

# Decorated Linear Order Types and the Theory of Concatenation

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## **Abstract**

We study the interpretation of Grzegorzczuk's Theory of Concatenation TC in structures of decorated linear order types satisfying Grzegorzczuk's axioms. We show that TC is incomplete for this interpretation. What is more, the first order theory validated by this interpretation interprets arithmetical truth. We also show that every extension of TC has a model that is not isomorphic to a structure of decorated order types.

# 1 Introduction

In his paper [Grz05], Andrzej Grzegorzcyk introduces a theory of concatenation TC. The theory has a binary function symbol  $*$  for concatenation and two constants  $\mathbf{a}$  and  $\mathbf{b}$ . The theory is axiomatized as follows.

$$\text{TC1. } \vdash (x * y) * z = x * (y * z)$$

$$\text{TC2. } \vdash x * y = u * v \rightarrow ((x = u \wedge y = v) \vee \exists w ((x * w = u \wedge y = w * v) \vee (x = u * w \wedge y * w = v)))$$

$$\text{TC3. } \vdash x * y \neq \mathbf{a}$$

$$\text{TC4. } \vdash x * y \neq \mathbf{b}$$

$$\text{TC5. } \vdash \mathbf{a} \neq \mathbf{b}$$

Axioms TC1 and TC2 are due to Tarski. Grzegorzcyk calls axiom TC2: the editor axiom. We will consider two weaker theories. The theory  $\text{TC}_0$  has the signature with just concatenation, and is axiomatized by TC1,2. The theory  $\text{TC}_1$  is axiomatized by TC1,2,3. We will also use  $\text{TC}_2$  for TC.<sup>1</sup>

Andrzej Grzegorzcyk and Konrad Zdanowski have shown that TC is essentially undecidable. This result can be strengthened by showing that Robinson's Arithmetic  $\mathbf{Q}$  is mutually interpretable with TC. Note that  $\text{TC}_0$  is undecidable —since it has an extension that parametrically interprets TC— but that  $\text{TC}_0$  is not essentially undecidable: it is satisfied by a one-point model. Similarly  $\text{TC}_1$  is undecidable, but it has as an extension the theory of finite strings of  $\mathbf{a}$ 's, which is a notational variant of Presburger Arithmetic and, hence decidable.

We will call models of  $\text{TC}_0$ : concatenation structures. We will call models of  $\text{TC}_i$ : concatenation  $i$ -structures.

We will be interested in a special class of concatenation structures: those whose elements are *decorated linear order types* with as operation *addition, or concatenation of decorated order types*. Let a non-empty class  $A$  be given. An  $A$ -decorated linear ordering is a structure  $\langle D, \leq, f \rangle$ , where  $D$  is a non-empty domain,  $\leq$  is a linear ordering on  $D$ , and  $f$  is a function from  $D$  to  $A$ . A mapping  $\phi$  is an *isomorphism* between  $A$ -decorated linear order types  $\langle D, \leq, f \rangle$  and  $\langle D', \leq', f' \rangle$  iff it is a bijection between  $D$  and  $D'$  such that, for all  $d, e$  in  $D$ , we have  $d \leq e \Leftrightarrow \phi d \leq' \phi e$ , and  $f d = f' \phi d$ . Our notion of isomorphism gives us a notion of  $A$ -decorated linear order type. We have an obvious notion of sum or concatenation between  $A$ -decorated linear orderings which induces a corresponding notion of sum or concatenation for  $A$ -decorated linear order types.

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<sup>1</sup>The theories  $\text{TC}_i$  are theories for concatenation without the empty string, i.o.w. without the unit element. One can show that TC is *bi-interpretable* with a corresponding theory  $\text{TC}^\varepsilon$  via one dimensional interpretations without parameters.<sup>2</sup> The theory  $\text{TC}_1$  is *bi-interpretable* via two-dimensional interpretations with parameters with a corresponding theory  $\text{TC}_1^\varepsilon$ . The situation for  $\text{TC}_0$  seems to be more subtle.

We use  $\alpha, \beta, \dots$  to range over such linear order types. We write  $\text{DLOT}(A)$  for the universe of  $A$ -decorated linear order types with concatenation. Since, linear order types are classes we have to follow one of two strategies: either to employ Scott's trick to associate a set object to any decorated linear order type or to simply refrain from dividing out isomorphism but to think about decorated linear orderings modulo isomorphism. We will employ the second strategy.

We will call a concatenation structure whose domain consists of (representatives of)  $A$ -decorated order types, for some  $A$ , and whose concatenation *is* concatenation of decorated order types: a *concrete concatenation structure*. It seems entirely reasonable to stipulate that e.g. the interpretation of  $\mathbf{a}$  in a concrete concatenation structure is an decorated linear order type of a one element order. However, for the sake of generality we will refrain from making this stipulation.

Grzegorzcyk conjectured that every concatenation 2-structure is isomorphic to a concrete concatenation structure. We prove that this conjecture is false. (i) Every extension of  $\text{TC}_1$  has a model that is not isomorphic to a concatenation 1-structure and (ii) The set of principles of valid in all concrete concatenation 2-structures interprets arithmetical truth.

The plan of the paper is as follows. We show, in Section 2, that we have, for all decorated order types  $\alpha, \beta$  and  $\gamma$ , the following principle:  $(\dagger) \beta * \alpha * \gamma = \alpha \Rightarrow \beta * \alpha = \alpha * \gamma = \alpha$ .<sup>3</sup> It is easy to see that every group is a concatenation structure and that  $(\dagger)$  does not hold in the two element group. We show, in Section 5, that every concatenation structure can be extended to a concatenation structure with any number of atoms. It follows that there is a concatenation structure with at least two atoms in which  $(\dagger)$  fails. Hence,  $\text{TC}$  is incomplete for concrete concatenation structures. In Section 3, we provide a counterargument of a different flavour. We provide a tally interpretation that defines the natural numbers (with concatenation in the role of addition) in every concrete concatenation 2-structure. It follows that for every extension of  $\text{TC}_1$  is satisfied by a concatenation 1-structure that is not isomorphic to any concrete concatenation 1-structure, to wit any model of that extension that contains a non-standard element. In Section 4, we strengthen the result of Section 3, by showing that in concrete concatenation 2-structures we can add multiplication to the natural numbers. It follows that the set of arithmetically true sentences is interpretable in the concretely valid consequences of  $\text{TC}_2$ .

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<sup>3</sup>This fact was already known. See: [KT06], problem 6.13. Our proof, however, is different.

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## 2 A Principle for Decorated Order Types

In this section we prove a universal principle that holds in all concatenation structures, which is not provable in TC. There is an earlier proof of this principle. See: [KT06], problem 6.13. Our proof, however, is different.

**Theorem 2.1** *Let  $\alpha_0, \alpha_1, \alpha_2$  be decorated order types. Suppose  $\alpha_1 = \alpha_0 * \alpha_1 * \alpha_2$ . Then  $\alpha_1 = \alpha_0 * \alpha_1 = \alpha_1 * \alpha_2$ .*

### Proof

Suppose  $\alpha_1 = \alpha_0 * \alpha_1 * \alpha_2$ . Consider a decorated linear ordering  $\mathcal{A} := \langle A, \leq, f \rangle$  of type  $\alpha_1$ . By our assumption, we may partition  $A$  into  $A_0, A_1, A_2$ , such that:

$$\langle A, \leq, f \rangle = \langle A_0, \leq \upharpoonright A_0, f \upharpoonright A_0 \rangle * \langle A_1, \leq \upharpoonright A_1, f \upharpoonright A_1 \rangle * \langle A_2, \leq \upharpoonright A_2, f \upharpoonright A_2 \rangle,$$

where  $\mathcal{A}_i := \langle A_i, \leq \upharpoonright A_i, f \upharpoonright A_i \rangle$  is an instance of  $\alpha_i$ . Let  $\phi : \mathcal{A} \rightarrow \mathcal{A}_1$  be an isomorphism.

Let  $\phi^n \mathcal{A}_{(i)} := \langle \phi^n[A_{(i)}], \leq \upharpoonright \phi^n[A_{(i)}], f \upharpoonright \phi^n[A_{(i)}] \rangle$ . We have:  $\phi^n \mathcal{A}_i$  is of order type  $\alpha_i$  and  $\phi^n \mathcal{A}$  is of order type  $\alpha_1$ .

Clearly,  $\phi \mathcal{A}_0$  is an initial substructure of  $\phi \mathcal{A} = \mathcal{A}_1$ . So,  $\mathcal{A}_0$  and  $\phi \mathcal{A}_0$  are disjoint and  $\phi \mathcal{A}_0$  adjacent to the right of  $\mathcal{A}_0$ . Similarly, for  $\phi^n \mathcal{A}_0$  and  $\phi^{n+1} \mathcal{A}_0$ . Take  $A_0^\omega := \bigcup_{i \in \omega} \phi^i \mathcal{A}_0$ . We find that  $\mathcal{A}_0^\omega := \langle A_0^\omega, \leq \upharpoonright A_0^\omega, f \upharpoonright A_0^\omega \rangle$  is initial in  $\mathcal{A}$  and of decorated linear order type  $\alpha_0^\omega$ . So  $\alpha_1 = \alpha_0^\omega * \rho$ , for some  $\rho$ . It follows that  $\alpha_0 * \alpha_1 = \alpha_0 * \alpha_0^\omega * \rho = \alpha_0^\omega * \rho = \alpha_1$ . The other identity is similar.  $\square$

So, every concrete concatenation structure validates that  $\alpha_1 = \alpha_0 * \alpha_1 * \alpha_2$  implies  $\alpha_1 = \alpha_0 * \alpha_1 = \alpha_1 * \alpha_2$ . We postpone the proof that this principle is not provable in TC to Section 5.

## 3 Definability of the Natural Numbers

In this section, we show that the natural numbers can be defined in every concrete concatenation 1-structure. We define:

- $x \subseteq y : \leftrightarrow x = y \vee \exists u (u * x = y) \vee \exists v (x * v = y) \vee \exists u, v (u * x * v = y)$ .
- $x \subseteq_{\text{ini}} y : \leftrightarrow x = y \vee \exists v (x * v = y)$ .
- $x \subseteq_{\text{end}} y : \leftrightarrow x = y \vee \exists u (u * x = y)$ .
- $n : \tilde{\mathcal{N}}_{\mathbf{a}} : \leftrightarrow \forall m \subseteq_{\text{ini}} n (m = \mathbf{a} \vee \exists k \subseteq_{\text{ini}} m (k \neq m \wedge m = k * \mathbf{a}))$ .

We write  $m, n : \tilde{\mathcal{N}}_{\mathbf{a}}$  for:  $m : \tilde{\mathcal{N}}_{\mathbf{a}} \wedge n : \tilde{\mathcal{N}}_{\mathbf{a}}$ . Etc.

We prove the main theorem of this section.

**Theorem 3.1** *In any concrete concatenation structure, we have:*

$$\tilde{\mathbb{N}}_{\mathbf{a}} = \{\mathbf{a}^{n+1} \mid n \in \omega\}.$$

*I.o.w,  $\tilde{\mathbb{N}}_{\mathbf{a}}$  is precisely the class of natural numbers in tally representation (starting with 1). Note that  $*$  on this set is addition.*

### Proof

Consider any concrete concatenation 1-structure  $\mathfrak{A}$ . It is easy to see that every  $\mathbf{a}^{n+1}$  is in  $\tilde{\mathbb{N}}_{\mathbf{a}}$ .

Clearly, every element of  $\tilde{\mathbb{N}}_{\mathbf{a}}$  is either  $\mathbf{a}$  or it has a predecessor. The axioms of  $\text{TC}_1$  guarantee that this predecessor is unique. This justifies the introduction of the partial predecessor function  $\text{pd}$  on  $\tilde{\mathbb{N}}_{\mathbf{a}}$ . Let  $\alpha$  be the order type corresponding to  $\mathbf{a}$ . Let  $\beta_0$  be any element of  $\tilde{\mathbb{N}}_{\mathbf{a}}$ . If, for some  $n$ ,  $\text{pd}^n \beta_0$  is undefined, then  $\beta_0$  is clearly of the form  $\alpha^{k+1}$ , for  $k$  in  $\omega$ .

We show that the other possibility cannot obtain. Suppose  $\beta_n := \text{pd}^n \beta_0$  is always defined. Let  $\mathcal{A}$  be a decorated linear ordering of type  $\alpha$  and let  $\mathcal{B}_i$  be a decorated linear ordering of type  $\beta_i$ . We assume that the domain  $A$  of  $\mathcal{A}$  is disjoint from the domains  $B_i$  of the  $\mathcal{B}_i$ . Thus, we may implement  $\mathcal{B}_{i+1} * \mathcal{A}$  just by taking the union of the domains.

Let  $\phi_i$  be isomorphisms from  $\mathcal{B}_{i+1} * \mathcal{A}$  to  $\mathcal{B}_i$ . Let  $\mathcal{A}_i := (\phi_0 \circ \dots \circ \phi_i)(\mathcal{A})$ . Then, the  $\mathcal{A}_i$  are all of type  $\alpha$  and, for some  $\mathcal{C}$ , we have  $\mathcal{B}_0 = \mathcal{C} * \dots * \mathcal{A}_1 * \mathcal{A}_0$ . Similarly  $\mathcal{B}_1 = \mathcal{C} * \dots * \mathcal{A}_2 * \mathcal{A}_1$ . Let  $\tilde{\omega}$  be the opposite ordering of  $\omega$ . It follows that  $\beta_0 = \gamma * \alpha^{\tilde{\omega}} = \beta_1 = \text{pd}(\beta_0)$ . Hence,  $\beta_0$  is not in  $\tilde{\mathbb{N}}_{\mathbf{a}}$ .<sup>4</sup> A contradiction.  $\square$

We call a concatenation structure *standard* if  $\tilde{\mathbb{N}}_{\mathbf{a}}$  defines the tally natural numbers. Since, by the usual argument, any any extension of  $\text{TC}_1$  has a model with non-standard numbers, we have the following corollary.

**Corollary 3.2** *Every extension of  $\text{TC}_1$  has a model that is not isomorphic to a concrete concatenation 1-structure. In a different formulation: for every concatenation 1-structure there is an elementarily equivalent concatenation 1-structure that is not isomorphic to a concrete concatenation 1-structure.*

Note that the non-negative tally numbers with addition form a concrete concatenation 1-structure. Thus, the concretely valid consequences of  $\text{TC}_1 + \forall x (x : \tilde{\mathbb{N}}_{\mathbf{a}})$ , i.e., the principles valid in every concrete concatenation 1-structure satisfying  $\forall x (x : \tilde{\mathbb{N}}_{\mathbf{a}})$  are decidable.

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<sup>4</sup>Note that we are not assuming that  $\gamma$  is in  $\mathfrak{A}$ .

## 4 Definability of Multiplication

If we have two atoms to work with, we can add multiplication to our tally numbers. This makes the set of concretely valid consequences of TC non-arithmetical. The main ingredient of the definition of multiplication is the theory of relations on tally numbers. In TC, we can develop such a theory. We represent the relation  $\{\langle x_0, y_0 \rangle, \dots, \langle x_{n-1}, y_{n-1} \rangle\}$ , by:

$$\mathbf{bb} * x_0 * \mathbf{b} * y_0 * \mathbf{bb} * x_1 * \dots * \mathbf{bb} * x_{n-1} * \mathbf{b} * y_{n-1} * \mathbf{bb}.$$

We define:

- $r : \text{REL} \leftrightarrow \mathbf{bb} \subseteq_{\text{end}} r$ ,
- $\emptyset := \mathbf{bb}$ ,
- $x[r]y \leftrightarrow x, y : \tilde{\mathbf{N}}_{\mathbf{a}} \wedge \mathbf{bb} * x * \mathbf{b} * y * \mathbf{bb} \subseteq r$ .
- $\text{adj}(r, x, y) := r * x * \mathbf{b} * y * \mathbf{bb}$ .

Clearly, we have:  $\text{TC} \vdash \forall u, v \neg u[\emptyset]v$ . To verify that this coding works we need the adjunction principle.

**Theorem 4.1** *We have:*

$$\text{TC} \vdash (r : \text{REL} \wedge x, y, u, v : \tilde{\mathbf{N}}_{\mathbf{a}}) \rightarrow (u[\text{adj}(r, x, y)]v \leftrightarrow (u[r]v \vee (u = x \wedge v = y))).$$

We can prove this result by laborious and unspicuous case splitting. However, it is more to do the job with the help of a lemma. Consider any model of  $\text{TC}_0$ . Fix an element  $w$ . We call a sequence  $(w_0, \dots, w_k)$  a *partition* of  $w$  if we have that  $w_0 * \dots * w_k = w$ . The partitions of  $w$  form a category with the following morphisms.  $f : (u_0, \dots, u_n) \rightarrow (w_0, \dots, w_k)$  iff  $f$  is a surjective and weakly monotonic function from  $n + 1$  to  $k + 1$ , such that, for any  $i \leq k$ ,  $w_i = u_s * \dots * u_\ell$ , where  $f(j) = i$  iff  $s \leq j \leq \ell$ . We write  $(u_0, \dots, u_n) \leq (w_0, \dots, w_k)$  for:  $\exists f f : (u_0, \dots, u_n) \rightarrow (w_0, \dots, w_k)$ . In this case we say that  $(u_0, \dots, u_n)$  is a *refinement* of  $(w_0, \dots, w_k)$ .

**Lemma 4.1** *Consider any concatenation structure. Consider a  $w$  in the structure. Then, any two partitions of  $w$  have a common refinement.*

### Proof

Fix any concatenation structure. We first prove that, for all  $w$ , all pairs of partitions  $(u_0, \dots, u_n)$  and  $(w_0, \dots, w_k)$  of  $w$  have a common refinement, by induction of  $n + k$ .

If either  $n$  or  $k$  is 0, this is trivial. Suppose  $(u_0, \dots, u_{n+1})$  and  $(w_0, \dots, w_{k+1})$  are partitions of  $w$ . By the editor axiom, either (a)  $u_0 * \dots * u_n = w_0 * \dots * w_k$  and  $u_{n+1} = w_{k+1}$ , or there is a  $v$  such that (b)  $u_0 * \dots * u_n * v = w_0 * \dots * w_k$  and  $u_{n+1} = v * w_{k+1}$ , or (c)  $u_0 * \dots * u_n = w_0 * \dots * w_k * v$  and  $v * u_{n+1} =$

$w_{k+1}$ . We only treat case (b), the other cases being easier or similar. By the induction hypothesis, there is a common refinement  $(x_0, \dots, x_m)$  of  $(u_0, \dots, u_n, v)$  and  $(w_0, \dots, w_n)$ . Let this be witnessed by  $f$ , resp.  $g$ . It is easily seen that  $(x_0, \dots, x_m, w_{k+1})$  is the desired refinement with witnessing functions  $f'$  and  $g'$ , where  $f' := f[m+1 : n+1]$ ,  $g' := g[m+1 : k+1]$ .  $\square$

We turn to the proof of Theorem 4.1. The verification proceeds more or less as one would do it for finite strings.

## Proof

Consider any concatenation 2-structure. Suppose  $\text{REL}(r)$ . The right-to-left direction is easy, so we treat left-to-right. Suppose  $x, y, u$  and  $v$  are tally numbers. and  $u[\text{adj}(r, x, y)]v$ . There are two possibilities. Either  $r = \mathbf{bb}$  or  $r = r_0 * \mathbf{bb}$ . We will treat the second case. Let  $s := \text{adj}(r, x, y)$ . One the following four partitions is a partition of  $s$ : (i)  $(\mathbf{b}, \mathbf{b}, u, \mathbf{b}, v, \mathbf{b}, \mathbf{b})$ , or (ii)  $(w, \mathbf{b}, \mathbf{b}, u, \mathbf{b}, v, \mathbf{b}, \mathbf{b})$ , or (iii)  $(\mathbf{b}, \mathbf{b}, u, \mathbf{b}, v, \mathbf{b}, \mathbf{b}, z)$ , or (iv)  $(w, \mathbf{b}, \mathbf{b}, u, \mathbf{b}, v, \mathbf{b}, \mathbf{b}, z)$ . We will treat cases (ii) and (iv).

Suppose  $\sigma := (w, \mathbf{b}, \mathbf{b}, u, \mathbf{b}, v, \mathbf{b}, \mathbf{b})$  is a partition of  $s$ . We also have that  $\tau := (r_0, \mathbf{b}, \mathbf{b}, x, \mathbf{b}, y, \mathbf{b}, \mathbf{b})$  is a partition of  $s$ . Let  $(t_0, \dots, t_k)$  be a common refinement of  $\sigma$  and  $\tau$ , with witnessing functions  $f$  and  $g$ . The displayed  $\mathbf{b}$ 's in these partitions must have unique places among the  $t_i$ . We define  $m_\sigma$  to be the unique  $i$  such that  $f(i) = m$ , provided that  $\sigma_m = \mathbf{b}$ . Similarly, for  $m_\tau$ . (To make this unambiguous, we assume that if  $\sigma = \tau$ , we take  $\sigma$  as the common refinement with  $f$  and  $g$  both the identity function.)

We evidently have  $7_\sigma = 7_\tau = k$  and  $6_\sigma = 6_\tau = k - 1$ . Suppose  $4_\sigma < 4_\tau$ . It follows that  $\mathbf{b} \subseteq v$ . So,  $v$  would have an initial subsequence that ends in  $\mathbf{b}$ , which is impossible. So,  $4_\sigma \not\prec 4_\tau$ . Similarly,  $4_\tau \not\prec 4_\sigma$ . So  $4_\sigma = 4_\tau$ . It follows that  $v = y$ . Reasoning as in the case of  $4_\sigma$  and  $4_\tau$ , we can show that  $2_\sigma = 2_\tau$  and, hence  $u = x$ .

Suppose  $\rho := (w, \mathbf{b}, \mathbf{b}, u, \mathbf{b}, v, \mathbf{b}, \mathbf{b}, z)$  is a partition of  $s$ . We also have that  $\tau := (r_0, \mathbf{b}, \mathbf{b}, x, \mathbf{b}, y, \mathbf{b}, \mathbf{b})$  is a partition of  $s$ . Let  $(t_0, \dots, t_k)$  be a common refinement of  $\rho$  and  $\tau$ , with witnessing functions  $f$  and  $g$ . We consider all cases, where  $1_\tau < 6_\rho$ . Suppose  $6_\rho = 1_\tau + 1 = 2_\tau$ . Note that  $7_\rho = 6_\rho + 1$ , so we find:  $\mathbf{b} \subseteq x$ , quod non, since  $x$  is in  $\tilde{\mathbf{N}}_{\mathbf{a}}$ . Suppose  $2_\tau < 6_\rho < 4_\tau$ . In this case we have a  $\mathbf{b}$  as substring of  $x$ . Quod non. Suppose  $6_\rho = 4_\tau$ . Since  $7_\rho = 6_\rho + 1$ , we get a  $\mathbf{b}$  in  $y$ . Quod non. Suppose  $4_\tau < 6_\rho < 6_\tau$ . In this case, we get a  $\mathbf{b}$  in  $y$ . Quod impossibile. Suppose  $6_\rho \geq 6_\tau = k - 1$ . In this place there is no place left for  $z$  among the  $t_i$ . So, in all cases, we obtain a contradiction. So the only possibility is  $6_\rho \leq 1_\tau$ . Thus, it follows that  $u[r]v$ .  $\square$

We can now use our relations to define multiplication of tally numbers in the usual way. See e.g. Section 2.2 of [Bur05]. In any concrete concatenation 2-structure, we can use induction to verify the defining properties of multiplication

as defined. It follows that we can interpret all arithmetical truths in the set of concretely valid consequences of TC.

**Corollary 4.2** *We can interpret true arithmetic in the set of all principles valid in concrete concatenation 2-structures.*

## 5 The Sum of Concatenation Structures

In this section we show that concatenation structures are closed under sums. This result will make it possible to verify the claim that the universal principle of Section 2 is not provable in TC. The result has some independent interest, since it provides a good closure property of concatenation structures.

Consider two concatenation structures  $\mathfrak{A}_0$  and  $\mathfrak{A}_1$ . We write  $\star$  for concatenation in the  $\mathfrak{A}_i$ . We may assume, without loss of generality, that the domains of  $\mathfrak{A}_0$  and  $\mathfrak{A}_1$  are disjoint. We define the sum  $\mathfrak{B} := \mathfrak{A}_0 \oplus \mathfrak{A}_1$  as follows.

- The domain of  $\mathfrak{B}$  consists of non-empty sequences  $w_0 \cdots w_{n-1}$ , where the  $w_j$  are alternating between elements of the domains of  $\mathfrak{A}_0$  and  $\mathfrak{A}_1$ . In other words, if  $w_j$  is in the domain of  $\mathfrak{A}_i$ , then  $w_{j+1}$ , if it exists, is in the domain of  $\mathfrak{A}_{1-i}$ .
- The concatenation  $\sigma \star \tau$  of  $\sigma := w_0 \cdots w_{n-1}$  and  $\tau := v_0 \cdots v_{k-1}$  is  $w_0 \cdots w_{n-1} v_0 \cdots v_{k-1}$ , in case  $w_{n-1}$  and  $v_0$  are in the domains of different structures  $\mathfrak{A}_i$ . The concatenation  $\sigma \star \tau$  is  $w_0 \cdots (w_{n-1} \star v_0) \cdots w_{k-1}$ , in case  $w_{n-1}$  and  $v_0$  are in in the same domain.

In case  $\sigma \star \tau$  is obtained via the first case, we say that  $\sigma$  and  $\tau$  are *glued together*. If the second case obtains, we say that  $\sigma$  and  $\tau$  are *clicked together*.

**Theorem 5.1** *The structure  $\mathfrak{B} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$  is a concatenation algebra.*

### Proof

Associativity is easy. We check the editor property TC2. Suppose  $\sigma_0 \star \sigma_1 = z_0 \cdots z_{m-1} = \tau_0 \star \tau_1$ . We distinguish a number of cases.

**Case 1.** Suppose both of the pairs  $\sigma_0, \sigma_1$  and  $\tau_0, \tau_1$  are glued together. Then, for some  $k, n > 0$ , we have  $\sigma_0 = z_0 \cdots z_{k-1}$ ,  $\sigma_1 = z_k \cdots z_{m-1}$ ,  $\tau_0 = z_0 \cdots z_{n-1}$ , and  $\tau_1 = z_n \cdots z_{m-1}$ .

So, if  $k = n$ , we have  $\sigma_0 = \tau_0$  and  $\sigma_1 = \tau_1$ .

If  $k < n$ , we have  $\tau_0 = \sigma_0 \star (z_k \cdots z_{n-1})$  and  $\sigma_1 = (z_k \cdots z_{n-1}) \star \tau_1$ . The case that  $n < k$  is similar.

**Case 2.** Suppose  $\sigma_0, \sigma_1$  is glued together and that  $\tau_0, \tau_1$  is clicked together. So, there are  $k, n > 0$ ,  $u_0$ , and  $u_1$  such that  $\sigma_0 = z_0 \cdots z_{k-2} u_0$ ,  $\sigma_1 = u_1 z_k \cdots z_{m-1}$ ,  $u_0 \star u_1 = z_{k-1}$ ,  $\tau_0 = z_0 \cdots z_{n-1}$ , and  $\tau_1 = z_n \cdots z_{m-1}$ .



Suppose  $k \leq n$ . Then,  $\tau_0 = \sigma_0 * (u_1 z_k \cdots z_{n-1})$  and  $\sigma_1 = (u_1 z_k \cdots z_{n-1}) * \tau_1$ . Note that, in case  $k = n$ , the sequence  $z_k \cdots z_{n-1}$  is empty. The case that  $k \geq n$  is similar.

**Case 3.** This case, where  $\sigma_0, \sigma_1$  is clicked together and  $\tau_0, \tau_1$  is glued together, is similar to case 2.

**Case 4.** Suppose that  $\sigma_0, \sigma_1$  and  $\tau_0, \tau_1$  are both clicked together. So, there are  $k, n > 0$ ,  $u_0, u_1, v_0, v_1$  such that  $\sigma_0 = z_0 \cdots z_{k-2} u_0$ ,  $\sigma_1 = u_1 z_k \cdots z_{n-1}$ ,  $u_0 * u_1 = z_{k-1}$ ,  $\tau_0 = z_0 \cdots z_{n-2} v_0$ ,  $\tau_1 = v_1 z_n \cdots z_{m-1}$  and  $v_0 * v_1 = z_{n-1}$ .

Suppose  $k = n$ . We have  $u_0 * u_1 = z_{k-1} = v_0 * v_1$ . So, we have either (a)  $u_0 = v_0$  and  $u_1 = v_1$ , or, for some  $w$ , either (b)  $u_0 * w = v_0$  and  $u_1 = w * v_1$ , or (c)  $u_0 = v_0 * w$  and  $w * u_1 = v_1$ . In case (b), we have:  $\sigma_0 * w = \tau_0$  and  $\sigma_1 = w * \tau_1$ . We leave (a) and (c) to the reader.

Suppose  $k < n$ . We have:

$$\sigma_0 * (u_1 z_k \cdots z_{n-2} v_0) = \tau_0 \text{ and } \sigma_1 = (u_1 z_k \cdots z_{n-2} v_0) * \tau_1.$$

The case that  $k > n$  is similar. □

It is easy to see that  $\oplus$  is a sum or coproduct in the sense of category theory. The following theorem is immediate.

**Theorem 5.2** *If  $a$  is an atom of  $\mathfrak{A}_i$ , then  $a$  is an atom of  $\mathfrak{A}_0 \oplus \mathfrak{A}_1$ .*

Finally, we have the following theorem.

**Theorem 5.3** *Let  $A$  be any set and let  $\mathfrak{B} := \langle B, * \rangle$  be any concatenation structure. We assume that  $A$  and  $B$  are disjoint. Then, there is an extension of  $\mathfrak{B}$  with at least  $A$  as atoms.*

## Proof

Let  $A^*$  be the free semi-group on  $A$ . We can take as the desired extension of  $\mathfrak{B}$ , the structure  $A^* \oplus \mathfrak{B}$ . □

**Remark 5.4** The whole development extends with only minor adaptations, when we replace axiom TC2 by:

- $\vdash x * y = u * v \rightarrow ((x = u \wedge y = v) \dot{\vee} (\exists! w (x * w = u \wedge y = w * v) \vee \exists! w (x = u * w \wedge y * w = v)))$

Here  $\dot{\vee}$  is *exclusive or*. □

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