A CUT-FREE SEQUENT SYSTEM FOR TWO-DIMENSIONAL MODAL LOGIC, AND WHY IT MATTERS

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Abstract: The two-dimensional modal logic of Davies and Humberstone [3] is an important aid to our understanding the relationship between *actuality*, *necessity* and *a priori knowability*. I show how a cut-free hypersequent calculus for 2D modal logic not only captures the logic precisely, but may be used to address issues in the epistemology and metaphysics of our modal concepts. I will explain how use of our concepts motivates the inference rules of the sequent calculus, and then show that the completeness of the calculus for Davies–Humberstone models explains why those concepts have the structure described by those models. The result is yet another application of the completeness theorem.

MOTIVATION

The 'two-dimensional modal logic' of Davies and Humberstone [3] is an important aid to our understanding the relationship between *actuality*, *necessity* and

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a priori knowability. It's is widely used in philosophical discussions of these notions, but it is by no means uncontroversial [2, 4, 5, 7, 13]. Models for the logic are well understood. It is a standard modal logic, but instead of evaluating statements at worlds, we double index, and evaluate at pairs of worlds. A holds at $\langle w, v\rangle$ when were w to be the actual world, then A would have held had v been the case. Then $\Box A$ holds at $\langle w, v\rangle$ if A holds at $\langle w, v'\rangle$ for each different world v' (A is necessary if it holds at every alternative world), $@A$ holds at $\langle w, v\rangle$ iff A holds at $\langle w, w\rangle$ (A is actually the case if it holds back at the actual world), and $\mathsf{APK}\,A$ holds at $\langle w, v\rangle$ iff A holds at $\langle w', w'\rangle$ for each world w' (if A holds in every circumstance considered as actual, it holds however things could actually be). In these models, we can consider another world as a subjunctive alternative (had things gone differently, that would have been the case) or as an indicative alternative (we might be wrong and that might actually be the case). $\Box$ is the modal logic corresponding to subjunctive alternatives and APK is the modal logic corresponding to indicative alternatives.

The two notions of necessity fall apart rather radically, as you can see with the presence of the actuality operator $@$. In any model $p \equiv @p$ is true at each pair $\langle w, w\rangle$. Suppose p is true at $\langle w, w\rangle$, but that p is false at a subjunctive alternative $\langle w, v\rangle$. Now, at $\langle w, v\rangle$, $@p$ is still true (since p holds at $\langle w, w\rangle$) so at $\langle w, v\rangle$, $p \equiv @p$ is *false*. It follows that it is not *necessary* (in the sense of $\Box$) back at $\langle w, w\rangle$. $\Box(p \equiv @p)$ fails at $\langle w, w\rangle$.

However, there is another sense in which $p \equiv @p$ is 'necessary'—we need not know anything about the nature of the world (in particular, we need not know anything about the truth or falsity of p) to know that $p \equiv @p$ is, in fact, true. At whatever world w we choose to evaluate $p \equiv @p$, we have $p \equiv @p$ true at $\langle w, w\rangle$. In this sense (at any *indicative* alternative), $p \equiv @p$ is true. It is knowable *a priori*. $\mathsf{APK}\,(p \equiv @p)$ is a theorem of two-dimensional modal logic, while $\Box(p \equiv @p)$ is not.

APK and $\Box$ come apart in the other direction, too. Sometimes claims of the form $\Box A$ can be true where $\mathsf{APK}\,A$ is not. If p is a contingent truth, knowable only *a posteriori*, holding at $\langle w, w\rangle$, but not at every pair $\langle v, v\rangle$, then not only is $@p$ true at $\langle w, w\rangle$, it is true at $\langle w, v\rangle$ for every world v, and hence, $\Box @p$ is true at $\langle w, w\rangle$. For the contingent, *a posteriori* truth p, it is *necessarily true* that it is actually the case. However, it is not *a priori knowable* that it is actually the case. $\mathsf{APK}\,@p$ can fail to be true at $\langle w, w\rangle$ since $@p$ fails at $\langle v, v\rangle$, for any v where p is untrue. So, APK and $\Box$ come apart in both directions. Some things are necessary but not *a priori* knowable, and other things are *a priori* knowable but not necessary, and Davies–Humberstone models for these concepts have given philosophers a degree of clarity in the discussion of the concepts of necessity, actuality and *a priority* previously unavailable to us.[1]

However, not everybody is happy with this picture. For one thing, the application to modelling *a priori* knowability seems to require the distinction between conceiving of a possible world as a subjunctive alternative (a way things could have been) and as an indicative alternative (a way things might actually be). There seems to be no doubt that we do indeed conceive of matters in this way: there are different ways things could have been, and there are different ways that things might actually be for all we know. However, in these models the one and the same set of possible worlds have to do both different jobs. This seems unproblematic when it comes to the interaction between necessity and actuality (and its parallel distinction between temporal operators and the analogous notion of 'now', which also is modelled by double indexing). While we might agree that the different possible worlds can be pressed into service as providing indicative alternatives (we can try 'placing ourselves' in different possible circumstances and consider what things would have seemed like to us were we there), there is less of a reason to think that the indicative alternatives are exhausted by all of the different possible worlds.[2] You must do a great deal of work to begin to show that the models of two-dimensional modal logic can do this kind of epistemic heavy lifting required by this interpretation [1] if we are going to read APK as *truly* providing an account of some of the conceptual structure of *a priori* knowledge. The problem is stark if we ask whether or not *fatalism* (the claim that $(\forall p)(p \equiv \Box p)$) is *a priori* conceivable. (Something is *a priori* conceivable iff its negation is not *a priori* knowable. APC stands to APK as possibility stands to necessity.) On the standard two-dimensional picture,

[1] There on one detail concerning Davies–Humberstone models I have not mentioned. In these models, the truth of an atomic formula varies only with respect to the second coordinate in a pair, and not the first coordinate. An atomic formula is taken to be a raw description of the alternative world, with no recourse to which world is considered as actual. This feature of two-dimensional modal logic is interesting, but will play no further role here. This condition means that Davies–Humberstone models do not satisfy a uniform substitution rule: $\mathsf{APK}\,p \equiv \Box p$ holds for all atomic formulas p, but not for all formulas featuring APK or @. If you think of atomic formulas as judgements which we have not analysed using the resources at our disposal, there is no reason to think that these judgements have the special logical properties of atomic formulas in Davies–Humberstone models. For example, if you think that the concept *water* is equivalent to "the actual watery stuff" then even though "my bath is filled with water" does not contain "@" or "APK" as explicit constituents, it may have the same logical behaviour as formulas which do have those constituents.

[2] Davies and Humberstone were careful to make clear that their formal semantics elucidates the logical structure of necessity, actuality and what they call 'fixity' (where APK is fixed actuality). None of the argument of that paper requires taking the indicative alternatives in a 2D model to exhaust our indicative alternatives. Davies and Humberstone's fixed actuality need not entail *a priori* knowability. It is the later appropriation of this framework by people such as David Chalmers [1].

fatalism is *a priori* conceivable, only if there is only one possible world, for *every* indicative alternative $\langle v, v\rangle$ has a row of exactly the same length as any other (the alternatives $\langle v, w\rangle$ for each world w). At no indicative alternative do the *subjunctive* alternatives shrink down to one possible world. But *why*? What powers of modal reasoning do we have to render non-fatalism (if it is actually the case) *a priori* knowable? None that I can see. We have not seen how the indicative alternatives allowed by the two-dimensional picture *exhaust* the indicative alternatives available.

Another question about these models is more fundamental. It lies with the order of explanation. It is one thing to say that Kripke models of modal logics give us an account of the logical structure of the concepts modelled. It is another to say that this logical structure is *explained* by modal models. Arthur Prior expressed this sentiment many years ago:

> ... possible worlds, in the sense of possible states of affairs are not really individuals (just as numbers are not really individuals). To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i. e. if something else were the case ... We understand 'truth in states of affairs' because we understand 'necessarily'; not vice versa. [9, p. 243–244]

This problem is exacerbated with two-dimensional modal logic. We can say we understand possible subjunctive alternatives and indicative alternatives because we understand possibility and *a priori* knowability, but if we do have this understanding, then why is it that these possible worlds arrange themselves in the way described by two-dimensional models? And how is it that we gain this understanding of the two different concepts of possibility? It is one thing to say, as Gödel did with mathematics, that we have access to the concepts by a kind of intuition. It is another thing to say something about why the intuition has the structure that allows for the kind of logical articulation provided by these models. In this paper, I will address these concerns. Prior's parallel between numbers and possible worlds is illuminating. Talk of numbers makes sense because of our intuitive ability to count. Our intuitive access to facts concerning numbers is not passive receptivity, it is (as Kant taught us) bound up with the activity of counting. I will take the parallel seriously, and examine the connection between our modal concepts and the activity of supposing.

In the rest of this paper I will show how this connection between our modal concepts and the acts of supposing—together with the fact that we can suppose in two distinct ways—motivates a sequent calculus for the concepts $\Box$, @ and

APK, which requires no prior contact with possible worlds. I will show that this calculus is not only intuitively motivated, but it is also logically well-behaved. It is a hypersequent calculus of a relatively familiar kind, and the *Cut* rule is admissible in the system. Furthermore, it is not only sound and complete for a class of Kripke models, but the structure of these models is explained in terms of the prior structure of sequents in the sequent calculus, and the means of constructing models by idealising invalid sequents. This completeness construction is common to a range of sequent systems [12]. The result is an explanation of possible worlds in terms of possibility (as Prior wanted), but further, an explanation of why our concepts of possibility and necessity have the structure described by such models. The models that result include Davies–Humberstone two-dimensional ('square') models as examples, but they also allow for a broader class of indicative alternatives than is possible in square models. Finally, I will show that this more general class of models makes no difference to the logic in the vocabulary of $\Box$, @ and APK, by showing that any sequent invalid in a general model can be shown to be invalid in a square model.

What has this to do with Gödel's groundbreaking work? There are three distinct connections. First, the major result of this will be the application of a completeness theorem. Our core question in modal logic, metaphysics and epistemology—how it is that our concepts of necessity, actuality and *a priori* knowability might have the structure given in 2D models, and how is it that we could have knowledge of the things so described?—is answered by way of the completeness theorem. Our understanding of the concepts is manifest in our governing those concepts by means of the *inference rules*. Soundness and Completeness relating the proof rules to the models then tells us why these models are appropriate for the concepts we have.

The second connection is Gödel's work in mathematical epistemology, and his defence of Platonism and mathematical intuition as the answer to the question of how we have access to mathematical reality. The same sort of answer is given here for modal reality, we have intuitive access to necessity, actuality and *a priori* knowability, not by a faculty like perception, but through our abilities to make different kinds of supposition and to manage our reasoning in the scope of these suppositions.

The third connection is in subject matter: Gödel's interest in logic was not narrow-minded. He worked not only in classical logic and set theory, but also in the interpretation of intuitionistic logic (with the double negation translation and the Dialectica interpretation) as well as in the application of modal logic to the ontological argument. Gödel was able to shed light on a range of different logical concepts by means of careful mathematisation combined with deep insight into what is truly fundamental. I attempt here to make the same kinds of advances for the structure, epistemology and metaphysics of our modal concepts.

SEQUENTS

Our starting point is the hypersequent calculus for the necessity $\Box$ of the simple modal logic S5 [11]. I will informally motivate this sequent calculus, and the additions to deal with @ and APK. Then, once all the concepts are introduced, I we will stand back and formally define the sequent calculus.

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A hypersequent calculus for the one-dimensional modal logic of necessity employs 'hypersequents' involving more than one sequent of formulas.

$$X_0 \vdash Y_0 \mid X_1 \vdash Y_1 \mid \cdots \mid X_n \vdash Y_n$$

Each sequent is a pair of multisets of formulas (we keep track of repetitions of formulas, but not their order). In the traditional sequent calculus we can read a derivation of a sequent $X \vdash Y$ as telling us that asserting each formula in X precludes denying each formula in Y [10]. In this hypersequent calculus, we generalise, by noticing that when we reason with possibility and necessity, we not only *assert* and *deny* flat out, expressing our commitments, we also assert and deny under the scope of various suppositions. I might suppose that things had been different. Assertions and denials under the scope of these suppositions can be manipulated in the same way as assertions and denials which straightforwardly express our commitments. However, they are, to some extent, prophylactically separated from one another. Usually, asserting p and asserting $\neg p$ clash. However, asserting p, and then—under a different supposition—asserting $\neg p$ does not clash at all. The different 'zones' in a discourse are insulated from each other. However this insulation does not keep everything out. Asserting that something is not only true, but *necessarily* true, makes it the kind of thing that jumps across zones. As we've seen, we indicate the different zones in a hypersequent by a vertical line. The following sequent is a canonically *valid* modal hypersequent.

$$\Box p \vdash \ \mid \ \vdash p$$

Asserting $\Box p$ in one zone of a discourse clashes with denying p in another zone in that discourse. In general, if asserting A induces a clash somewhere, then asserting $\Box A$ (in the same zone, or in another zone) will also induce that clash. We will have the following inference principle for $\Box$.

$$\frac{X' \vdash Y' \mid X, A \vdash Y \mid \Delta}{X', \Box A \vdash Y' \mid X \vdash Y \mid \Delta}\,[\Box L]$$

which introduces a $\Box$ on the left of a turnstile. (Here, "Δ" is a placeholder for the rest of the hypersequent in question.) Similarly, if denying A in its own zone, all by itself, is ruled out, then denying $\Box A$ is ruled out. For if we wish to deny $\Box A$, we must admit that A is deniable. The other rule introduces $\Box$ in the right.

$$\frac{\vdash A \mid X \vdash Y \mid \Delta}{X \vdash \Box A, Y \mid \Delta}\,[\Box R]$$

Zones in this sequent calculus feature analogously to names or variables in the calculus for predicate logic, and $\Box$ is like a universal quantifier. To prove $\Box A$ we need to prove A *arbitrarily*—with no other assumptions in that zone.

If we add to this calculus standard rules, we may derive all the features of the standard modal logic S5 (a system notoriously difficult to give a sequent calculus). For example, here is a derivation of $\neg p \vdash \Box\neg\Box p$, using the two rules we have seen, the two standard negation rules ($\neg$ flips a formula from one side of a turnstile to another within a zone: asserting [or denying] $\neg A$ has the same force as denying [or asserting] A), and we start from an axiomatic sequent: one with a formula on both sides of a turnstile (asserting A precludes denying A).

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash \mid p \vdash p}{\Box p \vdash \mid \vdash p}\,[\Box L]}{\Box p \vdash \mid \neg p \vdash}\,[\neg L]}{\vdash \neg\Box p \mid \neg p \vdash}\,[\neg R]}{\neg p \vdash \Box\neg\Box p}\,[\Box R]$$

This sequent derivation tells us how we can derive $\Box\neg\Box p$ from $\neg p$. In "Proofnets for S5" [11] I show how these sequent derivations can be unwound into a proofnet system, giving proof structures in which we derive the conclusion $\Box\neg\Box p$ from the premise $\neg p$ by means of introduction and elimination rules in a relatively standard way. Here, we will stick to sequents, in order to keep this paper of a manageable length. Our next stop is actuality.

— @ —

Not all zones are equal. We introduced a multiplicity of zones in order to allow assertion and denial under distinct suppositions. However, there is still assertion and denial *tout court*, under no supposition. Let's reflect this in our hypersequent structure and mark one of our sequents with an '@.' We have hypersequents that look like this:

$$X_0 \vdash_@ Y_0 \mid X_1 \vdash Y_1 \mid \cdots \mid X_n \vdash Y_n$$

Given this structure, we may reflect it in the language. @ is an operator, just like $\Box$, except now assertion of $@A$ has the same effect as assertion of A in the special *actual* zone, and denial of $@A$ has the same effect as denial of A in that zone. So, canonically valid hypersequents with the actuality operator are these:

$$@p \vdash \ \mid\ \vdash_@ p \qquad \vdash @p \mid p \vdash_@$$

Asserting $@p$, in any zone whatsoever, clashes with denying p in the @-zone. Similarly, asserting p in the @-zone clashes with denying $@p$ in any zone. So, the rules for the @ operator look like this:

$$\frac{X \vdash Y \mid X', A \vdash_@ Y' \mid \Delta}{X, @A \vdash Y \mid X' \vdash_@ Y' \mid \Delta}[@L] \qquad \frac{X \vdash Y \mid X' \vdash_@ A, Y' \mid \Delta}{X \vdash @A, Y \mid X' \vdash_@ Y' \mid \Delta}[@R]$$

Here is an example derivation, using these two rules for @ and combining them with an instance of $\Box R$.

$$\cfrac{\cfrac{\cfrac{p \vdash_@ p \mid \vdash}{p \vdash_@ \ \mid \vdash @p}[@R]}{p \vdash_@ \ \mid \vdash \Box @p}[\Box R]}{\vdash_@ \ \mid @p \vdash \Box @p}[@L]$$

This tells us that asserting $@p$ and denying $\Box @p$—in any context whatever—are inconsistent.

Given these resources, it's clear that we have two different kinds of consequence. If we have a derivation of $A \vdash B$ then asserting A clashes with denying B, no matter what suppositions are involved. If we have a derivation of $A \vdash_@ B$ then asserting A clashes with denying B, in the actual zone. We have, for example, $p \vdash_@ @p$ and $@p \vdash_@ p$, but we don't have $p \vdash @p$ and $@p \vdash p$. Sequents with this labelling allow us two kinds of logical consequence.

— APK —

Now consider *a priori* knowability. Here we move to consider two different ways we may suppose. For $\Box$ the different suppositions are different ways things might have been, had things gone differently. The different contexts are non-actual, not marked by @. On the other hand, when we consider what might be the case if we are wrong—if we consider *indicative* alternatives—we are considering different ways we may take the actual world to be.

> "... the ur-distinction, accounting for all the others, is between two practices of reasoning: indicative reasoning, which functions in the first instance to update our doxastic state in the face of new evidence, and subjunctive reasoning, the root context of which is deliberation about what to

> do and about the propriety of actual or potential actions. These different practices of reasoning are expressed in the differences between indicative and subjunctive conditionals, but in our view it is the conditionals that are to be understood in terms of the practices of reasoning rather than the other way around." — Mark Lance and W. Heath White [6]

Lance and White argue in this paper that these two abilities are deeply embedded in our nature as creatures who *act* on the basis of a *perspective*. We both correct our perspectives (hence the need for indicative update) and to reason about different plans (hence the need for subjunctive update). So, we have not only subjunctive alternatives, but we may also have indicative alternatives, which introduce into the discourse assertions with other zones taken as actual—they are alternative ways to take actual matters to be. The structure of a hypersequent is now more complex:

DEFINITION 1 [2D HYPERSEQUENTS]: A 2D hypersequent is a multiset of multisets of sequents. Each inner multiset has a single sequent marked with an @, its ACTUAL SEQUENT. The sequents in each inner multiset are SUBJUNCTIVE ALTERNATIVES to one another. Each of the actual sequents in hypersequent are the *indicative alternatives* to each sequent in the whole hypersequent. We use the following notation for a 2D hypersequent, marking off subjunctive alternatives with a single bar (|), indicative alternatives with a double bar (||) and actual sequents with a subscripted @. The general structure is this.

$$X^0_0 \vdash_@ Y^0_0 \;\mid\; X^0_1 \vdash Y^0_1 \;\mid\; \cdots \;\mid\; X^0_{n^0} \vdash Y^0_{n^0} \;\|$$
$$X^1_0 \vdash_@ Y^1_0 \;\mid\; X^1_1 \vdash Y^1_1 \;\mid\; \cdots \;\mid\; X^1_{n^1} \vdash Y^1_{n^1} \;\|\; \cdots \;\|$$
$$X^m_0 \vdash_@ Y^m_0 \;\mid\; X^m_1 \vdash Y^m_1 \;\mid\; \cdots \;\mid\; X^m_{n^m} \vdash Y^m_{n^m}$$

in which each $X^i_0 \vdash_@ Y^i_0$ is an indicative alternative of every sequent, and each $X^i_j \vdash Y^i_j$ is a subjunctive alternative of $X^i_k \vdash Y^i_k$.

The APK operator exploits indicative alternatives in just the same way as $\Box$ exploits subjunctive alternatives. Asserting $\mathsf{APK}\,p$ in one zone clashes with denying p in its indicative alternatives. The following sequent is derivable:

$$\mathsf{APK}\,p \vdash \quad \| \quad \vdash_@ p$$

In fact, we will have the following sort of derivation:

$$\dfrac{\dfrac{\dfrac{\vdash \;\|\; p \vdash_@ p}{\vdash \;\|\; p \vdash_@ @p}\,[@R]}{\vdash \;\|\; \vdash_@ p \supset @p}\,[\supset R]}{\vdash \mathsf{APK}\,(p \supset @p)}\,[\mathsf{APK}\,R]$$

Now we have the resources to define the sequent system. We will use one more convention for notation. When we write

$$\mathcal{H}[X \vdash Y]$$

this is a 2D hypersequent, in which $X \vdash Y$ occurs as a particular component sequent. This component sequent may be marked as actual, it may not. When we write

$$\mathcal{H}[X' \vdash Y']$$

this is the 2D hypersequent which results from taking $\mathcal{H}[X \vdash Y]$ and replacing the component sequent $X \vdash Y$ with the sequent $X' \vdash Y'$. In addition, if the indicated $X \vdash Y$ was marked as actual in $\mathcal{H}[X \vdash Y]$, so is the indicated $X' \vdash Y'$ in $\mathcal{H}[X' \vdash Y']$. When we write

$$\mathcal{H}[X \vdash Y \ | \ X' \vdash Y']$$

this is a hypersequent in which the sequents $X \vdash Y$ and $X' \vdash Y'$ occur as subjunctive alternatives—they are members of the same inner multiset. (Again, either $X \vdash Y$ or $X' \vdash Y'$ may be marked as actual.) In the particular case where we write

$$\mathcal{H}[X \vdash Y \ | \ X \vdash Y]$$

this may include *both* the case where there are *two* distinct subjunctive alternative instances of $X \vdash Y$ inside the hypersequent,[3] and the case in which there is just one.[4] The same goes for

$$\mathcal{H}[X \vdash Y \ \| \ X' \vdash_@ Y']$$

Here, $X' \vdash_@ Y'$ is an indicative alternative of $X \vdash Y$, and this will include the case where the indicated $X \vdash Y$ is the *same* instance as $X' \vdash_@ Y'$.[5] Now we have the resources to state the rules of the sequent calculus, smoothly and efficiently.

DEFINITION 2 [THE RULES OF THE 2D SEQUENT SYSTEM]: First, we will start with the STRUCTURAL RULES, of *identity*, *cut* and *contraction*.

$$\mathcal{H}[X, A \vdash A, Y] \ [Id] \qquad\qquad \frac{\mathcal{H}[X \vdash A, Y] \qquad \mathcal{H}[X, A \vdash Y]}{\mathcal{H}[X \vdash Y]} \ [Cut]$$

[3] This includes the case where one is marked as actual and one is not.

[4] This fussiness now pays off when it comes to state the rules. It allows us to uniformly state each rule, where it would need a number of different cases in the instances of each rule for $\Box$, @ and APK.

[5] In which case $X' = X$, $Y' = Y$, and $X \vdash Y$ is marked as actual.

$$\frac{\mathcal{H}[X, A, A \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\,[WL] \qquad \frac{\mathcal{H}[X \vdash A, A, Y]}{\mathcal{H}[X \vdash A, Y]}\,[WR]$$

Now the CLASSICAL connective rules. We will use just *negation* and *conjunction* as examples:

$$\frac{\mathcal{H}[X \vdash A, Y]}{\mathcal{H}[X, \neg A \vdash Y]}\,[\neg L] \qquad \frac{\mathcal{H}[X, A \vdash Y]}{\mathcal{H}[X \vdash \neg A, Y]}\,[\neg R]$$

$$\frac{\mathcal{H}[X, A, B \vdash Y]}{\mathcal{H}[X, A \wedge B \vdash Y]}\,[\wedge L] \qquad \frac{\mathcal{H}[X \vdash A, Y] \quad \mathcal{H}[X \vdash B, Y]}{\mathcal{H}[X \vdash A \wedge B, Y]}\,[\wedge R]$$

Now the MODAL RULES. First, *necessity*:

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\,[\Box L] \qquad \frac{\mathcal{H}[\;\vdash A \mid X \vdash Y]}{\mathcal{H}[X \vdash \Box A, Y]}\,[\Box R]$$

Second, *actuality*:

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash_{@} Y']}{\mathcal{H}[X, @A \vdash Y \mid X' \vdash_{@} Y']}\,[@L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash_{@} A, Y']}{\mathcal{H}[X \vdash @A, Y \mid X' \vdash_{@} Y']}\,[@R]$$

and finally, *a priori knowability*:

$$\frac{\mathcal{H}[X \vdash Y \parallel X', A \vdash_{@} Y']}{\mathcal{H}[X, \mathsf{APK}\,A \vdash Y \parallel X' \vdash_{@} Y']}\,[\mathsf{APK}\,L] \qquad \frac{\mathcal{H}[\;\vdash_{@} A \parallel X \vdash Y]}{\mathcal{H}[X \vdash \mathsf{APK}\,A, Y]}\,[\mathsf{APK}\,R]$$

These complete the rules of the sequent system.

LEMMA 1 [DERIVED RULES—WEAKENING]: *If $\mathcal{H}[X \vdash Y]$ has a derivation with n steps, so does $\mathcal{H}[X, X' \vdash Y', Y]$, for arbitrary extra formulas X' and Y' to add to the sequent, and so does $\mathcal{H}[X \vdash Y \mid X' \vdash Y']$, with an extra subjunctive alternative sequent and $\mathcal{H}[X \vdash Y \parallel X' \vdash Y']$, with an extra indicative alternative sequent. In other words, if a hypersequent is derivable, so is any weaker hypersequent, with extra formlulas added in the left or right of a component sequent, or whole extra component sequents added—and this weaker sequent is derivable in a derivation of exactly the same length.*

Proof: An induction on the construction of the derivation of $\mathcal{H}[X \vdash Y]$. Notice that if $\mathcal{H}[X \vdash Y]$ is an identity sequent, so is any of its weakenings—the only constraint is that some component sequent has a formula in both sides of the turnstile.

For the induction step, notice that in each rule use to derive $\mathcal{H}[X \vdash Y]$ it may be used to derive the appropriate weakening of $\mathcal{H}[X \vdash Y]$ too, in terms of weakenings of the premise hypersequents in the rule. The only subtlety occurs in $\Box R$ and $\mathsf{APK}\,R$, in which case the component sequent $\vdash \Box A$ and $\vdash \mathsf{APK}\,A$ in the premise is not affected by weakening. For example, if the original derivation ends in

$$\frac{\mathcal{H}[\ \vdash_@ A \ \parallel\ X \vdash Y]}{\mathcal{H}[X \vdash \mathsf{APK}\,A, Y]}\ [\mathsf{APK}\,R]$$

and we want a derivation of the weaker hypersequent $\mathcal{H}[X, X' \vdash \mathsf{APK}\,A, Y, Y']$ instead, we use the inference

$$\frac{\mathcal{H}[\ \vdash_@ A \ \mid\ X, X' \vdash Y, Y']}{\mathcal{H}[X, X' \vdash \mathsf{APK}\,A, Y, Y']}\ [\mathsf{APK}\,R]$$

which is still an instance of $\mathsf{APK}\,R$ and whose premise $\mathcal{H}[\ \vdash_@ A \ \mid\ X, X' \vdash Y, Y']$ is a weakening of $\mathcal{H}[\ \vdash_@ A \ \mid\ X \vdash Y]$ in which the component sequent $\vdash A$ is untouched, so the side conditions for the rule $\mathsf{APK}\,R$ are unaffected. ■

DEFINITION 3 [MERGING SUBSEQUENTS]: If $\mathcal{H}[X \vdash Y \ \mid\ X' \vdash Y']$ is a hypersequent in which two component sequents $X \vdash Y$ and $X' \vdash Y'$—subjunctive alternatives of each other—are displayed, its MERGE is the hypersequent $\mathcal{H}[X, X' \vdash Y, Y']$ in which those two component sequents are removed, and the sequent $X, X' \vdash Y, Y'$ is inserted to replace both. (This component sequent is a subjunctive alternative of each sequent that is a subjunctive alternative of the original sequents.)

Similarly, $\mathcal{H}[X \vdash Y \ \parallel\ X' \vdash Y']$ is a hypersequent in which two component sequents $X \vdash Y$ and $X' \vdash Y'$—indicative alternatives of each other—are displayed, its MERGE is the hypersequent $\mathcal{H}[X, X' \vdash Y, Y']$ in which those two component sequents are removed, and the sequent $X, X' \vdash Y, Y'$ is inserted to replace both, but now, the subjunctive alternatives of each original sequent are subjunctive alternatives of each other in the new hypersequent.

For example, if we merge $p \vdash_@ q$ and $p' \vdash_@ q'$ in the hypersequent below

$$p \vdash_@ q \ \mid\ \vdash r \ \parallel\ p' \vdash_@ q' \ \mid\ s \vdash \ \parallel\ t \vdash u \ \mid\ \cdots$$

the result is

$$p, p' \vdash_@ q, q' \ \mid\ \vdash r \ \mid\ s \vdash \ \parallel\ t \vdash u \ \mid\ \cdots$$

in which $\vdash r$ and $s \vdash$ become subjunctive alternatives. Now we may state the next lemma.

LEMMA 2 [DERIVED RULES—MERGE]: *If $\mathcal{H}[X \vdash Y \ \mid\ X' \vdash Y']$ has an n-step derivation, so does its merge $\mathcal{H}[X, X' \vdash Y, Y']$. Similarly, if $\mathcal{H}[X \vdash Y \ \parallel\ X' \vdash Y']$ has an n-step derivation, so does its merge $\mathcal{H}[X, X' \vdash Y, Y']$.*

Proof: A similar induction on the derivation of $\mathcal{H}[X \vdash Y \mid X' \vdash Y']$. The 'merge' of an identity sequent is still an identity sequent, and any instance of an inference rule used to derive $\mathcal{H}[X \vdash Y \mid X' \vdash Y']$ from premises may be manipulated into another instance of the same rule, in order to derive its merge $\mathcal{H}[X, X' \vdash Y, Y']$ from premises which are merges of the premises of the other rule. The only subtlety occurs in $\Box R$ and $\mathsf{APK}\,R$, in which case the component sequent $\vdash \Box A$ and $\vdash \mathsf{APK}\,A$ in the premise is not affected by the merge. For example, if the original derivation ends in

$$\frac{\mathcal{H}[\vdash A \mid X \vdash Y \mid X' \vdash Y]}{\mathcal{H}[X \vdash \Box A, Y \mid X' \vdash Y']}\,[\Box R]$$

and we want a derivation of the merged hypersequent $\mathcal{H}[X, X' \vdash \Box A, Y, Y']$ instead, just as with the case of weakening, we use the inference

$$\frac{\mathcal{H}[\vdash A \mid X, X' \vdash Y, Y']}{\mathcal{H}[X, X' \vdash \Box A, Y, Y']}\,[\Box R]$$

which is still an instance of $\Box R$ and whose premise $\mathcal{H}[\vdash A \mid X, X' \vdash Y, Y']$ is a merge of $\mathcal{H}[\vdash A \mid X \vdash Y \mid X' \vdash Y]$ in which the component sequent $\vdash A$ is untouched, so the side conditions for the rule $\Box R$ are unaffected. ∎

This proof system is quite well behaved. The left and right rules of each connective are harmonious: if I *Cut* an intermediate formula which is introduced in both inferences preceding that *Cut*, that *Cut* may be traded in for *Cuts* on subformulas. Take the example for APK. Our derivation ending in *Cut* has the following shape.

$$\frac{\dfrac{\begin{matrix}\vdots\,\delta_1\\ \mathcal{H}[\vdash_@ A \parallel X \vdash Y \parallel X' \vdash_@ Y']\end{matrix}}{\mathcal{H}[X \vdash \mathsf{APK}\,A, Y \parallel X' \vdash_@ Y']}\,[\mathsf{APK}\,R] \quad \dfrac{\begin{matrix}\vdots\,\delta_2\\ \mathcal{H}[X \vdash Y \parallel X', A \vdash_@ Y']\end{matrix}}{\mathcal{H}[X, \mathsf{APK}\,A \vdash Y \parallel X' \vdash_@ Y']}\,[\mathsf{APK}\,L]}{\mathcal{H}[X \vdash Y \parallel X' \vdash_@ Y']}\,[Cut]$$

By Lemma 2, we have a new derivation δ_1' (of the same shape and height as δ_1) of the merged hypersequent $\mathcal{H}[X \vdash Y \parallel X' \vdash_@ A, Y']$ in which the indicative alternatives $\vdash_@ A$ and $X' \vdash_@ Y'$ are merged. Then we may *Cut* on A, as follows

$$\frac{\begin{matrix}\vdots\,\delta_1'\\ \mathcal{H}[X \vdash Y \parallel X' \vdash_@ A, Y']\end{matrix} \quad \begin{matrix}\vdots\,\delta_2\\ \mathcal{H}[X \vdash Y \parallel X', A \vdash_@ Y']\end{matrix}}{\mathcal{H}[X \vdash Y \parallel X' \vdash_@ Y']}\,[Cut]$$

and avoid going through the step with APK. In the same way, all *Cut*s may be eliminated in a relatively standard way. However, I will not go through the cut elimination theorem in any more detail, for the admissibility of *Cut* will be an immediate corollary of the completeness theorem proved in the next section.

MODELS

For a completeness theorem, we need *models*. We could, in fact, provide models just by proving the completeness theorem using the technique of taking the limit of invalid hypersequents [12], and seeing what kinds of structures result. The kinds of structures that result are these:

DEFINITION 4 [2D FRAMES]: A Kripke frame for our logic is a structure $\langle W, \approx, @\rangle$ consisting of a non-empty set W, which we'll call the WORLDS, which are

- ▷ partitioned by an equivalence relation $\approx$ SUBJUNCTIVE ACCESSIBILITY, and
- ▷ with a distinguished set @ consisting of one representative from every $\approx$-class.

We will call the members of @ the ACTUAL WORLDS. Given a world w, we will call the unique $v \in @$ where $v \approx w$, the 'actual world according to w,' and we'll refer to this as '$@(w)$.'

A frame is a structure upon which we may evaluate formulas in our language in the usual way.

DEFINITION 5 [2D MODELS]: Given a 2D FRAME $\langle W, \approx, @\rangle$, a MODEL on that frame is a relation $\Vdash$ between W and the set of propositional atoms, extended to all formulas of the language as follows:

- ▷ $w \Vdash \neg A$ iff $w \nVdash A$.
- ▷ $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$.
- ▷ $w \Vdash \Box A$ iff $v \Vdash A$ for each $v \approx w$.
- ▷ $w \Vdash @A$ iff $@(w) \Vdash A$.
- ▷ $w \Vdash \mathsf{APK}\,A$ iff $v \Vdash A$ for each $v \in @$.

We could define truth in a model in the usual sorts of ways, as truth at some particular point in that model, or—more specifically—as truth at some given *actual* point in that model. We could define validity as well, as preservation of truth. However, we have a much more general way to relate statements in our language to models. We may interpret entire hypersequents in models.

DEFINITION 6 [COUNTEREXAMPLES TO 2D HYPERSEQUENTS]: The hypersequent

$$X^0_0 \vdash_@ Y^0_0 \mid X^0_1 \vdash Y^0_1 \mid \cdots \mid X^0_{n^0} \vdash Y^0_{n^0} \parallel$$
$$X^1_0 \vdash_@ Y^1_0 \mid X^1_1 \vdash Y^1_1 \mid \cdots \mid X^1_{n^1} \vdash Y^1_{n^1} \parallel \cdots \parallel$$
$$X^m_0 \vdash_@ Y^m_0 \mid X^m_1 \vdash Y^m_1 \mid \cdots \mid X^m_{n^m} \vdash Y^m_{n^m}$$

fails in the model $\Vdash$ on the frame $\langle W, \approx, @\rangle$ if and only if there is a collection of worlds $w^i_j \in W$ where

- ▷ $w^i_0 \in @$ for each $i = 0, \ldots, m$;
- ▷ $w^i_j \approx w^i_k$; for each $i = 0, \ldots, m$ and $j, k = 0, \ldots, n^i$, and
- ▷ for each world w^i_j, each member of X^i_j is true at w^i_j and each member of Y^i_j is false at w^i_j.

Given that definition of what it is for a hypersequent to FAIL in the model, if it doesn't fail, we say that the hypersequent HOLDS. If a hypersequent fails in a model we also say that this model is a COUNTEREXAMPLE to the hypersequent.

Now we begin to connect derivability and what hypersequents hold in models.

THEOREM 3 [SOUNDNESS]: *If $\mathcal{H}$ is derivable, it holds in every model.*

Proof: A straightforward induction on the structure of the derivation. Axioms clearly hold in each model (they have no counterexamples) and if the premises of a rule hold in a model, so does the conclusion. ∎

The soundness proof is so straightforward because the structure of the sequent so readily corresponds to the structure of models. Each structural feature of 2D hypersequents maps onto a feature of the 2D models. The left/right polarity of a sequent corresponds to truth/falsity at a world. An individual sequent corresponds to a world, subjunctive alternativeness for sequents is subjunctive accessibility for worlds, and actuality corresponds to actuality. However, a sequent structure is not *identical* to a modal model. Each sequent in a derivation is a finite thing. Kripke models need not be finite, and the set of formulas true at a world in a model is always infinite. Sequents are grounded in what we *do*. The left and the right hand sides of a sequent turnstile correspond nicely to truth and falsity, but they are not to be identified with truth and falsity. Much more is true or is false at a world than is mentioned in any sequent. A derivable sequent tells us a fact about all models. An underivable sequent—as we shall see—does something else. It points us to the existence of a particular kind of model. An underivable sequent is a description of how things could be, of a permissible, coherent position in a discourse, in which certain things are asserted and certain other things are denied, in different zones connected with each other as subjunctive and indicative alternatives. In the rest of this section we will see how such a position, partial and incomplete, may be filled out in a systematic way so as to describe a model. The technique is general. An earlier version of the construction is presented in "Truth Values and Proof Theory" [12]. The first crucial definition is the notion of extension.

DEFINITION 7 [EXTENDING HYPERSEQUENTS]: $\mathcal{H}'$ EXTENDS $\mathcal{H}$ iff there is some map f from the formulas of $\mathcal{H}$ to the formulas of $\mathcal{H}'$ that preserves all of the hypersequent structure. This means the following three things: *First*, for each formula occurrence A in $\mathcal{H}$ its corresponding occurrence $f(A)$ in $\mathcal{H}'$, shares its shape (it is an instance of the same formula), its position in its sequent (on the left or on the right) and lastly, if A is in an actual sequent in $\mathcal{H}$, $f(A)$ is in $\mathcal{H}'$. *Second*, if A and B are in the same sequent in $\mathcal{H}$ so are $f(A)$ and $f(B)$ in $\mathcal{H}'$, so given a sequent $X \vdash Y$ in $\mathcal{H}$, it makes sense to talk of its corresponding sequent $f(X \vdash Y)$ in $\mathcal{H}'$, though this sequent may contain *more* formulas than the original sequent $X \vdash Y$. *Third*, the map f sends subjunctive alternatives in $\mathcal{H}$ to subjunctive alternatives in $\mathcal{H}'$—if $X \vdash Y$ and $X' \vdash Y'$ are subjunctive alternatives in $\mathcal{H}$ then so are $f(X \vdash Y)$ and $f(X' \vdash Y')$ in $\mathcal{H}'$. It also sends indicative alternatives in $\mathcal{H}$ to indicative alternatives in $\mathcal{H}'$ in exactly the same way.

So, for example, the hypersequent

$$p \land q \vdash_@ r \;\|\; \vdash_@ \Box p \;|\; \neg s \vdash$$

is extended by

$$p \land q, p \vdash_@ r \;|\; p \land q, r \vdash q \;\|\; \neg s, r \vdash_@ \Box p$$

by the mapping marked here. The relation of extension is reflexive and transitive (but not antisymmetric). It is a preorder but not a partial order.[6] In the remainder of this section we will consider the structure of the collection of *underivable* sequents. We will show that maximal directed sets of underivable sequents correspond to models.

DEFINITION 8 [DIRECTED SETS OF HYPERSEQUENTS]: A set $\mathfrak{D}$ of hypersequents is DIRECTED if and only if it is (1) *closed under inclusion*: whenever $\mathcal{H}$ is in $\mathfrak{D}$, and $\mathcal{H}$ extends $\mathcal{H}'$ then $\mathcal{H}'$ is in $\mathfrak{D}$ too and (2) *contains upper bounds* if $\mathcal{H}$ and $\mathcal{H}'$ are in $\mathfrak{D}$ there is some hypersequent in $\mathfrak{D}$ extending both $\mathcal{H}$ and $\mathcal{H}'$.

LEMMA 4 [MODELS DETERMINE DIRECTED SETS]: *The set of all hypersequents failing in some given model is directed.*

[6] $p \vdash q$ is extended by $p, p \vdash q$ which is extended in turn by $p \vdash q$.

Proof: That the set of 2D hypersequents failing in model is directed is straightforward. If $\mathcal{H}$ fails in a model, then so does any hypersequent $\mathcal{H}$ extends. If $\mathcal{H}$ and $\mathcal{H}'$ fail in some model, then the disjoint union of the two hypersequents extends both and also fails in that model. ∎

A directed set $\mathfrak{D}$ of hypersequents will determine a frame in the following way.

DEFINITION 9 [THE FRAME OF A DIRECTED SET]: Given a directed set $\mathfrak{D}$ of hypersequents, a component sequent in a hypersequent in $\mathfrak{D}$ determines a directed set of sequents: those to which this sequent is *extended* and each sequent that also extends to those sequents. This directed set is a *world* in the frame. Two worlds are related by $\approx$ if they contain sequents which are subjunctive alternatives. A world is in @ if it contains sequents marked by @.

DEFINITION 10 [TRUTH AND FALSITY]: Given a world w in a frame of a directed set $\mathfrak{D}$ we will say that a formula A is true in w if it appears in the left of a sequent in w (once it appears in the left of a sequent in w, it appears in the left of all extending sequents), and it is false in w if it appears in the right of a sequent in w.

In some directed sets, truth and falsity in the frame will work like truth and falsity in *models*. In the construction of "Truth Values and Proof Theory" we show that *maximal* directed sets (of all unprovable hypersequents extending some given hypersequent) determine a model. This makes great use of *Cut*.

$$\frac{\mathcal{H}[X \vdash A, Y] \quad \mathcal{H}[X, A \vdash Y]}{\mathcal{H}[X \vdash Y]}\,[Cut]$$

If $\mathcal{H}[X \vdash Y]$ is an underivable hypersequent, then it may be extended by a hypersequent in which A is true at the $X \vdash Y$ world, or one in which A is false at this world. So, in a *maximal* directed set, every statement is either true or false at each world. In this paper, we will use a more modest construction in which we will not appeal to *Cut*. The directed sets will be smaller and a corollary will be the admissibility of *Cut*.

DEFINITION 11 [DOWNWARD CLOSURE]: A directed family of hypersequents is said to be CLOSED DOWNWARDS if and only if the following closure conditions are satisfied.

Negation Closure: If $\neg A$ is true at a world, then A is false at that world. If $\neg A$ is false at a world, then A is true at that world. Given an underivable hypersequent $\mathcal{H}$ featuring a negation $\neg A$ as true (resp. false) at some world, it may be extended into an underivable hypersequent where A is false (resp. true) at that world, because we have the following derivations, which show that if that wasn't the case, $\mathcal{H}$ would be derivable.

$$\dfrac{\dfrac{\mathcal{H}[X, \neg A \vdash A, Y]}{\mathcal{H}[X, \neg A, \neg A \vdash Y]}[\neg L]}{\mathcal{H}[X, \neg A \vdash Y]}[WL] \qquad \dfrac{\dfrac{\mathcal{H}[X, A \vdash \neg A, Y]}{\mathcal{H}[X \vdash \neg A, \neg A, Y]}[\neg R]}{\mathcal{H}[X \vdash \neg A, Y]}[WR]$$

Conjunction Closure: If $A \wedge B$ is true at a world, then A and B are true at that world. If $A \wedge B$ is false at a world, then either A or B is false at that world. Given an underivable hypersequent $\mathcal{H}$ featuring $A \wedge B$ as true (resp. false) at some world, it may be extended into an underivable hypersequent where A and B are true (resp. either A is false or B is false) at that world, because we have the following derivations, which show that if that wasn't the case, $\mathcal{H}$ would be derivable.

$$\dfrac{\dfrac{\mathcal{H}[X, A, B, A \wedge B \vdash Y]}{\mathcal{H}[X, A \wedge B, A \wedge B \vdash Y]}[\wedge L]}{\mathcal{H}[X, A \wedge B \vdash Y]}[WL] \qquad \dfrac{\dfrac{\mathcal{H}[X \vdash A, A \wedge B, Y] \quad \mathcal{H}[X \vdash B, A \wedge B, Y]}{\mathcal{H}[X \vdash A \wedge B, A \wedge B, Y]}[\wedge R]}{\mathcal{H}[X \vdash A \wedge B, Y]}[WR]$$

Necessity Closure: If $\Box A$ is true at a world, then A is true at each subjunctive alternative to A. If $\Box A$ is false at a world, then A is false at some subjunctive alternative to A. is false at that world. Given an underivable hypersequent $\mathcal{H}$ featuring $\Box A$ as true at some world, and featuring some subjunctive alternative to that world, $\mathcal{H}$ may be extended into an underivable hypersequent where A is true at that subjunctive alternative; and if $\Box A$ is false at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where A is false at some subjunctive alternative to that world, because of the following derivations:

$$\dfrac{\dfrac{\mathcal{H}[X, \Box A \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A, \Box A \vdash Y \mid X' \vdash Y']}[\Box L]}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}[WL] \qquad \dfrac{\dfrac{\mathcal{H}[\vdash A \mid X \vdash \Box A, Y]}{\mathcal{H}[X \vdash \Box A, \Box A, Y]}[\Box R]}{\mathcal{H}[X \vdash \Box A, Y]}[WR]$$

Actuality Closure: If @A is true at a world, then A is true at that world's actual subjunctive alternative. If @A is false at a world, then A is false at that world's actual subjunctive alternative. Given an underivable hypersequent $\mathcal{H}$ featuring @A as true (resp. false) at some world, it may be extended into an underivable hypersequent where A is true (resp. false) at that world's actual subjunctive alternative, because we have the following derivations, which show that if that wasn't the case, $\mathcal{H}$ would be derivable.

$$\dfrac{\dfrac{\mathcal{H}[X, @A \vdash Y \mid X', A \vdash_@ Y']}{\mathcal{H}[X, @A, @A \vdash Y \mid X' \vdash_@ Y']}[@L]}{\mathcal{H}[X, @A \vdash Y \mid X' \vdash_@ Y']}[WL] \qquad \dfrac{\dfrac{\mathcal{H}[X \vdash @A, Y \mid X' \vdash_@ A, Y']}{\mathcal{H}[X \vdash @A, @A, Y \mid X' \vdash_@ Y']}[@R]}{\mathcal{H}[X \vdash @A, Y \mid X' \vdash_@ Y']}[WR]$$

A Priori Knowability Closure: If $\mathsf{APK}\,A$ is true at a world, then A is true at each indicative alternative to A. If $\mathsf{APK}\,A$ is false at a world, then A is false at some subjunctive alternative to A. Given an underivable hypersequent $\mathcal{H}$ featuring a $\mathsf{APK}\,A$ as true at some world, and featuring some subjuctive alternative to that world, $\mathcal{H}$ may be extended into an underivable hypersequent where A is true at that indicative alternative; and if $\mathsf{APK}\,A$ is false at some world, $\mathcal{H}$ may be extended into an underivable hypersequent where A is false at some indicative alternative to that world, because of the following derivations:

$$\dfrac{\dfrac{\mathcal{H}[X, \mathsf{APK}\,A \vdash Y \parallel X' \vdash_@ A, Y']}{\mathcal{H}[X, \mathsf{APK}\,A, \mathsf{APK}\,A \vdash Y \parallel X' \vdash_@ Y']}[\mathsf{APK}\,L]}{\mathcal{H}[X, \mathsf{APK}\,A \vdash Y \parallel X' \vdash_@ Y']}[WL] \qquad \dfrac{\dfrac{\mathcal{H}[\vdash_@ A \parallel X \vdash \mathsf{APK}\,A, Y]}{\mathcal{H}[X \vdash \mathsf{APK}\,A, \mathsf{APK}\,A, Y]}[\mathsf{APK}\,R]}{\mathcal{H}[X \vdash \mathsf{APK}\,A, Y]}[WR]$$

This completes the definition of downward closure.

So, if we start with an underivable sequent (even a sequent that cannot be derived in the *Cut*-free sytem), we may close under these conditions to get a downard closed, directed family $\mathfrak{D}$ of hypersequents.

LEMMA 5 [DOWNWARD CLOSED DIRECTED FAMILIES]: *Given any underivable (or Cut-free underivable) hypersequent $\mathcal{H}$, there is a directed family $\mathfrak{D}$ of underivable (or Cut-free underivable) hypersequents, satisfying the downward closure conditions.*

Now we have all the raw materials for our completeness theorem.

THEOREM 6 [COMPLETENESS]: *Any hypersequent which has no Cut-free derivation has a counterexample in some model.*

Proof: Take an underivable hypersequent. By Lemma 5, there is a downward closed directed family $\mathfrak{D}$ containing our starting hypersequent. Consider the frame of $\mathfrak{D}$ and choose an evaluation $\Vdash$ such that if p is true at w, $w \Vdash p$ and if p is false at w, $w \nVdash p$. Since for no sequent do we have p both on the left and on the right (if we did, that hypersequent would be an axiom, and hence derivable without *Cut*), this stipulation is consistent.

Now we check that for *every* formula A, and every world w in the model, if A is true at w then $w \Vdash A$ and if A is false at w then $w \nVdash A$. This is a proof by induction on the construction of A. The atomic case is given by stipulation. The downward closure conditions, and the definition of the frame elements: $\approx$ and @, are exactly what we need to show the induction steps.

Take conjunction. If $A \wedge B$ is true at w, then $A \wedge B$ is in the left part of some sequent inside a hypersequent in $\mathfrak{D}$, say $\mathcal{H}[X, A \wedge B \vdash Y]$. By the Conjunction Closure condition, this hypersequent is extended by another sequent $\mathcal{H}[X, A, B, A \wedge B \vdash Y]$ in $\mathfrak{D}$ in which A and B are true at the same world w. By hypothesis, $w \Vdash A$ and $w \Vdash B$ and by the definition of $\Vdash$, it follows that $w \Vdash A \wedge B$ as desired.

If $A \wedge B$ is false at w, then $A \wedge B$ is in the right part of some sequent inside a hypersequent in $\mathfrak{D}$, say $\mathcal{H}[X \vdash A \wedge B, Y]$. By the Conjunction Closure condition, this hypersequent is extended by another sequent, either $\mathcal{H}[X \vdash A, A \wedge B, Y]$ or $\mathcal{H}[X \vdash B, A \wedge B, Y]$, one of which is in $\mathfrak{D}$. So, it follows that either A or B is false at w. So, by induction, either $w \nVdash A$ or $w \nVdash B$, and in either case, $w \nVdash A \wedge B$ as desired.

Take APK. If $\mathsf{APK}\, A$ is true at w, then $\mathsf{APK}\, A$ is in the left part of some sequent in w inside a hypersequent in $\mathfrak{D}$. We wish to show that A is true in every indicative alternative of w. So, take an indicative alternative w'. This is a set of sequents, one of which occurs as an indicative alternative of a sequent in w inside some hypersequent: say $X' \vdash Y'$ in $\mathcal{H}[X, \mathsf{APK}\, A \vdash Y \;\|\; X' \vdash Y']$. By the *A Priori* Knowability Closure condition, this hypersequent is extended by another sequent $\mathcal{H}[X, \mathsf{APK}\, A \vdash Y \;\|\; X', A \vdash Y']$ in $\mathfrak{D}$ in which A is true at the world w'. This construction is general, and it holds for every indicative alternative w'. So, at every indicative alternative w', A is true, and by hypothesis, $w' \Vdash A$. It follows by the definition of $\Vdash$, that $w \Vdash \mathsf{APK}\, A$ as desired.

If $\mathsf{APK}\, A$ is false at w, then $\mathsf{APK}\, A$ is in the right part of some sequent inside a hypersequent in $\mathfrak{D}$, say $\mathcal{H}[X \vdash \mathsf{APK}\, A, Y]$. By the *A Priori* Knowability Closure condition, this hypersequent is extended by another sequent, $\mathcal{H}[\,\vdash_@ A \;\|\; X \vdash \mathsf{APK}\, A, Y]$, so take the world w' to be the world containing the sequent $\vdash_@ A$ indicated here. At this world, A is false, and it is an indicative alternative of w. By induction hypothesis, $w' \nVdash A$, and it follows that $w \nVdash \mathsf{APK}\, A$, as desired.

This construction is general: the cases for $\neg$, $\Box$ and @ are no more difficult. The result is a model in which if A occurs on the left of a sequent in w, then $w \Vdash A$ and if A occurs on the right of a sequent in w, then $w \nVdash A$.

But this is exactly what we need to show that our model provides a counterexample to the original hypersequent. Take the worlds in the frame to refute the starting hypersequent to be the worlds corresponding to the component sequents in this hypersequent. They are appropriately related, and the left parts of each sequent are *true* at the corresponding worlds and the right parts of each sequent are *false* at those worlds. We have the counterexample we want, and completeness is proved. ∎

COROLLARY 7 [CUT IS ADMISSIBLE]: *If a hypersequent is derivable with Cut, it is derivable without Cut too.*

Proof: We prove the contrapositive. If $\mathcal{H}$ is not derivable without *Cut*, then by Theorem 6 (COMPLETENESS), it has a counterexample in some model. By Theorem 3 (SOUNDNESS), it follows that this sequent is not derivable using *Cut*. So, contraposing, if $\mathcal{H}$ is derivable with *Cut*, it is also derivable without. ∎

We close this section, discharging our debt to show that while general models allow greater freedom in constructing counterexamples, it makes no difference to the logic in the vocabulary we have employed.

DEFINITION 12 [SQUARE MODELS]: Davies–Humberstone 2D models for $\Box, @, \mathsf{APK}$ are models where

- ▷ $W = V \times V$ for some set V.
- ▷ $\langle w, v\rangle \approx \langle w', v'\rangle$ iff $w = w'$.
- ▷ $@(\langle w, v\rangle) = \langle w, w\rangle$. So $@$ is the diagonal in $V \times V$.

THEOREM 8 [COMPLETENESS FOR SQUARE MODELS]: *Each hypersequent which has a counterexample in a general model also has a counterexample in a square model.*

Proof: Take an arbitrary model $\langle W, \approx, @, \Vdash\rangle$, we will add worlds to the structure (if that is necessary) to construct $\langle W', \approx', @', \Vdash'\rangle$, which will be isomorphic to some square model. Consider $|@|$, the cardinality of the set of actual worlds, and the supremum of the set of cardinalities of each $\approx$-equivalence class. If these are the same, excellent. If not, and if $|@|$ is smaller, we select a single $\approx$-equivalence class and *duplicate* it enough times to ensure that in the new structure there are as many $\approx$-equivalence classes as the supremum of the cardinalities of those classes. Call the new collection of actual members from the equivalence classes '$@'$.' Now, for each $\approx$-equivalence class with a smaller cardinality than the supremum, we add worlds by duplicating the *actual* world of that equivalence class as many times as required to ensure that each new class now contains as many worlds as $|@'|$ in this new frame. Call the new equivalence relation '$\approx'$.' So we have the new model $\langle W', \approx', @'\rangle$, and we define $\Vdash'$ by assigning the atomic propositions at a world present in the old model in exactly the same way in the new, and at an added world, we take the atomic propositions to be evaluated in exactly the same way as the world which has been 'duplicated.' The result is a model in which the truths at the original worlds W are unchanged.

Now to show that this is isomorphic to a square model, show how we can replace W' by a cartesian product. We will take the underlying 'worlds' to be the members of the cardinal $|@'|$, since we have one for each 'row' of the table, and each row is exactly the same length. So, the second coordinate of world in a row will be its world in the enumeration of all the rows. For first coordinate, we take the enumeration of the $\approx'$-class, but we rearrange it just a little. Wherever in this enumeration the @-world is, we swap it with the item in the enumeration in the same position as the position of the row in the enumeration. So, the @-world will be on the diagonal. The result is a proper square model in which the original hypersequent fails. ▪

So, by starting off with a proof theory, grounded in the acts of subjunctive and indicative supposing, we have an explanation of why $\Box$, @ and APK have the logical structure predicted by Davies–Humberstone models for two-dimensional modal logic. We have not had to start with these models, but rather, general possible worlds models result as idealisations or completions of invalid hypersequents. They are models, marking out different possibilities for assertion and denial, consistent with the inference rules governing our target concepts, the classical connectives, $\Box$, @ and APK. We do not need to explain modality in terms of our access to possible worlds (whether thought of as counterfactual scenarios or epistemic scenarios), rather we can understand why possible world talk has the efficacy which it does, by way of the rules governing these modal concepts. An ontology of subjunctive and indicative alternatives is then a reification of what holds in zones, and this 'ersatz' construction of worlds is no longer circular (defining necessity in terms of possible worlds and vice versa) because necessity is defined in terms of the underlying rules of use and their deeper connections with the two different practices of indicative and subjunctive supposing.

The extra assumptions made in Davies–Humberstone models, at least when understood as giving the account of *a priori* knowability—that the space of indicative alternatives has the same size and structure in every row of the model—is a harmless extra condition on models. Imposing it makes no difference in this vocabulary at least. Anything refutable in a general model is refutable in a square model as well. It is only when we extend the vocabulary, to include devices like propositional quantification, that the difference between general models and square models makes a difference to what holds.

BEYOND

The results of this paper open up new lines of research into modality and this construction. I will indicate a few.

- ▷ *The Epistemology and Ontology of Indicative & Subjunctive Modalities*: This construction tells us something about how a model respects meanings. An invalid sequent is a position in a discourse, with assertions and denials, in various zones, structured in such a way that the meaning postulates governing $\Box$, APK and @ are respected. This connects meaning, proofs and models. Further work must be done to connect proofs and models to the epistemology and ontology of modality.
- ▷ *Accessibility Relations*: We can generalise the account here to examine richer modal logics, governed by accessibility relations of various kinds. The tree hypersequent structures of Poggiolesi are appropriate here [8].
- ▷ *Defaults and Shifting*: What about conditionality (more restricted subjunctive shifts) and knowledge (more restricted indicative shifts)? Is there a way to give a similar kind of proof theory for counterfactuals and other more restricted modalities with default properties?
- ▷ *Objects and names*: The simplest treatment of quantification in a proof theory like this will result in models which are constant domain. We can prove the Barcan formulae $(\forall x)\Box Fx \supset \Box(\forall x)Fx$ and $(\forall x)\text{APK}\,Fx \supset \text{APK}\,(\forall x)Fx$. Is this appropriate? Is there a way to impose restrictions on the use of names and variables in different zones of a hypersequent in such a way as to give a different natural quantified modal logic?
- ▷ *Montague 'Grammar'*: We must generalise this to languages richer than epistemic modal first-order predicate logic, to allow quantification with higher types to allow for greater expressive power.
- ▷ *Context*: What about the treatment of temporal and indexical operators? Temporal operators and 'now' have a similar structure to the subjunctive modalities and 'actually.'
- ▷ *Hyperintensionality*: APK is highly idealised. What about the hyperintensionality of genuine knowledge claims? How does this change the picture?

REFERENCES

[1] DAVID CHALMERS. "Epistemic Two-Dimensional Semantics". *Philosophical Studies*, 118(1):153–226, 2004.

[2] MARTIN DAVIES. "Reference, Contingency, and the Two-Dimensional Framework". *Philosophical Studies*, 118(1):83–131, 2004.

[3] MARTIN DAVIES AND LLOYD HUMBERSTONE. "Two notions of necessity". *Philosophical Studies*, 38(1):1–30, 1980.

[4] LLOYD HUMBERSTONE. "Two-Dimensional Adventures". *Philosophical Studies*, 118(1):17–65, 2004.

[5] FRANK JACKSON. "Why We Need A-Intensions". *Philosophical Studies*, 118(1):257–277, 2004.

[6] MARK LANCE AND H. HEATH WHITE. "Stereoscopic Vision: Persons, Freedom, and Two Spaces of Material Inference". *Philosophers' Imprint*, 7(4):1–21, 2007.

[7] DIEGO MARCONI. "Two-dimensional semantics and the articulation problem". *Synthese*, 143(3):321–349, 02 2005.

[8] FRANCESCA POGGIOLESI. "The Method of Tree-Hypersequents for Modal Propositional Logic". In DAVID MAKINSON, JACEK MALINOWSKI, AND HEINRICH WANSING, editors, *Towards Mathematical Philosophy*, volume 28, pages 31–51. Springer Netherlands, 2009.

[9] ARTHUR N. PRIOR. "Worlds, Times and Selves". In PER HASLE, PETER ØHRSTRØM, TORBEN BRAÜNER, AND JACK COPELAND, editors, *Papers on Time and Tense*, pages 241–256. Oxford University Press, 2003. New edition.

[10] GREG RESTALL. "Multiple Conclusions". In PETR HÁJEK, LUIS VALDÉS-VILLANUEVA, AND DAG WESTERSTÅHL, editors, *Logic, Methodology and Philosophy of Science: Proceedings of the Twelfth International Congress*, pages 189–205. KCL Publications, 2005. http://consequently.org/writing/multipleconclusions.

[11] GREG RESTALL. "Proofnets for S5: sequents and circuits for modal logic". In COSTAS DIMITRACOPOULOS, LUDOMIR NEWELSKI, AND DAG NORMANN, editors, *Logic Colloquium 2005*, number 28 in Lecture Notes in Logic. Cambridge University Press, 2007. http://consequently.org/writing/s5nets/.

[12] GREG RESTALL. "Truth Values and Proof Theory". *Studia Logica*, 92(2):241–264, 2009. http://consequently.org/writing/tvpt/.

[13] SCOTT SOAMES. *Reference and Description: The Case against Two-Dimensionalism*. Princeton University Press, Princeton, N. J., 2005.