Extending Intuitionistic Logic with Subtraction

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This paper is an exercise in formal and philosophical logic. I will show how intuitionistic propositional logic can be extended with a new two-place connective, not expressible in the traditional language of intuitionistic logic (the language of conjunction, disjunction, negation and implication). The new system will be shown to be a conservative extension of intuitionistic logic. After examining the formal properties of this extension, the task will be to consider whether this is an 'acceptable' extension of intuitionistic logic. It will turn out that on *some* intuitionistic considerations the extension is acceptable, and on others it is not.

In this paper I presume that the reader has some idea both of the formal properties of intuitionistic logic, and some motivating philosophical principles which inform the development of intuitionistic logic. Readers wanting such an introduction can do no better than look at some of the excellent, extensive literature on intuitionism [3, 5, 8, 12].

Other work has been done on extending intuitionistic propositional logic with new connectives. Gabbay [6] considers extending the logic with propositional connectives. Our new connective is not one he considers. De Jongh too considers extending intuitionistic logic by adding arbitrary conjunction and disjunction. In his interesting paper [2] he shows that once you make such an addition to the basic logic, you end up with a *proper class* of propositional connectives expressible. In what follows we will consider adding just one new binary connective to the language of intuitintionistic propositional logic.

1 Defining The Extension

Intuitionistic logic can be introduced in a number of ways. You can define a Hilbert-style axiomatisation [8] or a Prawitz-style natural deduction system [11]. It can be characterised algebraically through Heyting Lattices, or either Beth or Kripke-style frame semantics will characterise the logic.

We will examine each of these in turn in what follows. To start, however, we will present intuitionistic logic by way of a Gentzen-style sequent calculus. Instead of the usual formulation, however, with sequents of the form '$\Sigma \vdash A$', we will use sequents are of the form $\Sigma \vdash \Delta$, where Σ and Δ are sets of formulae. (We write 'Σ, A, Δ' as a shorthand for '$\Sigma \cup \{A\} \cup \Delta$' as usual). The calculus has one axiom scheme:

$$\Sigma \vdash \Delta \quad \text{if } \Sigma \cap \Delta \neq \emptyset$$

This is read 'the conjunction of Σ entails the disjunction of Δ', if Σ and Δ share a formula.

For each binary connective there are introduction rules for both the left and the right of the turnstile.

$$\frac{\Sigma, A, B \vdash \Delta}{\Sigma A \wedge B \vdash \Delta} \qquad \frac{\Sigma \vdash A, \Delta \quad \Sigma \vdash B, \Delta}{\Sigma \vdash A \wedge B, \Delta}$$

$$\frac{\Sigma, A \vdash \Delta \quad \Sigma, B \vdash \Delta}{\Sigma, A \vee B \vdash \Delta} \qquad \frac{\Sigma \vdash A, B, \Delta}{\Sigma \vdash A \vee B, \Delta}$$

$$\frac{\Sigma, A \vdash B}{\Sigma \vdash A \supset B} \qquad \frac{\Sigma, B \vdash \Delta \quad \Sigma \vdash A, \Delta}{\Sigma, A \supset B \vdash \Delta}$$

Instead of a separate rule for negation, we define $\neg A$ to be $A \supset \bot$, where $\bot$ is the *falsum*. *Verum* and *falsum* have the following rules.

$$\Sigma \vdash \top, \Delta \qquad \Sigma, \bot \vdash \Delta$$

The rules are quite simple. The only feature which distinguishes the system from a Gentzen calculus for classical propositional logic is the *implication right* rule. Note that for this rule the consequent must be a single formula. This ensures that we cannot prove the intuitionistically undesirable

$$\vdash A \vee (A \supset B)$$

which would be provable if we allowed more than one formula in the consequent in the implication rule

$$\frac{\dfrac{A \vdash A, B}{\vdash A, A \supset B}}{\vdash A \vee (A \supset B)}$$

This is an example proof. In general, a proof of a sequent $\Sigma \vdash \Delta$ is a finite tree with $\Sigma \vdash \Delta$ at the root, with axioms as leaves, and with transitions instances of the rules.

Gentzen's famous *Hauptstatz* is the *Cut Admissibility* (or *Elimination*) Theorem, which states that the rule *Cut*

$$\frac{\Sigma_1, A \vdash \Delta_1 \quad \Sigma_2 \vdash A, \Delta_2}{\Sigma_1, \Sigma_2 \vdash \Delta_1, \Delta_2}$$

is admissibile, in the sense that if the premises are provable, so is the conclusion [7]. The result holds in our system of intuitionistic logic. The restrictions on the implication rule do not complicate the proof.

Our extension to intuitionistic logic will be motivated as follows. Consider the implication connective. It is related to conjunction by way of the following rule

$$A \wedge B \vdash C \text{ if and only if } A \vdash B \supset C$$

Is there any connective related to *disjunction* in a similar way? If the disjunction remains on the left of the turnstile, there is no such connective, since were

$$A \vee B \vdash C \text{ if and only if } A \vdash B * C$$

to hold for all A, B and C, then we would have, if A and C are $\bot$, then

$$\bot \vee B \vdash \bot \text{ if and only if } \bot \vdash B * \bot$$

The right hand expression is a tautology, since $\bot$ entails anything. The left hand expression is equivalent to $B \vdash \bot$, so if it is a tautology, then B entails the falsum. This cannot happen for all B, so you cannot add a connective such as $*$ to any propositional logic containing $\bot$, on pain of triviality. However, the definition of a two-place connective $-$ with the rule

$$A \vdash B \vee C \text{ if and only if } A - B \vdash C$$

faces no such problems. We read '$A-B$' as 'A without B' or 'A minus B' and we call the connective "subtraction" since that is roughly the notion it expresses. If A entails either B or C, then from A *without* B you can infer C. Conversely, if A without B gives you C, then if you have A you can infer either A or B. In the context of classical propositional logic, $A - B$ is definable as $A \wedge \neg B$. In intuitionistic logic no such definition is available. $A \wedge \neg B$ will not do for $A - B$, since you have $\top \wedge \neg B \vdash \neg B$, but certainly not $\top \vdash B \vee \neg B$. We will see later that not only is subtraction not definable in intuitionistic propositional logic from the standard connectives, but the addition of propositional quantification to intuitionistic logic is not enough to define it in terms of resources available in the language. However, it can be defined with the addition of two new rules to our Gentzen calculus. The rules are the obvious analogues of the implication rules.

$$\frac{\Sigma \vdash A, \Delta \quad \Sigma, B \vdash \Delta}{\Sigma \vdash A - B, \Delta} \qquad \frac{A \vdash B, \Delta}{A - B \vdash \Delta}$$

The Cut Admissibility proof works as before in our new extended system. The only significant new piece of checking is that a cut on a formula $A - B$ immediately introduced on both the right and left of the turnstile can be replaced by a cut on its subformulae. So, we wish to show that the cut in

$$\frac{\dfrac{A \vdash B, \Delta_1}{A - B \vdash \Delta_1} \qquad \dfrac{\Sigma_2 \vdash A, \Delta_2 \quad \Sigma_2, B \vdash \Delta_2}{\Sigma_2 \vdash A - B, \Delta_2}}{\Sigma_2 \vdash \Delta_1, \Delta_2}\text{(Cut)}$$

can be replaced by cuts on A and B. But this is simple

$$\frac{\dfrac{A \vdash B, \Delta_1 \quad \Sigma_2, B \vdash \Delta_2}{A, \Sigma_2 \vdash \Delta_1, \Delta_2}\text{(Cut)} \qquad \Sigma_2 \vdash A, \Delta_2}{\Sigma_2 \vdash \Delta_1, \Delta_2}\text{(Cut)}$$

The rest of the proof of admissibility is just as simple. We leave the details to the interested reader. It follows that the system with the addition of subtraction remains well behaved.

The admissibility of cut shows us that subtraction so introduced satisfies our defining condition. If $A - B \vdash C$ is provable, then we can show that $A \vdash B \vee C$ is provable too.

$$\dfrac{\dfrac{\dfrac{A \vdash A, B \quad A, B \vdash B}{A \vdash B, A - B} \qquad A - B \vdash C}{A \vdash B, C}\text{(Cut)}}{A \vdash B \vee C}$$

Similarly (dually) from $A \vdash B \vee C$ we can get $A - B \vdash C$.

$$\dfrac{\dfrac{A \vdash B \vee C \qquad \dfrac{B \vdash B, C \quad C \vdash B, C}{B \vee C \vdash B, C}}{A \vdash B, C}\,(\text{Cut})}{A - B \vdash C}$$

So, the Gentzen calculus gives us an axiomatisation of our notion. The formalisation gives us one result immediately.

Theorem 1 *Adding subtraction conservatively extends intuitionistic logic.*

Proof Suppose we had a proof of $\Sigma \vdash \Delta$ in our new system, where Σ and Δ do not involve subtraction. It follows that any proof will not use the subtraction rules, for our system has the subformula property. Any formula appearing in the proof must appear in the conclusion of the proof. So, any proof of $\Sigma \vdash \Delta$ in the new system is a proof in our original system for intuitionistic logic. ◁

So, adding subtraction does not disturb what consecutions can be proved in the original language. In this minimal sense, at least, the addition is intuitionistically acceptable.

We can use the Gentzen calculus to prove a powerful result, which will be very useful in exploring the behaviour of subtraction. First, we need to formalise the notion of *duality* in our logic.

Definition 1 The *duality* function ^{d} maps formulae to formulae, sets of formulae to sets of formulae and sequents to sequents as follows:

$$\begin{array}{rcl}
p^d & = & p \\
\top^d & = & \bot \\
\bot^d & = & \top \\
(A \wedge B)^d & = & A^d \vee B^d \\
(A \vee B)^d & = & A^d \wedge B^d \\
(A \supset B)^d & = & B^d - A^d \\
(A - B)^d & = & B^d \supset A^d \\
\{A_1, \ldots, A_n\}^d & = & \{A_1^d, \ldots, A_n^d\} \\
(\Sigma \vdash \, , \,)^d & = & , \, ^d \vdash \Sigma^d
\end{array}$$

Note that ^{d} is of period two.

Lemma 2 (Duality) *$\Sigma \vdash \Delta$ is provable if and only if its dual is provable.*

Proof Note that the dual of an axiom is an axiom, and dualising an instance of one of the rules results in an instance of another rule. For example, an instance of the subtraction right rule

$$\dfrac{\Sigma \vdash A, \Delta \qquad \Sigma, B \vdash \Delta}{\Sigma \vdash A - B, \Delta}$$

dualised becomes

$$\dfrac{A^d, \Delta^d \vdash \Sigma^d \qquad \Delta^d \vdash \Sigma^d, B^d}{B^d \supset A^d, \Delta^d \vdash \Sigma^d}$$

which is an instance of the implication left rule. The other rules are similar. So, given a proof of $\Sigma \vdash \Delta$, the result of dualising every node in the tree is a proof of $(\Sigma \vdash \Delta)^d$. The fact that ^{d} is of period two ensures that if $(\Sigma \vdash \Delta)^d$ is provable, so is $\Sigma \vdash \Delta$, so our result is completed. ◁

The Duality Lemma is very powerful. We can use it to convert familiar results about intuitionistic implication to new results about subtraction. For instance, it is well known that intuitionistic logic is *prime*, in the sense that if $\vdash A \vee B$ is provable, then either $\vdash A$ or $\vdash B$ is provable. This does *not* extend to the system with subtraction, since we can prove $\vdash p \vee (\top - p)$ but neither $\vdash p$ nor $\vdash \top - p$.

$$\dfrac{\dfrac{\vdash \top, p \quad p \vdash p}{\vdash p, \top - p}}{\vdash p \vee (q - p)}$$

Even though primeness fails, the Duality Lemma shows us that there is an analogue to primeness which holds in the implication-free part of the language.

Lemma 3 *If $A \wedge B \vdash$, and A and B are implication-free then either $A \vdash$ or $B \vdash$.*

Proof Suppose A and B are implication-free. Then their duals A^d and B^d are subtraction-free and so, if $\vdash A^d \vee B^d$ then either $\vdash A^d$ or $\vdash B^d$. It follows that if $A \wedge B \vdash$, then by duality, $\vdash (A \wedge B)^d$, i .e. $\vdash A^d \vee B^d$. It follows that either $\vdash A^d$ or $\vdash B^d$, or equivalently, $A \vdash$ or $B \vdash$. ◁

The Hilbert-style axiomatisation of intuitionistic logic is a recursive enumeration of the theorems of the logic. That is, it is an enumeration of the formulae A such that $\vdash A$. For example, ignoring connectives other than $\supset$ for the moment, the axioms are the formulae of the forms

$$\big(A \supset (B \supset C)\big) \supset \big((A \supset B) \supset (A \supset C)\big) \qquad A \supset (B \supset A)$$

and the rule is *modus ponens*

From $A \supset B$ and A to derive B

Duality can give us an axiomatisation of the *co-theorems* of the subtraction fragment of the logic. That is, we can enumerate the formulae A such that $A \vdash$, by dualising the axioms above. You get the *co-axioms*

$$\big((C - A) - (B - A)\big) - \big((C - B) - A\big) \qquad (A - B) - B$$

and the rule

From $B - A$ and A to derive B

There are no formulae constructed out of subtraction alone which are theorems, dually to the fact that there are no co-theorems constructed out of implication alone. Why is this? Substitute verum for the propositional constants in an implication-only formula, and the result is equivalent to verum. So, the original implication only formula cannot be a co-theorem, lest verum be a co-theorem. Dually, substitute the falsum for the propositional constants in a subtraction-only formula, and the result is equivalent to the falsum. So, the original subtraction-only formula cannot be a theorem. It follows that the subtraction-only calculus is another example of a logic (along with Kleene's three-valued logics) which has derivable rules but no theorems. (Unlike Kleene's systems, however, the subtraction calculus has co-theorems.)

Just as intuitionistic negation $\neg A$ can be defined as $A \supset \bot$, a dual form $\sim A$ can be defined as $\top - A$. This negation has all the dual properties of $\neg$. For example, $\vdash A \vee \sim A$ is probable, but $A \wedge \sim A \vdash$ is not. $\sim\sim A \vdash A$ is provable, but $A \vdash \sim\sim A$ is not.

2 Kripke Models

In the next sections we will examine some models of intiuitionistic logic, to see to what extent subtraction is "at home" there. We will start with frames.

Definition 2 A *frame* is a set P of points together with a partial order $\leq$ of *inclusion* on P. The set $\text{Prop}(P, \leq)$ of *propositions* on a frame is the set of all the sets X of points which are closed upwards. That is, if $x \in X$ and $x \leq y$ then $y \in X$.

A *Kripke-evaluation* on a frame is a relation $\Vdash$ between points and atomic formulae satisfying the following *hereditary condition*

- If $x \Vdash p$ and $x \leq y$ then $y \Vdash p$, for atomic formulae p.

This condition ensures that the set of points at which an atomic formula is forced in the frame is indeed a proposition in the frame. The forcing relation is then extended to relate points to arbitrary formulae as follows:

- $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$.
- $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$.
- $x \Vdash \top$ and $x \nVdash \bot$ for each x.
- $x \Vdash A \supset B$ iff for each $y \geq x$, if $y \Vdash A$ then $y \Vdash B$.
- $x \Vdash A - B$ iff for some $y \leq x$, $y \Vdash A$ and $y \nVdash B$.

A frame together with an evaluation is called a *Kripke model.*

As usual, it's not difficult to show that for any proposition A the set of points at which A is forced is a proposition on the frame. The verification is a routine induction on the complexity of the formula. The only new step is the verification for subtraction, but this step is simple. If $x \Vdash A - B$ and $x' \geq x$, then $x' \Vdash A - B$ too, since any $y \leq x$ must also be a $y \leq x'$.

Note that the defining clause for subtraction is dual to the clause for implication. $A - B$ is true at a point just when for some earlier point we have A without B. Note too that on any Kripke model for intuitionistic logic we can define subtraction. It's instructive to show that the defining clause for subtraction holds in these models. To do this we need to know how to evaluate a sequent $\Sigma \vdash \Delta$ in a model. We will say that Σ entails Δ in a model if the set of points which each force each element of Σ is a subset of the set of points which each force some element of Δ. Then note that if A entails $B \vee C$ in a model, if $x \Vdash A - B$ then there is some point $y \leq x$ where $y \Vdash A$ but $y \nVdash B$. It follows that since A entails $B \vee C$, $y \Vdash B \vee C$, and since $y \nVdash B$ we must have $y \Vdash C$. Since the hereditary condition holds for all formulae, $x \Vdash C$ too, as desired.

Conversely, if $A - B$ entails C in the model, and $x \Vdash A$, we wish to show that $x \Vdash B \vee C$. If $x \not\Vdash B$ we have $x \Vdash A - B$, since $x \not\Vdash B$, $x \Vdash A$ and $x \leq x$. This, with the fact that $A - B$ entails C gives us $x \Vdash C$, so we know $x \Vdash B \vee C$ as desired.

It follows that these models are sound for our extension of intuitionistic logic. The completeness proof involves showing that the canonical model for our extension of intuitionistic logic satisfies the model conditions. The canonical model is simply the set of all prime theories — the sets , such that if $, \vdash \Delta$ then $, \cap \Delta \neq \emptyset$. The relation $\Vdash$ is defined by setting $, \Vdash p$ if and only if $p \in ,$. The inclusion relation $\leq$ is subsethood of theories.

The complexity arises in showing that the model conditions are satisfied by $\Vdash$ so defined. The distributive lattice operators are trivial to verify. One part of the implication clause is simple. If $A \supset B \in ,$ and $, \subseteq \Theta$ then if $A \in \Theta$ it follows that $B \in \Theta$, since $A, A \supset B \vdash B$. The other part is more complex. It involves showing that if $A \supset B \notin ,$ then there is a prime $\Theta \supseteq ,$ such that $A \in \Theta$ and $B \notin \Theta$. We use the *pair extension lemma* to do this.

Lemma 4 (Pair Extension (Belnap, Gabbay) [1]) *If $\vdash$ is a suitable consequence relation on a well-ordered language, then if $\Sigma \not\vdash \Delta$, then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\Sigma' \not\vdash \Delta'$, and $\Sigma' \cap \Delta'$ is the whole language.*

For our details it is sufficient for me to assure you that $\vdash$ in our language is suitable in Belnap and Gabbay's sense. The proof of this lemma is not too difficult. In involves well ordering the language, and at each step adding a formula whichever of Σ or Δ allow us to keep $\Sigma \not\vdash \Delta$. The logic is "suitable" if it allows us to do one or the other at each step. The endpoint of this process is the required Σ', Δ' pair.

Once we have such a pair, Σ' must be a prime theory, for if $\Sigma' \vdash \Psi$ for some Ψ, then we cannot have $\Psi \subseteq \Delta'$ since $\Sigma' \not\vdash \Delta'$, so Σ' and Ψ must intersect, as Σ' and Δ' exhaust the language.

We can apply this technique to our completeness theorem, since if $A \supset B \notin ,$ then $, \not\vdash A \supset B$, and hence $, , A \not\vdash B$. Therefore we can extend $, , A$ and B to an exclusive pair, the first element of which is a prime $\Theta \supseteq ,$, where $A \in \Theta$ and $B \notin ,$.

The same technique applies to our subtraction condition. We want to show that $A - B \in ,$ iff for some $\Theta \subseteq ,$, $A \in \Theta$ and $B \notin \Theta$. One half is easy. If $A \in \Theta$ and $B \notin \Theta$, then $A \vdash A - B, B$ tells us that $A - B \in \Theta$, and hence $A - B \in ,$ as required. For the more difficult half, suppose $A - B \in ,$. We wish to show that there is a $\Theta \subseteq ,$ such that $A \in \Theta$ and $B \notin \Theta$. We know that $A \not\vdash B, \overline{,}$, where $\overline{,}$ is the set of all formulae not in , , for if $A \vdash B, \overline{,}$, then $A - B \vdash \overline{,}$, which would give $A - B \notin ,$, as , is a prime theory. So, since $A \not\vdash B, \overline{,}$, we can extend A and $B, \overline{,}$ to an exclusive pair. The first part of this pair will be a prime theory (call it Θ) containing A, the second part will include B and everything not in , , so Θ must be a subset of , as desired.

So, we have shown that the canonical model is indeed a model for our extension of intuitionistic logic. This, with the simpler soundess proof gives us the following result.

Theorem 5 *$\Sigma \vdash \Delta$ is provable if and only if Σ entails Δ in every Kripke model.*

Now that we have this theorem, we can consider *extensions* of the basic system given by adding conditions on frames. Again, duality aids in our investigations.

DEFINITION 3 Given a model $\langle P, \leq, \Vdash \rangle$ its *dual* is the triple $\langle P, \geq, \Vdash^d \rangle$, where $x \Vdash^d p$ iff $x \nVdash p$, for each atomic formula p.

We use $\Vdash^d$ rather than $\nVdash$, since $\Vdash^d$ is extended to relate arbitrary formulae using the usual rules. For example, $x \Vdash^d A \wedge B$ if and only if $x \Vdash^d A$ and $x \Vdash^d B$. It would be confusing to write this '$x \nVdash A \wedge B$ if and only if $x \nVdash A$ and $x \nVdash B$,' thinking of $\nVdash$ as a forcing relation and not just as the negation of $\Vdash$, for this is not true, in general, when reading '$\nVdash$' as 'does not force'. It's better to have a new notation, like '$\Vdash^d$' for the relation which agrees with $\nVdash$ on atomic formulae, but diverges for more complex ones.

Note that the dual of a model is indeed a model (the hereditary condition is satsfied by $\Vdash^d$). The dual of a model is related to the original model by the following dualtiy lemma

LEMMA 6 (DUALITY FOR KRIPKE MODELS) *For any model $\langle P, \leq, \Vdash \rangle$, for any point $x \in P$, and for any proposition A, $x \Vdash A$ if and only if $x \nVdash^d A^d$. Furthermore, Σ entails Δ in a model if and only if Δ^d entails Σ^d in the dual of that model.*

The first result is a straightforward induction on the complexity of the proposition A. The second result is a simple corollary of the first. The lemma brings with it correspondence results for extensions of our basic logic.

Here is one example. The validity of $\top \vdash (A \supset B) \vee (B \supset A)$ corresponds to *no forward branching*, in the sense that $\top \vdash (A \supset B) \vee (B \supset A)$ is valid in all models with no forward branching, and if you have a frame *with* forward branching, then there is an evaluation on that frame which invalidates $\top \vdash (A \supset B) \vee (B \supset A)$. The duality lemma shows us that the dual condition, $(A - B) \wedge (B - A) \vdash \bot$ corresponds to *no backward branching*. The reasoning is direct. Consider a model with no backwards branching. Its dual has no forward branching, so there's no point at which $(A \supset B) \vee (B \supset A)$ fails. As a result, in the original model, there's no point at which the dual $(A - B) \wedge (B - A)$ succeeds, by duality. Conversely, suppose you have a frame with backwards branching. It follows that its dual has forward branching, and you can find an evaluation on that frame which invalidates $(A \supset B) \vee (B \supset A)$ somewhere. Dualise that model to construct a model on the original frame. This is a model with a point at which $(A - B) \wedge (B - A)$ succeeds somewhere, invalidating $(A - B) \wedge (B - A) \vdash \bot$ as required.

This procedure is general, so it applies in other cases too. For example, $\top \vdash \neg A \vee \neg\neg A$ corresponds to *forwards confluence* (if $x \leq y, z$ then for some w, $y, z \leq w$) so *backwards confluence* (if $x \geq y, z$ then for some w, $y, z \geq w$) corresponds to $\sim A \wedge \sim\sim A \vdash \bot$.

Depending on your interpretation of Kripke models, you may prefer frames which allow backwards branching, or those which do not. For example, if you take the frames to model states of a particular reasoner, with different choices during the process of reasoning, you might wish to allow forwards branching but to disallow backwards branching. (You might think you could only get *here* through the particular path of reasoning you've actually taken.) If that is the case, then subtraction allows you to express that thought in your object language. You think that $(A - B) \wedge (B - A)$ entails $\bot$. That is, you can't get both A before B and B before A.

3 Propositional Quantification

Now, if we extend the system with $\forall p$ with the following rules:

$$\frac{\Sigma \vdash Ap, \Delta}{\Sigma \vdash \forall pA, \Delta} \quad \text{where } p \text{ is not free in } \Sigma \cup \Delta$$

$$\frac{\Sigma, Ap \vdash \Delta}{\Sigma, \forall pA \vdash \Delta}$$

it follows that all intuitionistic connectives can be defined in terms of implication and univeral propositional quantification. (These rules might be too strong for your taste. For example, you can prove $\forall p(A \vee B) \supset \forall pA \vee \exists pB$ this way. You can restrict the rules, so that the universal right rule allows no Δ in the consequent, to get a more intuitionistically palatable flavour of universal propositional quantification. That is irrelevant for our purposes, for we will show that subtraction cannot be defined using $\forall p$ as given here — it will follow that it cannot so be defined using any *weaker* connective.) The definition of the connectives are as follows:

$$\begin{array}{rcl} \bot & = & \forall pp \\ A \wedge B & = & \forall p\big((A \supset (B \supset p)) \supset p\big) \\ A \vee B & = & \forall p\big((A \supset p) \supset ((B \supset p) \supset p)\big) \end{array}$$

The natural question arises. Can subtraction be defined in these terms too?

THEOREM 7 *Subtraction cannot be defined in terms of propositional quantification and intuitionistic propositional logic.*

PROOF The universal quantifier rules given above are valid in Kripke models in which we interperet univeral propositional quantification with a constant domain of propositions. In other words, given a frame $\langle P, \leq \rangle$, the propositions quantified over are the members of the set $\text{Prop}(P, \leq)$. Given the set PV of propositional variables, an *assignment* is a function a from PV to $\text{Prop}(P, \leq)$. A for any propositional variable p and any proposition X, the *modification* on an assignment a, written '$a(p := X)$', is the function which returns X when given p, but agrees with a otherwise. A model is determined by a relation $\Vdash$ between atomic formulae (propositional constants) and points as before. However now the relation is extended to relate points, formulae and *assignments* in order to model formulae with free propositional variables. The recursive definition is as follows.

- $a, x \Vdash p$ iff $x \Vdash p$, if p is a propositional constant.
- $a, x \Vdash p$ iff $x \in a(p)$, if p is a propositional variable.
- $a, x \Vdash A \supset B$ iff for each $y \geq x$, if $y \Vdash A$ then $y \Vdash B$.
- $a, x \Vdash \forall pA$ iff $a(p := X), x \Vdash A$ for each $X \in \text{Prop}(P, \leq)$.

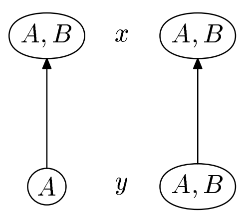
It is quite simple to show that the rules given above for the universal quantifier are valid in these models.

Note that the rules given above care only about the descendents of points (that is, they care only about the points appearing further on in the model). There is nothing which looks backwards down the relation $\leq$. We will formalise this notion, and use it to show that subtraction cannot be defined using these connectives.

Given the model $\langle P, \leq, \Vdash \rangle$, the submodel generated from x is the triple, $\langle P_x, \leq_x, \Vdash_x \rangle$ defined by restricting your attention to the set of descendents of x. We will show by induction on A, that for each $y \geq x$, $a, y \Vdash A$ iff $a_x, y \Vdash_x A$.

- The result holds true when A is either a propositional constant. If $y \geq x$ then $y \Vdash p$ iff $y \Vdash_x p$.
- If A is a propositional variable p, then $a, y \Vdash p$ iff $a_x, y \Vdash_x p$, since $y \in a(p)$ iff $y \in a_x(p)$, as $y \geq x$.
- For implication, whenever $y \geq x$, $a, y \Vdash A \supset B$ iff for each $z \geq y$, if $a, z \Vdash A$ then $a, z \Vdash B$, iff for each $z \geq y$, if $a_x, z \Vdash_x A$ then $a_x, z \Vdash_x B$ (by hypothesis) iff $a_x, y \Vdash_x A \supset B$, since $z \geq y$ only if $z \geq x$.
- For universal quantification, $a, y \Vdash \forall pA$ iff $a(p := X), y \Vdash A$ for each $X \in \text{Prop}(P, \leq)$, iff $a(p := X)_x, y \Vdash_x A$ for each $X \in \text{Prop}(P, \leq)$ (by hypothesis) iff $a_x(p := X), y \Vdash_x A$ for each $X \in \text{Prop}(P_x, \leq_x)$, iff $a_x, y \Vdash_x \forall pA$, as desired.

It follows that subtraction is not definable in the language of propositional quantification and implication. Consider the following two models on the one frame.



These models agree at point x (at which A and B are both true) but not at the earlier point y. By the generated submodel result, any proposition expressible in terms of the universal quantifier and implication will have the same value at x in both models. (Since the models restricted to x agree.) However, $A - B$ is true at x in the first model, but not the second. So, the subtraction cannot be expressed in terms of universal quantification and implication. ◁

What does this mean for subtraction? If your only resources at hand are those of implication and propositional quantification and those definable in terms of these notions, then you cannot define subtraction. Does this mean that subtraction is not an intuitionistic connective? That depends on what resources you have at hand.

4 Embeddings

It is possible to embed classical logic into intuitionistic logic by way of the famous *double negation translation.* A is a classical theorem if and only if $\neg\neg A$

is an intuitionistic theorem. By duality, $\neg\neg A$ is a theorem if and only if $\sim\sim A^d$ is a co-theorem. It is simple to show that A is a classical theorem if and only if A^d is a classical co-theorem. It follows that $\sim\sim A$ is a co-theorem of our logic if and only if A is a classical co-theorem. So, for example, $\sim\sim(A \wedge \sim A)$ is a co-theorem, since $A \wedge \sim A$ is a classical co-theorem, dually to the intuitionistic theoremhood of $\neg\neg(A \vee \neg A)$.

The embedding works in the other direction too. It is well known that intuitionistic logic can be modelled inside S4 by the translation

$$\begin{array}{rcl} p^t & = & \Box p \\ \top^t & = & \top \\ \bot^t & = & \bot \\ (A \wedge B)^t & = & A^t \wedge B^t \\ (A \vee B)^t & = & A^t \vee B^t \\ (A \supset B)^t & = & \Box(A^t \supset B^t) \end{array}$$

The verification is a straightforward mapping between the possible worlds models for S4 and those for intuitionistic logic. An inspection of the semantics for the implication-free fragment shows that a similar mapping will work here:

$$\begin{array}{rcl} p^t & = & \Diamond p \\ \top^t & = & \top \\ \bot^t & = & \bot \\ (A \wedge B)^t & = & A^t \wedge B^t \\ (A \vee B)^t & = & A^t \vee B^t \\ (A - B)^t & = & \Diamond(A^t \wedge \neg B^t) \end{array}$$

However this time, the intuitionistic accessiblity relation $\leq$ is modelled by the *converse* of the relation R in the S4 frame, since propositions of the form $\Diamond A$ are not closed upwards in frames but rather, closed *downwards*. If $x \Vdash \Diamond A$ and yRx, it follows that $y \Vdash \Diamond A$ (when R is transitive, at least). The verification that this translation models the implication-free fragment is routine. The more interesting result is the modelling of the whole language. For this we need two modal operators, a necessity which looks up the relation R, and a possibility which looks down the relation. In other words, we need the *temporal* logic TS4, which extends S4 with another modal operator $\blacklozenge$, with the following rule

$$\frac{\blacklozenge A \vdash B}{A \vdash \Box B}$$

The logic is sound and complete with respect to the expected class of models. The frame conditions are as you would expect:

- $x \Vdash \Box A$ if and only if for each y where xRy, $y \Vdash A$.
- $x \Vdash \blacklozenge A$ if and only if for some y where yRx, $y \Vdash A$.

The translation of our logic into TS4 is as follows:

$$\begin{array}{rcl} p^t & = & \Box p \\ \top^t & = & \top \\ \bot^t & = & \bot \\ (A \wedge B)^t & = & A^t \wedge B^t \\ (A \vee B)^t & = & A^t \vee B^t \\ (A \supset B)^t & = & \Box(A^t \supset B^t) \\ (A - B)^t & = & \blacklozenge(A^t \wedge \neg B^t) \end{array}$$

The intuitionistic propositions according to this translation are still the sets of formulae which are closed upwards. It is simple to show that not only are formulae of the form $\Box A$ closed upwards, but so are formulae of the form $\blacklozenge A$, so we could have decided to translate p by $\blacklozenge p$ instead of $\Box p$, the translation would have the same results.

5 Algebra and Topology

It's well known that logics can be studied using algebraic techniques. The algebras appropriate to intuitioinistic logic are well-known. *Heyting algebras* are distributive lattices with relative pseudocomplementation. That is, in addition to binary operators $\wedge$ and $\vee$ satisfying the usual distributive lattice axioms, there is a two-place operator $\supset$ such that for every elements a, b and c of the algebra, $a \wedge b \leq c$ if and only if $a \leq b \supset c$, where $\leq$ is the natural partial order defined in terms of the lattice operations.

Subtraction can be defined analagously — it is the operator defined on the algebra by setting $a - b \leq c$ iff $a \leq b \vee c$.

It is well known that in any topological space, the set of open sets, under the natural order, union and intersection is a Heyting algebra. In particular, $A \supset B$ can be defined as the interior of the set $\overline{A} \vee B$. It is simple to show that subtraction can not be defined on all topological spaces. Consider the standard topology on the real line, and take the two open sets $(0,1)$ and $(0,2)$. Can the subtraction $(0,2) - (0,1)$ be selected as an open set? We know that $(0,2) \subseteq (0,1) \cup (x,2)$ whenever $x < 1$. So, if $(0,2) - (0,1)$ is defined, we have $(0,2) - (0,1) \subseteq (x,2)$ whenever $x < 1$. So, $(0,2) - (0,1) \subseteq \bigcap_{x<1}(x,2) = [1,2)$. However, $(0,2) - (0,1) \not\subseteq (1,2)$, as $(0,2) \not\subseteq (0,1) \cup (1,2)$. Therefore the subtraction $(0,2) - (0,1)$ is not an open set.

However, dually, the implication-free logic is the logic of closed sets in a topological space, just as the subtraction-free logic (standard intuitionistic logic) is the logic of open sets. (This is a straightforward consequence of our duality lemma.)

It would be a mistake, however, to infer from this that the logic of *both* implication *and* subtraction was the logic of *clopen* sets in a topological space. The natural structure is a set with *two* topologies, $\langle X, \mathcal{O}_1, \mathcal{O}_2 \rangle$. We'll call such structures *bi-topologies*.

Let's talk about these in terms of closure and interior operators Cl_1 and Int_2. They are tied together by the following rules. For each $Y, Z \subseteq X$

$$\mathrm{Cl}_1 Y \subseteq Z \text{ if and only if } Z \subseteq \mathrm{Int}_2 Y$$

It follows that the closed$_1$ sets are the open$_2$ sets.

Consider frames. In frames, $\mathrm{Cl}_1 Y = \{z : \exists y \in Y, y \leq z\}$. It turns out that any bi-topology is isomorphic to such a frame. Why? Set $y \leq z$ to mean $z \in \mathrm{Cl}_1\{y\}$. It's transitive and reflexive. (Not necessarily antisymmetric, however.) We'll show that $\mathrm{Cl}_1 Y = \{z : \exists y \in Y, z \in \mathrm{Cl}_1\{y\}\}$. Why is this? The closed$_1$ sets are the open$_2$ sets. It follows that the closed$_1$ sets are closed under arbitrary unions. So, $\{z : \exists y \in Y, z \in \mathrm{Cl}_1\{y\}\}$ is closed$_1$, as it's a union of closed sets. So, it must be the closure of Y, as it's closed, and it's contained in any closure of Y.

This is a known topological result. The topology we have defined on a frame is the *Alexandrov topology* on a partially ordered set ([9] page 45). We have shown that a topology is appropriate for intuitionisitic logic with subtraction if and only if it is a frame. So frames are a very good model for our logic. A topology is a model for subtraction only if it is isomorophic to a frame.

6 Beth Models

Now that we have our results from topology and algebra, it's simple to see how subtraction behaves in Beth models.

DEFINITION 4 Given a frame $\langle P, \leq \rangle$, a *bar through a point* $x \in P$ is a set B of points such that any path through x must intersect B at least once. A set X is *closed under bars* if and only if for each $x \in P$ if X bars x then $x \in X$.

The set $\text{Prop}(P, \leq)$ of *Beth propositions* on a frame $\langle P, \leq \rangle$ is the set of all the sets X of points which are both closed upwards and closed under bars.

A *Beth-evaluation* on a frame is a relation $\Vdash$ between points and atomic formulae satisfying the following *hereditary* and *bar conditions*.

- If $x \Vdash p$ and $x \leq y$ then $y \Vdash p$, for atomic formulae p.
- If x is barred by a set Y such that for each $y \in Y$, $y \Vdash p$, then $x \Vdash p$ too.

These conditions ensure that the set of points at which an atomic formula is forced in the frame is indeed a proposition in the frame. The forcing relation is then extended to relate points to arbitrary formulae as follows:

- $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$.
- $x \Vdash A \vee B$ iff for some bar B of x, for each $y \in B$, $y \Vdash A$ or $y \Vdash B$.
- $x \Vdash \top$ and $x \nVdash \bot$ for each x.
- $x \Vdash A \supset B$ iff for each $y \geq x$, if $y \Vdash A$ then $y \Vdash B$.

A frame together with an evaluation is called a *Beth model*.

Beth frames differ from Kripke frames by introducing the notion of a bar, and the modified condition for disjunction. The idea is this: a disjunction can be true at a point without either disjunction being true at that point, provided that *no matter how things turn out*, one or other disjunct is eventually made true.

As in the Kripke case, it's not difficult to show that for any proposition A the set of points at which A is forced is a proposition on the frame. The verification is a routine induction on the complexity of the formula.

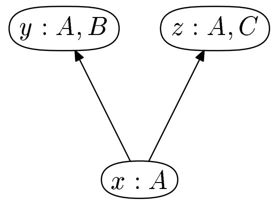
These frames can be extended to model subtraction, because propositions in Beth models are closed under infinite intersection. If you have a family of propositions (closed upwards and closed under bars) then the intersection must be closed upwards and under bars, so it is a proposition too. Therefore, we can define the subtraction of two propositions using intersection. Given two propositions X and Y, the subtraction of Y from X is the intersection of all propositions Z such that X entails (that it, is a subset of) the closure of $Y \cup Z$. In other words, X minus Y is the smallest proposition Z such that $Y \cup Z$ bars all elements of X.

To make this look a *little* more like the traditional evaluation clauses, $x \Vdash A - B$ iff $x \Vdash C$ for each C where $A \Vdash B \vee C$, or $A - B$ is the smallest proposition C where $A \Vdash B \vee C$. So, $x \Vdash A - B$ iff

$$\forall X \forall z (z \Vdash A \Rightarrow z \text{ is barred by } ||B|| \cup X) \Rightarrow x \in X)$$

(where $||B||$ is the set of all points y where $y \Vdash B$). This is by no means a first-order condition on points in the model, but then, neither is the disjunction condition in Beth models. Both conditions quantify over not only points in the model but also sets of points.

The more simple and perspicuous condition for subtraction in Kripke models, alas, won't do in Beth models, as there are simple counterexamples. Consider the following model



In this model, $x \Vdash B \vee C$. In particular we have $A \vdash B \vee C$. So, we'd wish to have $A - B \vdash C$. C is true only at z, so it follows that we cannot have $x \Vdash A - B$. The point x would support $A - B$ were we to use a Kripke evaluation (but then, x would not support $B \vee C$, so the subtraction condition is still satisfied). In this model, $A - B$ is true only at z, as the set $\{z\}$ is a proposition and it is the smallest such proposition that with $\{y\}$ bars $\{x, y, z\}$.

Before moving on to other topics, we should note that the dualisation results from Kripke frames do not transfer to Beth frames, because of the way disjunction and conjunction are treated differently. However, if you desired, you could dualise the whole construction anyway — but in the dual frame, disjunction has the traditional clause, and conjunction has a clasue referring to bars of points. Subtraction has the simple clause from Kripke frames, but implication has a very complex clause. The points in Beth models are not theories (closed under conjunction and logical consequence) but they are closed under disjunction (if $A \vee B \in a$ then $A \in a$ or $B \in a$) and logical consequence, but not conjunction (you can have $A \in a$ and $B \in a$ but not yet $A \wedge B \in a$). Do such strange models for intuitionistic logic have any use? That remains to be seen.

7 Further Work

There's more work to look at than could fit in this paper.

- The λ-calculus corresponds to intuitionistic implication quite nicely. The types of closed terms are the theorems of intuitionistic implication. Is there a subtraction analogue?

- Functional Completeness: Is there a sense in which the connectives of intuitionistic propositional logic capture all of the propositional connectives on some domain of propositions?

- Forcing: It is known that intuitionistic logic has a connection with forcing [4]. Does subtraction add anything to our understanding of this connection?

- Topos Theory: Intuitionistic logic is modelled in toposes, which are naturally occurring category-theoretic structures. Does subtraction have a place in these structures? Work in closed set logic in categories is no doubt relevant here [10], though of course you will need a combination of closed and open set logic to model subtraction in intuitionistic logic.
- Quantifiers: Of course, propositional logic is not the only game in town. We ought to consider first-order quantifiers. Are the universal and existential quantifiers of intuitionistic logic also naturally extended to include other quantifiers in an analagous way?

8 Is the Extension Acceptable?

Intuitionistic logic can be interpreted in many different ways. In this section we will consider just three different understandings of the formal system of intiuitionistic logic to see whether subtraction emerges as an appropriate connective.

8.1 Incomplete Information

One reason to be interested in intuitionistic logic might be a concern to appropriately incomplete information. If your interest is in formal semantics which allow incomplete information, then a natural way to model this is in structures like Kripke frames. We model propositions as upwardly closed sets of points — where these points can be stages of reasoning, theories, times, precisifications, or something else again. Given such a modelling, then subtraction seems altogether as appealing as intuitionistic implication. It is definable on such frames, and and has an obvious interpretation. ($A - B$ means "A came before B").

8.2 The Constructive Reasoner

One particular intepretation of incomplete interpretations is the notion of the constructive reasoner, which has been so important for the interpretation of intuitionistic logic. We think of points in models as stages of reasoning (or construction, or verification) of an ideal reasoner. In such an interpretation, subtraction may or may not be meaningful. If the model is a Kripke model, and if the reasoner is, in fact, an ideal reasoner, then it seems that subtraction is an acceptable connective. All that is necessary is that the reasoner can remember what she has reasoned. Not simply the facts that she has verified up until now, but also when one proposition came before another. If the reasoner is like this, then subtraction is an acceptable connective. A verification of $A - B$ is given by noting that in the past A came at some stage when B was not present. For constructivists, the relation of "A is verified at x" is a decidable one, so provided that at any stage it may be decided which stages came earlier, then the subtractions true at a stage can be decided too. So subtraction seems acceptable on this account.

However, the addition of subtraction brings with it the failure of an important intuitionistic meta-theorem — the primeness theorem. In the context of subtraction we no longer have $\vdash A \vee B$ only if $\vdash A$ or $\vdash B$. We can have a proof of $A \vee B$ without having a proof of A or a proof of B. This might be a serious problem, or it might not. On a Kripke frame, primeness is preserved in the sense that at any point in the model, a disjunction is true if and only if at least one disjunct is true. In this sense, a disjunction is verified (constructed, demonstrated, whatever) if and only if one disjunct is verified (or constructed or demonstrated). Primeness in this weak sense is preserved. Of course, primeness is preserved in completely classical models too — after all, $A \vee B$ is true in a possible world iff in that world either A is true or B is true. That isn't enough for the interpretation to be particularly *constructive*. What makes the Kripke models at least more of a model of construction is the fact that the models allow for increasing information. This is not possible in classical possible worlds models. The only prime theories (classically) are the complete theories. And these brook of no addition under pain of inconsistency. In intuitionistic logic with subtraction, prime theories can be increased as more information comes in. Intuitionistic logic with subtraction varies from its subtractionless parent because in our extended logic, the set of theorems is not a prime theory. You cannot construct a Kripke model in which the theorems of the logic alone are verified in a point in that model. If you want this to be the case, you must move to Beth models, where the propositions true at a point need no longer be a prime theory.

If, your favoured interpretation of the constructive reasoner is a Beth model, then it is not clear that subtraction is an acceptable connective. True, subtraction can be defined on Beth models, in the sense that for any two propositions on the frame, their subtraction is also a proposition on the frame. However, this does not mean in and of itself that subtraction is an acceptable connective on this interpretation. It depends on the way the model is read. Given a particular point x, and particular propositions A and B, it may not be decidable whether $A - B$ is true at x or not. This is not something which can be read off the modelling condition for subtraction — for the modelling condition for disjunction involves second-order quantification. There is no obvious way to determine whether or not $A \vee B$ is true at a point x. Similarly, there is no obvious way to determine whether $A - B$ is true at x or not. Subtraction may or may not be a meaningful connective in Beth models.

8.3 Propositions as Open Sets

Topology also provides a setting for the understanding of intuitionistic logic. Subtraction is definable on a topological model only if a dual topology can be defined on the intended topological space. While we have seen that there are many of examples where this can be done (the topology on any frame). However, these mightn't be the topologcal models you prefer. If the space is one of "possibilities" in some sense, and a topology is defined by some kind of metric of perceptual similiarity, then open sets on such a space are good models of "in principle discriminable" propositions. Every possible state of affairs included in a proposition is surrounded by a set of other possibilities which are like it. There is no case of a proposition which discriminates between a possibility and all others,*no matter how similar*. On this reading, the logic of open sets is natural. However, metric spaces (like, for example $\mathbb{R}^n$, but also less well known spaces) have no dual topologies whenever they are interesting topological spaces. Why is this? We know that bi-topologies are isomorphic to frames. In particular, $x \leq y$ can be defined as $y \in \mathrm{Cl}(\{x\})$. In a metric space, $y \in \mathrm{Cl}(\{x\})$ only when the disntace of y from x is zero, which happens only when $y = x$. So the ordering relation on frames is identity, and the logic is classical.

So, if the motivating idea for intuitionisim is that propositions are all open sets, because we do not have the powers to discriminate properly any set which is not open then only a more extensive understanding of the 'space' in which propositions reside will give us an answer on subtraction, one way or another. In particular, subtraction is definable only when there is a sense of "precisification" on the space, which will model the ordering $\leq$ on the frame on the topology. Otherwise, the open sets will be closed under the connectives of intuitionistic logic, but not subtraction.[1]

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