Envelopes and Indifference

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August 15, 2007

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1 The Problem

Consider this situation: Here are two envelopes. You have one of them. Each envelope contains some quantity of money, which can be of any positive real magnitude. One contains twice the amount of money that the other contains, but you do not know which one. You can keep the money in your envelope, whose numerical value you do not know at this stage, or you can exchange envelopes and have the money in the other. You wish to maximise your money. What should you do?[1]

Here are three forms of reasoning about this situation, which we shall call FORMS 1, 2 and 3, respectively.

Form 1 Let n be the minimum of the quantities in the two envelopes. Then there are two possibilities, which we may depict as follows:

	POSSIBILITY 1	POSSIBILITY 2
YOUR ENVELOPE	n	$2n$
OTHER ENVELOPE	$2n$	n

By the principle of indifference, the probability of each possibility is $\frac{1}{2}$. The expected value of keeping your envelope is

$$\frac{1}{2} \times 2n + \frac{1}{2} \times n = \frac{3}{2}n$$

The expected value of switching is

$$\frac{1}{2} \times n + \frac{1}{2} \times 2n = \frac{3}{2}n$$

Conclusion: SWITCHING IS A MATTER OF INDIFFERENCE.[2]

[1]We bracket, here, considerations of diminishing returns (you wish to truly *maximise* your monetary return), and the discrete nature of currency. Say, for the purposes of the discussion, the quantity is a cheque made out to you for some positive real quantity, which your bank will deposit into your account.

[2]would baulk at calling the quantities computed expectations, since they themselves contain a (random) variable. This is simply a matter of nomenclature. However, we will consider the status of the variables in these expressions in Section 4, below.

Form 2 Let x be the amount of money in your envelope. Then there are two possibilities, which we may depict as follows:

	Possibility 1	Possibility 2
Your Envelope	x	x
Other Envelope	$2x$	$x/2$

By the principle of indifference, the probability of each possibility is $\frac{1}{2}$. The expected value of keeping your envelope is

$$\frac{1}{2} \times x + \frac{1}{2} \times x = x$$

The expected value of switching is

$$\frac{1}{2} \times 2x + \frac{1}{2} \times \frac{x}{2} = \frac{5}{4}x$$

Conclusion: SWITCH.

Form 3 Let y be the amount of money in the other envelope. Then there are two possibilities, which we may depict as follows:

	Possibility 1	Possibility 2
Your Envelope	$2y$	$y/2$
Other Envelope	y	y

By the principle of indifference, the probability of each possibility is $\frac{1}{2}$. The expected value of switching is

$$\frac{1}{2} \times y + \frac{1}{2} \times y = y$$

The expected value of keeping is

$$\frac{1}{2} \times 2y + \frac{1}{2} \times \frac{y}{2} = \frac{5}{4}y$$

Conclusions: KEEP.

Prima facie, FORMS 1, 2 and 3 seem equally good as pieces of reasoning. Yet it seems clear that they cannot all be right. What should we say?

2 The Solution

In fact, all three answers give you the right solution, in three different circumstances. The relevant reasoning determining what you ought to do to maximise your outcome is *under-determined* by the original description of the situation. The correct way to reason, in the sense of maximising your return given the possibilities — which, after all, is the aim of each kind of reasoning — depends on the process by which the money ends up in the envelopes. For each form of reasoning there are mechanisms such that, if *that* mechanism was employed, the reasoning delivers the correct answer. Here are three examples:

Mechanism 1 A number, n, is chosen in any way one likes. One of the two envelopes is chosen by the toss of a fair coin, and n is put in that envelope; $2n$ is put in the other.

Mechanism 2 A number, x, is chosen in any way one likes. That is put in your envelope. Either $2x$ or $x/2$ is then put in the other envelope, depending on the toss of a fair coin.

Mechanism 3 A number, y, is chosen in any way one likes. This is put in the other envelope. Either $2y$ or $y/2$ is then put in your envelope depending on the toss of a fair coin.

That the three different forms of reasoning are correct for each of the corresponding mechanism is obvious once one has seen the three possibilities. For example, for Mechanism 1, let us suppose that the number n is chosen with a probability measure P. Since we have not specified any range we from which the amount is chosen, the most we can say is that the probability measure defines, for a range of measurable sets S of quantities, the probability $P(n \in S)$ — the probability that the number n is in the set S. So, $P(n \in [0, 1])$ is the probability that the number n is between zero and one, inclusive).[3] Then, since the quantities in the two envelopes are n and $2n$, decided at the toss of a fair coin, the probability that the quantity

[3] We need this delicacy when considering the probabilities, since we cannot in every case define the probability measure on the choice of n by considering the probabilities for the atomic events of each particular choice of n. For example, if the quantity n is chosen uniformly over $[0, 1]$, then $P(n = r)$ is *zero* for each $r \in [0, 1]$. Yet the measure is non-trivial: for example, $P(n \in [0, \frac{1}{3}]) = \frac{1}{3}$.

in *my* envelope is in a set S is $\frac{1}{2}P(n \in S) + \frac{1}{2}P(2n \in S)$, since there are two ways the content of my envelope could be in the set S. One way (with probability $\frac{1}{2}$) is that my the amount in my envelope *is*, n, and the probability that n is in S is $P(n \in S)$. The other way (also with probability $\frac{1}{2}$) is that the quantity in my envelope is $2n$, and the probability that this is in S is $P(2n \in S) = P(n \in S/2)$, where $S/2$ is the set of all members of S divided by 2. The probability that the quantity in *your* envelope is in set S is $\frac{1}{2}P(2n \in S) + \frac{1}{2}P(n \in S)$, which is the same quantity, so indifference in this is warranted, as the probabilities are identical.

On the other hand, given Mechanism 2, if the number x (the quantity in *your* envelope) is chosen with measure P' (so $P'(x \in S)$ is the probability that x is in the given set S), then the probability that the quantity in your envelope is in S is simply $P'(x \in S)$, whereas the probability that the quantity in my envelope is in that same set is $\frac{1}{2}P'(2x \in S) + \frac{1}{2}P'(\frac{x}{2} \in S)$, which may diverge significantly from $P'(x \in S)$. If P' is the uniform distribution on $[0, 1]$, than $P'(x \in [0, 1]) = 1$. On the other hand, $\frac{1}{2}P'(2x \in [0, 1]) + \frac{1}{2}P'(\frac{x}{2} \in [0, 1]) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 = \frac{3}{4}$. The reasoning in the case of Mechanism 3 is similar.

We have specified the state spaces sufficiently to determine enough of the probability measure on each space, in such a way that the probabalistic reasoning is valid. However, if the reader has any doubt about this, intuitions about the scenarios can checked by a series of trials. For example, a sequence of trials is generated employing Mechanism 1. Whether you adopt a policy of keeping or switching or doing either at random, makes no difference in the long run. Similarly, a sequence of trials is generated employing Mechanism 2: Adopting the policy of switching comes out 5/4 ahead of the policy of keeping in the long term (and changing at random comes out 9/8 ahead). The case is similar for Mechanism 3.

The two envelope paradox is well known.[4] It comes in different versions. The paradigm version is produced by giving reasoning of Form 2 in a context where Mechanism 1 is deployed. This, of course, gives the wrong results. We have just solved this paradox.

[4] The paradox first appeared in the philosophical literature in Cargyle (1992). It continues to generate a substantial literature. See, e.g., Jackson, Menzies and Oppy (1994), Broome (1995), Clark and Shackel (2000), Horgan (2000), Chase (2002). It should be noted that much of the literature appeals to the fact that money is discrete with a minimum or maximum. As is shown by the way that we have set things up, this does not get to the heart of the problem.

3 Bertrand's Paradox

There is a paradox concerning the principle of indifference usually called *Bertrand's Paradox*.[5] This can be put in many well-known ways. Here is one. A train leaves at noon to travel a distance of 300km. It travels at a constant speed of between 100km/h and 300km/h. What is the probability that it arrives before 2pm? We may reason in the following two ways.

1. If the train arrives before 2pm, its velocity must be greater than or equal to 150km/h. Given the range of possible velocities, by the principle of indifference, the probability of this is $\frac{3}{4}$. Hence, the probability is $\frac{3}{4}$.

2. The train must arrive between 1pm and 3pm. 2pm is half way between these two. By the principle of indifference, the train is as likely to arrive before as after. So the probability is $\frac{1}{2}$.

The two applications of the principle of indifference seem equally correct, but they result in inconsistent probabilities.

A standard solution to the paradox is to point out that the correct application of the principle of indifference depends upon the mechanism by which the velocity of the train is determined. If, for example, the velocity is determined by setting it to a number between 100 and 300, chosen at random, then reasoning 1 is correct. Suppose, on the other hand, that the velocity is chosen as follows. Choose a number of minutes, n, between 0 and 120, at random. Set the speed of the train to be $300/(1 + n/60)$ (distance/time). Then reasoning 2 is correct. In case it is not *a priori* clear that these are the right ways to reason in the contexts, the matter can be demonstrated by a sequence of appropriately designed trials.

As should now be clear, the paradigm two envelope paradox can be seen as a version of Bertrand's paradox. Perhaps what has prevented it from being seen as such is simply the fact that only one of the ways of applying the principle of indifference is standardly given.

4 Decision Theory and Designation

So far so good. But where, exactly—it may fairly be asked—does the reasoning in the paradigm two envelope paradox go wrong? The answer is that

[5] See, e.g., Kneale (1952).

it depends. It depends on how we conceptualise the designators employed. The reasoning proceeds as follows:

> Let x be the amount of money in your envelope; then there are two possibilities, $\langle x, 2x\rangle$ or $\langle x, x/2\rangle$ ...

Ask whether 'x' is a rigid designator or simply a definite description. Suppose, first, that it is a rigid designator—say it denotes \$10. Then in the second possibility, that which arises when the envelopes are switched, the amount of money in your envelope is precisely not x, that is, \$10. (It is either \$5 or \$20.) The values involved in the computations of the various expectations (particularly, the second summands) are therefore incorrect.

Suppose, instead that 'x' as it occurs in our reasoning is a definite description, such as 'the amount of money in your envelope,' and not a rigid designator. Then it certainly refers to the amount of money in your envelope in possibility 2. But now it refers to a different quantity than it referred to in possibility 1. To go on and compare the values of the expectations computed in this way is therefore a nonsense. This would be like reasoning as follows. The number of sons of the king (in some context) is 4; the number of daughters of the king (in some other context) is 3; hence the king has more sons than daughters. As is clear, both of the kings in question may have more daughters than sons.

Interpreted in one way, then, the reasoning is unsound; interpreted in the other way it is invalid. Similar considerations hold if one applies any of the methods for computing expectations together with a non-corresponding mechanism. To see how the mismatch occurs, consider a circumstance well-suited to the second form of reasoning: the mechanism we have called "mechanism 2." In *this* case, the term 'x' may rigidly refer to the amount of money in your envelope, or it may be a definite description abbreviating 'the number in your envelope.' In either case, the reasoning of form 2 is appropriate because the two possibilities countenanced in that form of reasoning ($\langle x, 2x\rangle$ and $\langle x, x/2\rangle$) match up precisely with the different outcomes of Mechanism 2, *given that interpretation of the term* 'x.' This cannot be said of Mechanism 1 or Mechanism 3.

The moral of the story is simple: in a computation of expectation, the designation of a variable must refer to the right quantity, and that reference must not vary as the reasoning encompasses different possibilities.

5 Opening the Envelope

There are, of course, other versions of the two envelope paradox. In another, one opens the envelope before deciding whether to exchange. Thus, let us suppose again that the money is distributed via Mechanism 1. But suppose that this time we open the envelope and find, say, \$10. The expectation of keeping is therefore \$10. The contents of the other envelope are either \$5 or \$20. So the expectation of switching is \$12.50. So one should switch. This seems equally paradoxical: the precise amount of money in the envelope seems to provide no significant new information. What is to be said about this? (Note that, in this case, the computation of expectation does not employ variables at all, just numerals—rigid designators.)

Note, for a start, that, relative to certain items of background information, new information provided by opening the envelope can make it rational to switch. Thus, suppose that you know that the minimum amount, n, is an odd integer. Then if you open the envelope and find, e.g., \$5, you should change; whereas if you find \$10, you should not. If you find \$5, then all the possibilities other than $\langle 5, 2.5\rangle$ and $\langle 5, 10\rangle$ have zero probability. And if you know that n is an odd integer, the first of these also has zero probability. If one computes the expectation of the two outcomes using this information, the expectation of keeping the envelope is 5, and that of switching is 10. So one should switch.

In general, if one knows a prior probability distribution for the value of n, or at least enough about it, then, by employing Bayes' Law

$$P(h/e) = \frac{P(e/h) \times P(e)}{P(h)}$$

one can compute a posterior probability distribution, given the evidence provided by opening the envelope. The posterior probability distribution thus generated provides the basis for the maximum-expectation computation.

However, if one has no such information, then there is no way one can compute posterior probabilities, and so use these in a computation of expectation. Thus, suppose one knows nothing more than that Mechanism 1 was deployed. One does not know the prior probability distribution, only the following constraint on it: for any n, the probabilities of $\langle n, 2n\rangle$ and $\langle 2n, n\rangle$ are identical (and all other pairs have zero probability). This is sufficient information to compute prior expectations as a function of the value n. Now if one opens the envelope and discovers, say, \$10, the only two possibilities left with non-zero probability are $\langle 10, 5\rangle$ and $\langle 10, 20\rangle$. But since one has no information about the prior probabilities of these two possibilities, one cannot compute their posterior probabilities. In particular, one cannot argue that, since there are two possibilities left, each has probability $\frac{1}{2}$. Thus, for example, if the prior probability distribution was such that $\langle 10, 5\rangle$ and $\langle 5, 10\rangle$ each had probability 1/2, whilst everything else had probability 0 (which is consistent with our information), then the posterior probabilities of these two options are 1 and 0, respectively. On the other hand, if it was such that $\langle 10, 20\rangle$ and $\langle 20, 10\rangle$ each had probability 1/2, whilst everything else had probability 0, then the posterior probabilities of these two options are 0 and 1, respectively. To claim that the relevant posterior probabilities are a half each is, therefore, fallacious.

6 Probabilistic Ignorance

In the situation we have just considered, we have insufficient information to compute the relevant expectations. The same situation arises, even before opening the envelope, if we have no information concerning the mechanism for distributing the money between envelopes. In this case, the information given so far in the characterisation of scenario seems to give us no good reason to switch. But how can one justify this view? The answer depends on how one conceptualises rational choice in such situations.

One possibility is to suppose that probability considerations are still relevant to choice. The probabilities in question cannot be objective, of course; not enough about such probabilities is known. So they must be subjective. Now, the prior probability distributions in question are over an infinite space. Since there is no uniform distribution over all the possibilities, there is no one distribution that recommends itself. There are infinitely many equally good distributions consistent with our knowledge. We may nullify any argument to the effect that one should switch or keep based on a probability distribution by pointing out that there are equally good distributions that recommend the opposite. We have already seen this in the case in which we open the envelope. In the case where we do not know the mechanism, all we know about the prior probability distribution is that all the possibilities with non-zero probability are of the form $\langle q_1, q_2\rangle$, where $q_1 = 2q_2$, or $2q_1 = q_2$. We may therefore nullify any argument to the effect that we ought to keep, based on a certain probability distribution, P, by pointing to its dual, P', obtained by swapping the values of q_1 and q_2. This will recommend the opposite choice. There is therefore nothing to break the symmetry, and so to give ground for anything other than indifference.[6]

The other possibility, one might suppose, is to abandon appeal to probability altogether, in favour of some other principle of decision making. But in this case, there also seems to be nothing to break symmetry in the epistemic situation at hand. Moreover, if any argument for switching or keeping is given, this can again be neutralised by pointing out that there are circumstances (mechanisms, distributions), in which the argument will give us the wrong answer. Again, anything except indifference has no rational ground.

In either case, then, as each case has been sketched, you have no reason to be anything other than indifferent: arguments attempting to justify some difference between switching and keeping get no grip. Does it follow that you *should* be indifferrent? Of course not: we have not ruled out any of the infinity of other considerations that might incline you to one envelope rather than another. (One envelope is red and the other is blue and you have promised to accept no blue gifts today. You like the shape of one of the envelopes and it would be a good addition to your collection. And so on.) As the case has been described, we have not ruled out *all* of the possible normative considerations that might apply in the circumstance before you. But when it comes to evaluating those, the probabilities we have discussed will not help. They give you no insight into what to do other than to be indifferent between switching and keeping.[7]

References

[1] Broome, J. (1995), 'The Two Envelope Paradox', *Analysis*, 5: 6–11.

[2] Cargyle, J. (1992), 'On a Problem about Probability and decision' *Analysis* 52, 211–216.

[6]One might, of course, be a *complete* subjectivist about the matter. Whatever probability distribution does, in fact, reflect your degrees of belief, go with that. If one is *such* a subjectivist, there is nothing more to say.

[7]Thanks go to Bryson Brown and Christine Parker for helpful discussions before we wrote this paper. Versions of the paper itself were given at the Australian National University and the University of Melbourne. Thanks for helpful comments are due to many members of the audiences, including Geoff Brennan, Christian Link, Wlodek Rabinowicz, Laura Schroeter, and especially Bruce Langtry.

[3] Chase, J. (2002), 'The Non-Probabilistic Two Envelope Paradox', *Analysis* 62: 157–160.

[4] Clarke, M. and Shackel, N. (2000), 'The Two Envelope Paradox', *Mind* 109: 415–442.

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[6] Jackson, F., Menzies, P. and Oppy, G. (1994), 'The Two Envelope "Paradox"', *Analysis* 54: 43-45.

[7] Kneale, W. (1952), *Probability and Induction*, Oxford: Oxford University Press.