lambda : p \wedge \neg p)$, where we treat $\wedge$ *extensionally* in the usual fashion. Indeed, if we treat conjunction *intensionally*, by setting $A \mathbin{\&} B$ true at a point c when there are a and b where $Rabc$, A holds at a and B holds at b (which is required for the *residuation* condition $A \mathbin{\&} B \to C$ if and only if $A \to (B \to C)$), then we *also* have, at b, $(\exists p)(\lambda : p \mathbin{\&} \neg p)$ since $Rbbb$. With either definition of falsity (using the intensional conjunction & or the extensional conjunction $\wedge$) $F\lambda$ holds at b.

Similarly, since $\lambda : p \to p$ fails at b (we have $Rbbb$), it follows that $T\lambda$ fails at b. So, at *every* point, $T\lambda$ fails, $\neg T\lambda$ holds and $F\lambda$ holds too. So, from any point b, $\neg T\lambda$ is true at b^{s_1}, so at b, both $\lambda : F\lambda$ and $\lambda : \neg T\lambda$ holds. On the other hand, from b, we have $T\lambda$ failing at b^{s_1}, so $\lambda : T\lambda$ fails at b. So, we have a *relevant* conterexample to the conclusion to Bradwardine's argument to the effect that if $\lambda : F\lambda$ then $\lambda : T\lambda$. We have an object that says of itself that it is not true (at each point of the model). It doesn't say of itself that it is true (at any point of the model).

What do these counterexamples mean for the Bradwardine's argument? The argument appeals to a principle – the step marked (∗) – that fails in these models. Consider the relevant model, though what I write holds in the s5 model as well. What is curious in this model is the way that what λ signifies varies from point to point. Let p hold at 2 only, and let q hold at 1 only. Then it follows that from the perspective of point 0, we have $\lambda : p$, but we do *not* have $\lambda : p$ at point 2 itself. At point 2 we have $\lambda : q$ instead, where q and p are jointly inconsistent (they hold together nowhere). So, at point 0 we have both

$$\lambda : q \qquad q \to (\lambda : p \land \neg(\lambda : q))$$

which, when you think about it, is a very odd combination. At point 0, λ says that q (it says "the world is like point 2"), but if that is the case, then λ is not true, since in *that* circumstance, λ no longer says that q, it says that p.

To consider the step (∗) in Bradwardine's deduction. At 0, in the model, we have $\lambda : F\lambda$, and indeed, $F\lambda$ holds at 0. It does follow that something λ says fails to obtain. In the argument, we are asked to "call it q." There are two different interpretations for any q that fails to obtain at 0 and is said by λ at 0. It can hold at $\{2\}$ or at $\{1, 2\}$. Let's take $\{2\}$, as it is the most specific statement, from which the most follows. Our sentence q is true at 2 only. We do have $(\lambda : q \land \neg q)$ at 0, and also $(\lambda : q \ \& \ \neg q)$. Then the argument continues:

> Then if it's not q that fails to hold, it must be Fs that fails to hold,
> i.e., $Fs \Rightarrow (q \to \neg Fs)$ (∗)

This is the step that fails in our model. At 0 we have $F\lambda$. But we do not have $q \to \neg F\lambda$ at 0, since $R022$ and q holds at 2, yet $\neg F\lambda$ fails at 2, since $F\lambda$ holds at 2 as it does everywhere in our model.

Why does this step fail in our models? It seems to me that I cannot say that if q then $\neg F\lambda$, since we have that if q then λ would not have said that q. It would have said something else, had q been the case. In other words, we have a counterexample to the principle $\lambda : q \to (q \to \lambda : q)$ in this model. The failure of this condition seems crucial, and it is quite possibly a principle one might implicitly assume – since it is valid when "→" is read materially. However, the principle seems suspicious to a relevantist. If λ signifies q, then how does this fact follow (relevantly) *from* q? If the only argument is an appeal to $p \to (q \to p)$, this will cut no ice for the relevantist, since it is relevantly invalid. Can this principle be motivated on relevantist grounds, without collapsing distinctions of relevance? There seem to be three options on the table, if we wish to keep a rich theory of signification.

1. Endorse a modal Bradwardine axiom but not a material Bradwardine axiom.
2. Keep the relevant Bradwardine axiom, but reject Bradwardine's conclusion that liars signify their own truth.
3. Motivate a principle, such as $t : p \to (p \to t : p)$, which might undergird step $(*)$.

I do not know which choice ought to be made. I hope it suffices to present them, to allow us all to investigate the options, for Bradwardine's intriguing and fruitful theory of truth.

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