HISTORY OF LOGICAL CONSEQUENCE

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1. Introduction

Consequence is a, if not *the*, core subject matter of logic. Aristotle's study of the syllogism instigated the task of categorising arguments into the logically good and the logically bad; the task remains an essential element of the study of logic. In a logically good argument, the conclusion follows validly from the premises; thus, the study of consequence and the study of validity are the same.

In what follows, we will engage with a variety of approaches to consequence. The following neutral framework will enhance the discussion of this wide range of approaches. Consequences are conclusions of valid arguments. Arguments have two parts: a conclusion and a collection of premises. The conclusion and the premises are all entities of the same sort. We will call the conclusion and premises of an argument the *argument's components* and will refer to anything that can be an argument component as a *proposition*. The class of propositions is defined functionally (they are the entities which play the functional role of argument components); thus, the label should be interpreted as metaphysically neutral. Given the platonistic baggage often associated with the label "proposition", this may seem a strange choice but the label is already used for the argument components of many of the approaches below (discussions of Aristotlean and Medieval logic are two examples). A consequence relation is a relation between collections of premises and conclusions; a collection of premises is related to a conclusion if and only if the latter is a consequence of the former.

Aristotle's and the Stoics' classes of arguments were different, in part, because their classes of propositions differed. They thought that arguments were structures with a single conclusion and two or more premises;[1] conclusions and premises (that is, propositions) were the category of things that could be true or false. In Aristotlean propositions, a predicate is applied to a subject; the Stoics allowed for the *recombination* of propositions with connectives. Later on, some medieval logicians restricted propositions to particular concrete tokens (in the mind, or spoken, or written).

Changing the class of realisers of the propositional functional role affects the consequence relation. A relation involving only abstract propositions must different from a relation which involves some of concrete realisers. Not every change in the composition of propositions, however, is equal. If there is a mapping that connects the abstract propositions with the concrete sentences, and the consequence relation on these collections respects this mapping, then the differences are more metaphysical than they are logical. If there is no such mapping, then the choice between these implementations is of serious logical importance.

[1]This is until Antipater, head of the Stoic school around 159 – 130 BCE, who "recognized inference from one premise, his usage was regarded as an innovation" [76, p 163].

Aristotle and the Stoics dealt with arguments with two or more premises. Without further investigation of historical details, this can be interpreted in two ways: (1) any argument with fewer than two premises is invalid, or (2) arguments cannot have fewer than two premises. On the first interpretation, there is some necessary requirement for validity that zero and one premise arguments always fail to satisfy. According to some schools: for a conclusion to be a consequence of the premises, it must be genuinely *new*. This makes all single premise arguments invalid. Similarly, a zero premise argument is not one where the conclusion *results from* the premises. This is a choice about what the consequence relation is: whether a consequence has to be new, whether it must result from the premises, and so on. Different approaches to this issue have be taken through the history of logical consequence. Sometimes a rigid adherence to the motivations of a consequence being *new* and *resulting from* premises is maintained; at other times, this is sacrificed for the sake of simplicity and uniformity.

The second interpretation limits how a collection of premises can be structured in an argument. The combination of two propositions (one as a premise and the other as a conclusion) isn't a good argument because it isn't an argument. Premise combination has often been treated rather naively. Recently, careful discussions of premise combination have come out of Gentzen's proof systems and substructural logic. In substructural logics, premises are not combined as unordered sets. Different structural restrictions on the combination of premises, and the ways one is able to manipulate them (structural rules), result in different consequence relations. There has also been a loosening in the forms that conclusions take. Typical arguments *seem* to have exactly one conclusion (see [95] for an argument against this). This lead to focussing on single conclusions as consequences of premises. More generally, however, we can investigate whether a *collection* of conclusions is a consequence of a collection of premises.

Any theorist of consequence needs to answer the following questions:

(1) What sort of entity can play the role of a premise or of a conclusion? That is, what are propositions?
(2) In what ways can premises combine in an argument? In what ways can conclusions combine in an argument?
(3) What connection must hold between the premises and the conclusion(s) for the conclusion(s) be a consequence of the premises?

An answer to the first question has two main parts. There is the form of propositions (for example, on Aristotle's view propositions always predicate something of a subject) and the composition of propositions (for example, on a medieval nominalist's theory of propositions they are concrete singulars).

There are two broad approaches to the third question. Some theorists focus on a property of propositions; some theorists focus on connections between conclusions and premises. In both cases, consequence is explicated in terms of something else. In the first approach, the conclusion is a consequence of the premises if and only if, whenever the premises have some specified property, so does the conclusion. This approach focusses on whether the premises and conclusion have the designated property or not, it doesn't *rely* on a strong connection between premises and conclusion. In the paradigmatic example, this property is truth. The second approach is more concerned with the relation between the premises and conclusion. The consequence relation is build on top of another relation between premises and conclusions. If the premises and conclusion of an argument are connected by any number of steps by the basic relation, then the conclusion is a consequence of the premises. Paradigmatic examples are based on proof theories. We will refer to the first type of approaches as property based approaches, and the second as transference based approaches. There are many hybrids of the two approaches. A truth preservation approach sounds like a property based approach, but this depends on what we make of *preservation*. If it is important that the truth of the conclusion is connected via a processes of transference to the truth of the premises, then the approach has both property and transference features.

Different answers to these three questions originate from a variety of sources. Sometimes answers (especially to the first question) come from metaphysics; sometimes answers (especially to the third question) come from epistemology. Importantly, different answers are connected to different properties that consequence relations are expected to have. In the next three sections, we will look at some features that have struck theorists as important properties for consequence relations. Different answers to the three questions often correspond to different emphases on these properties.

Theorists, like Tarski in the quote below, have been aware that there are many tensions in developing an account of consequence. There is usually a trade off between precision, adherence to everyday usage of the concept, and with adherence to past accounts. Any precise account will be, to some extent, revisionary. In [120, p 409] Tarski says,

> The concept of *logical consequence* is one of those whose introduction into the field of strict formal investigation was not a matter of arbitrary decision on the part of this or that investigator; in defining this concept, efforts were made to adhere to the common usage of the language of everyday life. But these efforts have been confronted with the difficulties which usually present themselves in such cases. With respect to the clarity of its content the common concept of consequence is in no way superior to other concepts of everyday language. Its extension is not sharply bounded and its usage fluctuates. Any attempt to bring into harmony all possible vague, sometimes contradictory, tendencies which are connected with the use of this concept, is certainly doomed to failure. We must reconcile ourselves from the start to the fact that every precise definition of this concept will show arbitrary features to a greater or less degree.

This leaves the theorist with a final question to answer: What is the point of the theory? Precision, accord with everyday usage, accord with the normative constraints on reasoning, and many other answers have been forthcoming in the history of logical consequence.

1.1. **Necessity and Counterexamples.** Aristotle categorised syllogisms into those that are deductions and those that are not. The distinguishing feature of a deduction is that the conclusion *necessarily* results from the premises. That consequences follow of necessity from premises was one of the earliest characteristic features of consequence to be emphasised. It is not always easy to determine, however, what substance theorists impart into this necessity. The way in which theorists categorise arguments provides insight into how they understand necessity.

Aristotle, the Stoics, the medievals, Leibniz, Kant, and many more of the logicians and philosophers dealt with in this entry discuss necessity and modal logic. Of particular importance is Leibniz's account of necessity. A proposition is necessary if it is true in all possible worlds. There are two important parts of this move. Firstly, the notion of *possible world* is introduced. Possible worlds can serve as a type of *counterexample*. If it is possible for the premises of an argument to be true, and the conclusion false, then this is taken to demonstrate that the argument is invalid, and thus that the conclusion is not a consequence of the premises. Secondly, necessity is fixed as truth in *every* possible world. Universal quantification over possible worlds is a genuine advancement: for example, consider the equivalence of $\Box(A \wedge B)$ with $\Box A \wedge \Box B$.

A conclusion that is a consequence of a collection of premises should hold in *any* situation in which the premises do. Logical consequence can be used to reason about hypothetical cases as well as the actual case; the conclusion of a good argument doesn't merely follow *given the way things are* but will follow *no matter how things are.*

A characterisation of logical consequence in terms of necessity can lead away from the transference approach to consequence. A demonstration that there are no counterexamples to an argument needn't result in a recipe for the connecting the premises and conclusion in any robust sense. Necessity is not, however, an anathema to the transference approach. If the appropriate emphasis is placed in "necessarily *results from*" and "consequences *follow* of necessity", and this is appropriately implemented, then transference can still be respected.

1.2. **Formality and Structure.** Necessity is not sufficient for *logical* consequence. Consider the argument:

> All logicians are blue.
> Some blue objects are coloured.
> Therefore, all logicians are coloured.

It seems that, if the premises of the argument are true, the conclusion *must* also be; the conclusion seems to follow of necessity from the first premise. This is not a *formally* valid argument. That the conclusion is necessitated *relies* on all blue objects being coloured. This reliance disqualifies it as a logical consequence. A conclusion is a formal consequence of a collection of premise not when there is merely no possibility of the premises being true and conclusion false, but when it has an *argument form* where there is no possibility of *any* instance of the form having true premises and a false conclusion. Counterexamples are not only counterexamples to arguments but to argument forms.

In this example, there are counterexamples to the argument form:

> All αs are βs.
> Some βs are γs.
> Therefore, all αs are γs.

If the argument is not an instance of any other valid argument form, it is not valid and the conclusion is not a formal consequence of the premises. Argument forms and instances of argument forms play a crucial role in logical consequence; in some ways they are more central than arguments. Logical consequence is *formal* in at least this respect.

Formal consequence is not the only relation of consequence that logicians have studied. Some logicians have placed a high level of importance on *material* consequence. A conclusion is a material consequence of a collection of premises if it follows either given the way things are (so not of necessity) or follows of necessity but not simply because of the form of the argument. In order to properly distinguish material and formal consequence we require a better characterisation of the forms of propositions and of arguments.

That logical consequence is schematic, and in this sense formal, is a traditional tenet of logical theory. There is far more controversy over other ways in which consequence may be formal. The use of schemata is not sufficient for ruling out the sample argument about blue logicians. The argument appears to be of the following form:

$(\forall x)(Lx \to x \text{ is blue})$
$(\exists x)(x \text{ is blue} \wedge x \text{ is coloured})$
Therefore, $(\forall x)(Lx \to x \text{ is coloured})$,

where L is the only schematic letter. There are no instances of this schema where it is possible for the premises of the argument to be true and the conclusion false. Whether this counts as a legitimate argument form depends on what must be, and what may be, treated schematically. This choice, in turn, rests on the other ways in which consequence is formal.

Sometimes logic is taken to be "concerned merely with the form of thought" [71, p 2]. This can be understood in a number of ways. Importantly, it can be understood as focussing on the general structure of propositions. If propositions have some general form (a predicate applied to a subject, has some recursive propositional structure, and so on) then consequence is formal in that it results from the logical connections between these forms. In MacFarlane's discussion of the formality of logic, this is described as (1) "logic provides constitutive norms for thought as such" [82, p ii]. The other two ways in which logic can be formal what MacFarlane points out are:

(2) logic is "indifferent to the particular identities of objects."
(3) logic "abstracts entirely from the *semantic content* of thought."

He argues, convincingly, that Kant's logic was formal in all three senses, but that later theorists found themselves pressured into choosing between them.

1.3. ***A Priori* and Giving Reasons.** Logical consequence is often connected to the practice of reason giving. The premises of a valid argument are reasons for the conclusion. Some transference approaches take logical consequence to rest on the giving of reasons: C is a consequence of the premises Δ if and only if a justification for C can be constructed out of justifications for the premises in Δ. Logical consequence, on this view, is about the transformation of reasons for premises into reasons for conclusions.

Most reason giving doesn't rely entirely on logical consequence. Lots of reasoning is *ampliative*; the conclusion genuinely says more than the combination of the premises. The common example is that *there is smoke* is a reason for that *there is fire*. The argument:

There is smoke.

Therefore, there is fire.

is invalid — the conclusion is not a logical consequence of the premise. It is a *material consequence* of the premise. In this entry, we are concerned with *logical* consequence. In logical reason giving, the reasons are *a priori* reasons for the conclusion. That the premises of a valid argument are reasons for the conclusion does not rely on any further evidence (in this example, regarding the connections between smoke and fire).

Some rationalists, the rationalistic pragmatists, hold that material consequence is also, in some sense, *a priori* (e.g. Sellars [104, especially p 26]). Material consequences are, however, not *necessary* in the same way. A counterexample to a material consequence does not immediately force a revision of our conceptual scheme on us. This is not true with logical consequence: either the purported counterexample must be rejected, or the purported logical consequence must be. This necessity is closely connected to the normativity of logical and material consequence. I can believe that there is smoke and that there isn't fire, *so long as* I also believe that this is an exceptional situation. There is no similar exception clause when I accept the premises of an instance of *modus ponens* and reject its conclusion.

The connection between logical consequence and the giving of reasons highlights the normative nature of consequence. If an argument is valid and I am permitted to accept the premises, then I am permitted to accept the conclusion. Some theoriests make the stronger claim that if one accepts the premises of a valid argument, then one ought to accept the conclusion. One of the many positions between these positions is that if one accepts the premises of a valid argument, then one ought not reject the conclusion.

A focus on the giving of reasons and the normativity of logical consequence is often the result of an aim to connect logical consequence to human activity — to concrete cases of reasoning. Logical consequence, from this perspective, is the study of a particular way in which we are obligated and entitled to believe, accept, reject and deny.

2. Aristotle [384 BCE–322 BCE]

Aristotle's works on logic are the proper place to begin any history of consequence. They are the earliest formal logic that we have and have been immensely influential. Kant is merely one example of someone who thought that Aristotle's logic required no improvement.

> It is remarkable also that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine. If some of the moderns have thought to enlarge it . . . , this could only arise from their ignorance of the peculiar nature of logical science.
>
> [69, Bviii–ix]

Aristotle categorised syllogisms based on whether they were *deductions*, where the conclusion is a consequence of the premises.

According to Aristotle, propositions are either simple — predicating a property of a subject in some manner — or can be analysed into a collection of simple propositions. There are three parts to any simple proposition: subject, predicate and kind. In non-modal propositions predicates are either affirmed or denied of the subject, and are affirmed or denied either in part or universally (almost everything is controversial in the modal cases).

Subjects and predicates are *terms*. Terms come in two kinds: universal and individual. Universal terms can be predicates and subjects (for example: children, parent, cat, weekend). Individual terms can only be the subject of a proposition (for example: Plato, Socrates, Aristotle). A proposition which seems to have a individual term in the predicate position is, according to Aristotle, not a genuine proposition but merely an accidental predication that depends on a genuine predication for its truth (for example, "The cat on the mat is Tully" depends on "Tully is on the mat").

A proposition can be specified by nominating a subject, a predicate and a kind. Here are some examples with universal terms and the four non-modal kinds: universal affirmation, partial affirmation, universal denial and partial denial:

EXAMPLE	KIND	CODE
All *children* are *happy*.	Universal Affirmative	A
No *weekends* are *relaxing*.	Universal Negative	E
Some *parents* are *tired*.	Particular Affirmative	I
Some *cats* are not *friendly*.	Particular Negative	O

Any collection of propositions is a syllogism; one proposition is the conclusion and the rest are premises. Aristotle gives a well worked out categorisation of a subclass of syllogisms: the *categorical* syllogisms. A categorical syllogism has exactly two premises. The two premises share a term (the middle term); the conclusion contains the other two terms from the premises (the extremes). There are three resulting figures of syllogism, depending on where each term appears in each premise and conclusion. Each premise and conclusion (in the non-modal syllogisms) can be one of the four kinds in the table above.

The syllogisms are categorised by whether or not they are deductions.

> A deduction is a discourse in which, certain things having been supposed, something different from the things supposed results of necessity because these things are so. By 'because these things are so', I mean 'resulting through them,' and by 'resulting through them' I mean 'needing no further term from outside in order for the necessity to come about.' [121, Prior Analytics A1:24b]

The following example is a valid syllogism in the second figure with E and I premises and an O conclusion (it has come to be called "Fest$i$no").

No weekends are relaxing.
Some holidays are relaxing.
Therefore some holidays are not weekends.

Aristotle categorises syllogisms based on their form. He justifies this particular argument's form:

No Bs are As.
Some Cs are As.
Therefore some Cs are not Bs.

in a two step procedure. Aristotle transforms the argument form by converting the premise "No Bs are As" into the premise "No As are Bs". This transforms the second figure Festino into the first figure Ferio. The justification of Festino rests on the justification of the conversion and the justification of Ferio. Here is Aristotle's justification of the former:

> Now, if A belongs to none of the Bs, then neither will B belong to any of the As. For if it does belong to some (for instance to C), it will not be true that A belongs to none of the Bs, since C is one of the Bs. [121, Prior Analytics A2:25a]

There is no justification for the latter: merely an assertion that the conclusion follows of necessity.

Aristotle uses a counterexample to show that the syllogistic form:

All Bs are As
No Cs are Bs
Therefore, All Cs are As

is invalid. He reasons in the following way:

> However, if the first extreme [A] follows all the middle [B] and the middle [B] belongs to none of the last [C], there will not be a deduction of the extremes, for nothing necessary results in virtue of these things being so. For it is possible for [A] to belong to all as well as to none of the last [C]. Consequently, neither a particular nor a universal conclusion becomes necessary; and, since nothing is necessary because of these, there will not be a deduction. Terms for belonging to every are animal, man, horse; for belonging to none, animal, man, stone. [121, Prior Analytics A4:26a]

Aristotle concludes that the argument form is not a deduction as the syllogism:

All men are animals
No stones are men
Therefore, All stones are animals

is of the same form but one has true premises and a false conclusion, so the conclusion of the other syllogism cannot follow of necessity.

3. Stoics [300 BCE–200 CE]

The Stoic school of logicians provided an alternative to Aristotle's logic. The Stoic school grew out of the Megarian and Dialectical schools.[2] The Megarians and the members of the Dialectical school contributed to the development of logic by their attention to paradoxes, a careful examination of modal logic and by debating the nature of the conditional (notably by Philo of Megara). Eubulides was particularly noted among the Megarians for inventing paradoxes, including the liar paradox, the hooded man (or the Electra), the sorites paradox and the horned man. As we will return to the lair paradox when discussing the medieval logicians, we will formulate it here. "A man says that he is lying. Is what he says true or false?" [76, p 114]. If the man says something true, then it seems that he is indeed lying — but if he is lying he is not saying something true. Similarly, if what the man says is false, then what he says is not true and, thus, he must be lying — but he says that he is lying and we have determined that he is lying, so what he says is

[2] There is some controversy regarding who belongs to which school. The members of the Dialectical group were traditionally thought of as Megarians. For our discussion, the most noticeable of these are Philo of Megara and Diodorus Cronus. In our limited discussion it will not hurt to consider the groups as one.

true. Diodorus Cronus is well know for his *master argument*. Diodorus' argument is, plausibly, an attempt to establish his definition of modal notions.

According to Epictetus:

> The Master Argument seems to have been formulated with some such starting points as these. There is an incompatibility between the three following propositions, "Everything that is past and true is necessary", "The impossible docs not follow from the possible", and "What neither is nor will be is possible". Seeing this incompatibility, Diodorus used the convincingness of the first two propositions to establish the thesis that nothing is possible which neither is nor will be true. [76, p 119]

The reasoning involved in the argument is clearly non-syllogistic and the modal notions involved are complex.

The Stoic school was founded by Zeno of Citium, succeeded in turn by Cleanthes and Chrysippus. The third of these was particularly important for the development on Stoic logic. Chrysippus produced a great many works on logic; we encourage the reader to look at the list of works that Diogenes Laertius attributes to him [66, pp 299 – 319].

A crucial difference between the Stoic and Aristotelean schools is the sorts of propositional forms they allowed. In Aristotle's propositions, a predicate is affirmed or denied of a subject. The Stoics allowed for complex propositions with a recursive structure. A proposition could be basic or could contain other propositions put together with propositional connectives, like the familiar *negation*, *conditional*, *conjunction* and *disjunction*, but also the connectives *Not both ... and ...*; *... because ...*; *... rather than ...* and others. The Stoics had accounts of the meaning and truth conditions of complex propositions. This come close to modern truth table accounts of validity but, while meaning and truth were sometimes dealt with in a truth-table-like manner, validity was not.

Chrysippus recognised the following five indemonstrable moods of inference, [76, pp 163] [22, *Outlines of Pyrrhonism* II. 157f]:

(1) If the first, then the second; but the first; therefore the second.
(2) If the first, then the second; but not the second; therefore not the first.
(3) Not both the first and the second; but the first; therefore not the second.
(4) Either the first or the second; but the first; therefore not the second.
(5) Either the first or the second; but not the second; therefore the first.

These indemonstrable moods could be used to justify further arguments. The arguments, like Aristotle's categorical syllogisms, have two premises. The first premise is always complex. Notice that, even though the Stoics had a wide range of propositional connectives, only the conditional, disjunction and negation conjunction and (possibly) negation appear in these indemonstrables. This is an example of a transference style approach to logical consequence.

4. Medievals [476 CE – 1453 CE]

Logic was a foundational discipline during the medieval period. It was considered to have intrinsic value and was also regarded as an important groundwork for other academic study. Medieval logic is often divided into two parts: the old and the new logic. The demarcation is based on which Aristotelian texts were available. The old logic is primarily based on Aristotle's *Categories* and *De interpretatione* (this includes discussions on propositions and the square of opposition, but importantly lacks the *prior analytics*, which deal with the syllogism) while the new logic had the benefit of the rest of Aristotle's *Organon* (in the second half of the 12^{th} century). Many medieval logicians refined Aristotle's theory of the syllogism, with particular attention to his theory of modal logic. The medieval period, however, was not confined to reworking ancient theories. In particular, the *terminist* tradition produced novel and interesting directions of research. In the later medieval period, great logicians such as Abelard, Walter Burley, William of Ockham, the Pseudo-Scotus, John Buridan, John Bradwardine and Albert of Saxony made significant conceptual advances to a range of logical subjects.

It is not always clear *what* the medieval logicians were doing, nor *why* they were doing it [112]. Nevertheless, it is clear that consequence held an important place in the medieval view of logic, both as a topic of invesitagation and as a tool to use in other areas. Some current accounts of logical consequence have remarkable similarities to positions from the medieval era. It is particularly interesting that early versions of standard accounts of logical consequence were considered *and rejected* by thinkers of this period (in particular, see Pseudo-Scotus and Buridan below).

The medievals carried out extensive logical investigations in a broad range of areas (including: inference and consequence, grammar, semantics, and a number of disciples the purpose of which we are still unsure). This section will only touch on three of these topics. We will discuss theories of consequentiæ, the medieval theories of consequence. We will describe how some medievals made use of consequence in solutions to insolubilia. Lastly, we'll discuss the role of consequence in the medieval area of obligationes. This third topic is particularly obscure; it will serve as an example of where consequence plays an important role but is not the focus of attention.

4.1. **Consequentiæ.** The category of consequentiæ was of fluctuating type. It is clear that in Abelard's work a consequentiæ was a true conditional but that in later thinkers there was equivocation between true conditionals, valid one premise arguments, and valid multiple premise arguments. This caused difficulties at times but what is said about consequentiæ is clearly part of the history of logical consequence.

The mediaevals broadened the range of inferences dealt with by accounts of consequentiæ from the range of consequences that Aristotle and the Stoics considered. In the following list, from [76, pp 294 – 295], items (3), (4), (9) and (10) are particularly worth noting:

(1) From a conjunctive proposition to either of its parts.
(2) From either part of a disjunctive proposition to the whole of which it is a part.
(3) From the negation of a conjunctive proposition to the disjunction of the negations of its parts, and conversely.
(4) From the negation of a disjunctive proposition to the conjunction of the negations of its parts, and conversely.
(5) From a disjunctive proposition and the negation of one of its parts to the other part.
(6) From a conditional proposition and its antecedent to its consequent.

(7) From a conditional proposition and the negation of its consequent to the negation of its antecedent.
(8) From a conditional proposition to the conditional proposition which has for antecedent the negation of the original consequent and for consequent the negation of the original antecedent.
(9) From a singular proposition to the corresponding indefinite proposition.
(10) From any proposition with an added determinant to the same without the added determinant.

Like Aristotle and the Stoics, the medievals investigated the logic of modalities. The connections they drew between modalities, consequentiæ and the "follows from" relation are interesting. Ockham gives us the rules [76, p 291]:

(1) The false never follows from the true.
(2) The true may follow from the false.
(3) If a consequentia is valid, the negative of its antecedent follows from the negative of its consequent.
(4) Whatever follows from the consequent follows from the antecedent.
(5) If the antecedent follows from any proposition, the consequent follows from the same.
(6) Whatever is consistent with the antecedent is consistent with the consequent.
(7) Whatever is inconsistent with the consequent is inconsistent with the antecedent.
(8) The contingent does not follow from the necessary.
(9) The impossible does not follow from the possible.
(10) Anything whatsoever follows from the impossible.
(11) The necessary follows from anything whatsoever.

Unlike the Stoics approach to "indemonstrables", the medievals provided analyses of consequentiæ. According to Abelard, consequentiæ form a sub-species of *inferentia*. An inferentia holds when the premises (or, in Abelard's case the antecendent) necessitate the conclusion (consequence) in virtue of their meaning (in modern parlance, an inferentia is an entailment, and the "in virtue of" condition makes the relation *relevant*). The inferentia are divided into the perfect and the imperfect. In perfect inferentia, the necessity of the connection is based on the structure of the antecendent — "if the necessity of the consecution is based on the arrangement of terms regardless of their meaning" [15, pp 306]. The characteristic features of perfect inferentia are remarkably close to Balzano's analysis of logical consequence.

4.1.1. *Buridan and Pseudo-Scotus.* Buridan, Pseudo-Scotus and other medieval logicians argued against accounts of consequence that were based on necessary connections. Pseudo-Scotus and Buridan provide apparent counterexamples to a range of definitions of consequence. In this section, we look at three accounts of consequence and corresponding purported counterexamples. (We rely heavily on [15] and [75].)

The first analysis we consider is:

(A) A proposition is a consequence of another if it is impossible for the premise (antecedent) to be true and conclusion (consequent) not to be true.

Buridan offers a counterexample:

> the following is a valid consequence: every man is running; therefore, some man is running; still, it is possible for the first proposition to be true and for the second not to be true, indeed, for the second not to be.[3] [75, p 95 – 96]

Buridan's argument is a counterexample because his propositions are contingent objects; the proposition "Some man is running" could fail to exist. What doesn't exist, can't be true; so, it is possible for the premise to be true and not the conclusion. The ontological status of propositions can be as important to an account of consequence as the forms of propositions. Whereas our introduction to this entry argued that necessity is not a *sufficient* condition for consequence, Buridan's example is purported to show that it is not even a *necessary* condition.

Pseudo-Scotus provides a counterexample in a similar style: "Every proposition is affirmative, therefore no proposition is negative" [15, p 308]. In this case, the premise is true in situations where all negative propositions (including the conclusion) are destroyed.

In order to avoid these counterexamples, a definition of consequence has to take the contingency of propositions into account. Pseudo-Scotus responds to the definition

(B) For the validity of a consequence it is necessary and sufficient that it be impossible for things to be as signified by the antecedent without also being as signified by the consequent. [15, p 308]

with the argument *No chimaera is a goat-stag; therefore a man is a jack-ass.* The argument is thought to be invalid (having a true premise and false conclusion), but not ruled out by the proposed definition.

Both Pseudo-Scotus and Buradin consider the definition

(C) For a consequence to be valid it is necessary and sufficient that it be impossible that if the antecedent and the consequent are formed at the same time, the antecedent be true and the consequent false.[4] [15, p 308]

Pseudo-Scotus gives the example: *God exists; therefore this consequence is not valid* [15, p 308], which is meant to be invalid but satisfies the definition. The argument cannot be valid — assuming that the argument is valid is self defeating. The premise and the conclusion are both, apparently, necessary propositions and so it is impossible that they are formed at the same time when the former is true and the latter false. Pseudo-Scotus ultimately accepts this definition, but allows for exceptions, in light of this example, and calls for further investigation.

Buridan's counterexample is "No proposition is negative; therefore, no donkey is running" [75, p 96]. In this example, the premise cannot be formed without falsifying itself. It is impossible for the premise to be formed and true, so it is impossible for premise and conclusion to be formed with the premise true and the conclusion false. The argument meets the definition but isn't valid. The invalidity of the argument is justified on the assumption that logical consequence supports contraposition (If the argument A therefore B is valid, so is the argument Not A therefore Not B), and the contrapositive of this argument is clearly invalid.

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[3] The validity of the argument depends on existential import for universal quantifiers, a topic which we need not go into here.

[4] Alternatively: "that proposition is the antecedent with respect to another proposition which cannot be true while the other is not true, when they are formed together." [75, p 96]

Buridan favours a blending of (B) and (C):

> Therefore, others define [antecedent] differently, [by saying that] that a proposition is antecedent to another which is related to it in such a way that it is impossible for things to be in whatever way the first signifies them to be without their being in whatever way the other signifies them to be, when these propositions are formed at the same time. [37, p 103]

His approach requires using both the notions of what a proposition signifies and a proposition being formed; for more details see [37].

4.2. **Self-Reference and Insolubilia.** The medieval logicians devoted considerable effort to paradoxes and *insolubilia*. There was no sense of danger in discussions of insolubilia [113]. There was no fear that logic might be unable to solve insolubilia nor any fear that insolubilia were signs of incoherences in logic. Discussions of insolubilia were aimed at discovering the right way to deal with these examples, not at vindicating logic. The medieval's primary example was the lair paradox (introduced in the Stoics section above); we also will focus on this insolubilia.

There was a range of approaches to insolubilia in the medieval period.[5] There are many similarities between the types of medieval and modern day responses to the lair. For example, some theories dealt with the paradox by restricting self-reference (e.g. Burley and Ockham). The approaches of Roger Swyneshead, and William Heytesbury, Gregory of Rimini, John Buridan and Albert of Saxony are all worth further discussion, but we will restrict ourselves to Bradwardine and Buridan (with a brief mention of Swyneshead).

Bradwardine's solution relies on the connection between signification and truth. A proposition is true if it only signifies things which are the case, a proposition is false it signifies anything which is not the case. Bradwardine held that a proposition signifies everything which, in some sense, *follows* from it (the closure principle); the detail, however, of the closure principle are currently contested. What is crucial is that this is meant to justify the thesis: Any proposition which signifies that it is false, signifies that it is true and false. From this it follows that the liar is false, but not that it is true. Spade's [110, 115] interpretation of the closure principle is that any consequence of s is signified by s (where $s : P$ is that s signifies P and $\Rightarrow$ is "follows from", this is $(\forall s)(\forall P)((s \Rightarrow P) \rightarrow s : P)$). Read's version [90], based on making sure that Bradwardine's proof of the thesis works, is that s signifies anything which is a consequence of something it signifies $((\forall s)(\forall P)(\forall Q)((s : P \wedge (P \Rightarrow Q)) \rightarrow s : Q))$. In both versions, a type of consequence is an important element of signification.[6] In the previous section, we saw that Buridan's account of consequentiæ relied on signification; this section shows that some theories of signification involve some sort of consequence.

Buridan, like may other medieval logicians, blocked the liar paradox in a similar way to Bradwardine. In these approaches, the liar is false but it doesn't follow from this that it is true. If the liar is true, then everything that it signifies follows, including that it is false. If the liar is false, it *doesn't* follow that it is true. Both Bradwardine's and Buradin's solutions have additional requirements for truth, and thus the paradox is blocked. In Bradwardine's case, the liar signifies its truth and

[5] For a survey of these types, and their connections to modern approaches, see [38].

[6] See [40] for further discussion.

its falsity. To show that it is true, one has to demonstrate that it is true and that it is false. Buridan's early solution was similar; he thought that *every* proposition signified its own truth. He rejected this solution on metaphysical grounds; his nominalism and propositional signification were irreconcilable. He replaced this with the principle that every proposition entails another that claims that the original proposition is true. The liar is simply false because, while what it claims is the case (it claims that it is false, and is false), no entailed proposition which claims the lair's truth can be supplied. A form of consequence is again crucial to the solution. Buridan's account of the lair is connected to complex metaphysical and semantic theories, and has we have already seen, this drives him to define logical consequence in a very particular way.

Medieval logicans were aware that their solutions to the lair paradox via theories of truth were connected to theories of consequence. Klima [75, Section 4] argues that, as Buridan's theory of consequence doesn't require an account of truth, his solution to the liar is not bound by the same requirements as Bradwardine's. One of the consequences of Roger Swineshead's solution to the liar is that the argument:

> The consequent of this consequence is false; therefore, the consequent of this consequence is false.
>
> [77, p 251]

is a valid argument which doesn't preserve truth (the premise is true, and the conclusion is false)! Swineshead's position is that while not all consequences preserve truth, they all preserve correspondence to reality.

4.3. **Obligationes.** Obligationes are among the more obscure areas in which medieval logicians worked [112, 117]. An obligationes is a stylised dispute between two parties: the *opponent* and the *respondent*. The name 'obligationes' seems to be drawn from the manner in which the parties are obligated to respond within the dispute.

There are a variety of types of obligationes, the most common of which are called *positio*. In this form of obligatione, the opponent begins by making a posit — the *positum*. The *positum* is either admitted or denied by the respondent. If the respondent admits the proposition, then the opponent continues with further propositions. This time the respondent has three options: they can concede, deny or doubt. Their responses have to be in accord with their obligations. How this is dealt with varies between authors.

The rules of Walter Burley's were standard among earlier authors:

> For positio, Burley gives three fundamental rules of obligations:
> (1) Everything which follows from (a) the positum, with (b) a granted proposition or propositions, or with (c) the opposite(s) of a correctly denied proposition or propositions, known to be such, must be granted.
> (2) Everything which is incompatible with (a) the positum, with (b) a granted proposition or propositions, or with (c) the opposite(s) of a correctly denied proposition or propositions, known to be such, must be denied.
> (3) Everything which is irrelevant (impertinens) [that is, every proposition to which neither rule (1) nor rule (2) applies] must be granted or denied or doubted according to its own quality,

> that is, according to the quality it has in relation to us [i.e., if we know it to be true, we grant it; if we know it to be false, we deny it; if we do not know it to be true or do not know it to be false, we doubt it].
>
> [117, p 322]

The positio ends in one of two ways. If the respondent is forced to a contradictory position, they lose. If the respondent doesn't lose in a predetermined amount of time, the opponent loses.

There are numerous suggestions as to what purpose obligationes served. One suggestion is that they were logic exercises for students. This is often dismissed on the grounds that some of highly respected logicians (for example, Burley and Ockham) seem to have put more effort into them than mere exercises deserve [37, p 147]. Spade [111] suggested that obligationes provided a framework for exploring counterfactuals (but later withdrew this suggestion). Stump has [117, 116] suggested that this was a framework for dealing with *insolubilia*. Catarina Dutilh Novaes (formalizing, and two papers) has suggested that obligationes correspond to game theoretic consistency maintenance.

Consequence plays an important role within obligationes. The parties must proceed without violating the rules, and the rules often depend on consequence and the closely related notions of incompatibility and relevance. If Dutilh Novaes' suggestion is correct, then obligationes are a different mechanism for determining consequences of some type; the disputes tell us what we ought to and may accept or reject, based on our prior commitments.

5. Leibniz [1646–1716]

Gottfried Leibniz contributions to philosophy, mathematics, science and other areas of knowledge are astonishing. He was, quite simply, a genius. He is, perhaps, best known for discovering the calculus (at roughly the same time as Newton), but his contributions to philosophy (metaphysics, epistemology, philosophy of religion), to physics and other mathematical achievements should not be ignored. His work in logical theory foreshadowed many later advancements [80, 86]. Here, we note two directions that his work took. First, Leibniz pushed for a mathematization of logic.

> ... it seems clear that Leibniz had conceived the possibility of elaborating a basic science which would be like mathematics in some respects, but would include also traditional logic and some studies as yet undeveloped. Having noticed that logic, with its terms, propositions, and syllogisms, bore a certain formal resemblance to algebra, with its letters, equations and transformations, he tried to present logic as a calculus, and he sometimes called his new science universal mathematics ... There might be calculi concerned with abstract or formal relations of a non-quantitative kind, e.g. similarity and dissimilarity, congruence, inclusion ... It would cover the theory of series and tables and all forms of order, and be the foundation of other branches of mathematics such as geometry, algebra, and the calculus of chances. But most important of all it would be an instrument of discovery. For according to his own statement

> it was the *ars combinatoria* which made possible his own achievements in mathematics ... [76, pp 336–337]

Leibniz's approach to logic and philosophy required representing language as a calculus. Complex terms (subjects and predicates) were analysed into parts and given numerical representations. The numerical representations could be used to determine the truth of certain propositions. Surprisingly, Leibniz thought that the calculus, if developed correctly, would not only determine logical or analytic truths but all universal affirmative propositions. According to Leibniz, a universal affirmative proposition was true only if the representation of the subject *contained* the representation of the predicate.

> From this, therefore, we can know whether some universal affirmative proposition is true. For in this proposition the concept of the subject ... always contains the concept of the predicate. ... [I]f we want to know whether all gold is metal (for it can be doubted whether, for example fulminating gold is still a metal, since it is in the form of powder and explodes rather than liquefies when fire is applied to it in a certain degree) we shall only investigate whether the definition of metal is in it. That is, by a very simple procedure ... we shall investigate whether the symbolic number of gold can be divided by the symbolic number of metal. [80, pp 22]

Leibniz's very strong notion of truth is closely connected with Kant's later notion of analytical truth.

Secondly, Leibniz provided a detailed account of necessity based on possible worlds. He had a theory of possible worlds as collections of individuals (or, more precisely, of individual concepts), the actual world being the only world with all and only the actual individuals. This possible world based approach to necessity is highly influential in current approaches to necessity. As necessity is a core feature of consequence, this was a remarkable advancement in understanding logical consequence. Leibniz recognized that possible worlds could be used in understanding consequence, as well as connecting it to probability theory and other areas.

6. Kant [1724–1804]

Kant's characterisation of logic was immensely important for the later development of philosophy of logic. Kant seems to have taken Aristotle's work to provide a "completed body of doctrine" to which nothing sensible could be added.

> That logic has already, from the earliest times, proceeded upon this sure path is evidenced by the fact that since Aristotle it has not required to retrace a single step, unless, indeed, we care to count as improvements the removal of certain needless subtleties or the clearer exposition of its recognised teaching, features which concern the elegance rather than the certainty of the science. It is remarkable also that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine. If some of the moderns have thought to enlarge it by introducing *psychological* chapters on the different faculties of knowledge (imagination, wit, etc.), *metaphysical* chapters on the origin of knowledge or on the different kinds of certainty

> according to difference in the objects (idealism, scepticism, etc.), or *anthropological* chapters on prejudices, their causes and remedies, this could only arise from their ignorance of the peculiar nature of logical science. We do not enlarge but disfigure sciences, if we allow them to trespass upon one another's territory. The sphere of logic is quite precisely delimited; its sole concern is to give an exhaustive exposition and strict proof of the formal rules of all thought, whether it be *a priori* or empirical, whatever be its origin or its object, and whatever hindrances, accidental or natural, it may encounter in our minds.
>
> [69, Bviii–ix]

Kant's main importance is what he took logic to be, rather than making changes within it. Logic, according to Kant, is the study of the forms of judgements and, as such, the study of the formal rules of all thought. This extended the characterisation of *formal* consequence beyond the study of schemata. The kantian forms are characterised by the *table of judgements*. The table is similar to Aristotle's characterisation of propositions. Propositions and judgements vary in their quantity (whether they are universal, particular or singular judgements), in their quality (whether they are affirmative, negative or infinite), in their relation (whether they are categorical, hypothetical or disjunctive) and in their modality.

> The table of judgments, in turn, captures a fundamental part of the science of pure general logic: *pure*, because it is *a priori*, necessary, and without any associated sensory content; *general*, because it is both universal and essentially formal, and thereby abstracts away from all specific objective representational contents and from the differences between particular represented objects; and *logic* because, in addition to the table of judgments, it also systematically provides normative cognitive rules for the truth of judgments (i.e., the law of non-contradiction or logical consistency) and for valid inference (i.e., the law of logical consequence)
>
> [64, (A52-55/B76-79) (9: 11-16)]

Logical consequence, as the relations which arises from the forms of judgements, inherits these characteristic properties.

Kant had a clear idea of what logic is and what logic isn't. Logic is the study of the formal rules of thought and the form of judgements. Logic is general: it is not concerned with whether a thought is "*a priori* or empirical", nor with what its origins or objects are, nor how it is processed in my minds. Psychological, metaphysical and anthropological considerations are not part of a pure general logic.

Kant famously drew two dichotomies on judgements: the *a priori* and the *a posteriori*; and the analytic and the synthetic. Pure general logic, according to Kant is *a priori* analytic while arithmetic and geometry are *a priori* synthetic. In the critique of pure reason, Kant sets out to provide grounds for synthetic *a priori* judgements. Frege's later intent of showing that arithmetic is in fact analytic *a priori* would result in a revolution in logic and the study of consequence.

7. Bolzano [1781–1848]

In the 19^{th} Century, mathematicians carefully examined the reasoning involved in infinite and infinitesimal numbers. Bolzano played an important role in clarifying these mathematical concepts, which were fraught with paradox and confusion. The Kantian account of intuition as the grounds for *a priori* synthetic judgements was coming into question. Bolzano managed to give a definition of continuity for real-valued functions. In the course of this, he made important philosophical and logical contributions. Part of Bolzano's project for clarity included an analysis of propositions and of consequence.

> We often take certain representations in a given proposition to be variable and, without being clearly aware of it, replace these variable parts by certain other representations and observe the truth values which these propositions take on ... Given a proposition, we could merely inquire whether it is true or false. But some very remarkable properties of propositions can be discovered if, in addition, we consider the truth values of all those propositions which can be generated from it, if we take some of its constituent representations as variable and replace them with any other representations whatever. [16, p 194]

Bolzano uses this account of propositions, where variable components can be replaced by other representations, to give an analysis of logical consequence.

> The 'follows of necessity' can hardly be interpreted in any other way than this: that the conclusion becomes true *whenever* the premises are true. Now it is obvious that we cannot say of one and the same class of propositions that one of them becomes true *whenever* the others are true, unless we envisage some of their parts as variable ... The desired formulation was this: as soon as the exchange of certain representations makes the premises true, the conclusion must also become true. [16, p 220]

Consider the following three premise argument:

- If Fred lives in New Zealand, then he is further away from Sally than if he lived in Australia.
- If Fred is further away from Sally than if he lived in Australia, then Fred is very sad.
- Fred isn't very sad
- Therefore, Fred doesn't live in New Zealand.

The argument is valid; the conclusion follows necessarily from the premises. Suppose that "Fred lives in New Zealand", "Fred is further away from Sally than if he lived in Australia", and "Fred is very sad" are the *variable* parts of the premises and conclusion. In this case, the argument is valid according to Bolzano's analysis of consequence if and only if, for any variation of these variable parts, the conclusion is true whenever all the premises are. That is, whenever p, q and r are uniformly replaced by propositions in the following form: either a premise is false or the conclusion is true.

- If p, then q.
- If q, then r.
- Not r

- Therefore, not p.

This is remarkably similar to current definitions of logical consequence and to accounts rejected by medieval logicians. Bolzano's account relies on substitution of representations into the variable parts of propositions. This differs from the later Tarskian definition of consequence where open sentences are satisfied, or not, by objects. Bolzano's account works on a single level, where Tarski's involves two levels. A single level definition can either be given at the level of the language (e.g. truth preservation from *sentences* to *sentences* across substitution of *names* for *names* and *predicates* for *predicates*) or at the level of the semantic values of sentences (truth preservation from *semantic values of sentences* to *semantic values* across substitution of *objects* for *objects*, *properties* for *properties*). With Bolzano's talk of replacement of representations for representations, his account seems to be of the latter sort (*pace* Etchemendy [42]).

Bolzano's analysis of consequence depends on an account of the variable components of propositions. The logical components must remain fixed while the other components must be suitably varied. This requires that a line is drawn between the logical and non-logical vocabulary. It is still a difficult task (some argue that it is impossible, [42]) to provide such a line. Bolzano provided examples of logical components (including *. . . has . . .*, *non-. . .* and *something*) but confessed that he had no sharp distinction to provide. This is a recurring theme in the history of logical consequence. Aristotle, the Stoics, Kant and others had theories on what the forms of propositions were. Providing reasons for why these forms exhausts the logical forms is a difficult endeavour. Bolzano is interesting here, as he says that the matter is entirely conventional, this is a foreshadow of Carnap's philosophy of consequence.

8. Boole [1815–1864]

In *The Laws of Thought* [18], Boole advanced the mathematization of logic. Boole's work applied the current accounts of algebra to thought and logic, resulting in algebraic techniques for determining formal logical consequences. Boole thought that an argument held in logic if and only if, once translated in the appropriate way into equations, it was true in the common algebra of 0 and 1 [18, pp 37 – 38] [21, section 4].

Before explaining what this *appropriate translation* is and why Boole choose 0 and 1 we need his account of the forms of propositions. Every proposition has a negation or denial (the negation of p is $\neg p$). Every pair of propositions has a disjunction and a conjunction ($p \vee q$ and $p \wedge q$).

A simple proposition is translated as, what we now call, a variable. The negation of the proposition p is translated as $1 - p$ (that is, the translation of p subtracted from one). From this it follows that $\neg\neg p = p$, as $1 - (1 - p) = p$. The disjunction of two propositions corresponds to the multiplication of their translation; for this reason Boole notated $p \vee q$ as pq. The important features of 1 and 0 are that they are idempotent: their squares are equal to themselves. This is required if a disjunction of a proposition with itself is equivalent to itself. The conjunction of two propositions is similar to addition (and Boole represents it with an addition sign $+$) but with the difference that $1 + 1 = 1$.

Boole provides a structure that is meant to characterise all systems of propositions. He gives their form in terms of negation, conjunction and disjunction, and he gives algebraic laws which characterise how the collection of these propositions is structured. The result is that we have algebraic devices for determining consequences. The resulting structures are called *Boolean Algebras* in his honour. The study of Boolean Algebras, related structures and algebraic approaches to logic now is a core tradition in the study of logic.

Boole's advancement of the mathematization of logic coincides with a waning interest in logical consequence. Emphasis is placed on axioms, tautologies and logical truths. This continues through the sections on Frege and Russell below. The mathematical techniques that were developed, and the philosophical insights gained, in this period were important for later studies of consequence but the focus on consequence that runs from the Greeks and medievals to Bolzano is diffused until Tarski and Carnap.

9. Frege [1848–1925]

Gottlob Frege is one of the fathers of modern logic. He profoundly influenced the disciplines of logic, the philosophy of mathematics and the philosophy of language. Frege developed a logical notation which was meant to clarify and improve on natural languages. The *begrifftschrift*, or concept script, is a precise regimentation of Frege's own natural language, German. His intention was to remove the ambiguities, inconsistencies and misleading aspects of natural language. For the project to succeed, Frege's logic had to be much more than a *mere* calculating device; thus, he rejected the boolean algebraic tradition.

> Frege devoted considerable effort to separating his own conceptions of "logic" from that of the mere computational logicians such as Jevons, Boole and Schroeder. Whereas these people, he explained, were engaged in the Leibnizian project of developing a calculus ratiocinator, his own goal was the much more ambitious one of designing a *lingua characteristica*. Traditional logicians were concerned basically with the problem of identifying mathematical algorithms aimed at solving traditional logical problems—what follows from what, what is valid, and so on. Frege's goal went far beyond what we now call formal logic and into semantics, meanings, and contents, where he found the ultimate foundation of inference, validity, and much more.
>
> [31, p 65]

Frege's intention was to show, in opposition to Kant, that arithmetic is analytic. According to both Kant and Frege, geometry is *a priori* synthetic, but Kant and Frege differed on the status of arithmetic. Frege's logicism aimed at a reduction of arithemtic to logic; Kant thought that arithmetic was synthetic. There is no direct opposition between Frege and Kant here. Kant and Frege's categories of *analytic* are different because they are based on different accounts of the forms of propositions; it is here that they are in opposition. The purely logical forms of propositions, according to Kant, are quite limited. Frege abandoned the Aristotlean forms of propositions. In Aristotle's and Kant's categorisations, the following are all propositions where a term is predicated of a subject.

- Socrates is mortal.
- Every human is mortal.
- No-one is mortal.

The forms of these propositions still differ, according to Aristotle and Kant (for example, according to Kant's table of judgements: the first is singular, the second is universal; the first two are positive, the third is negative). Frege, however, is quite clear that "a distinction of *subject* and *predicate* finds *no place* in [his] way of representing a judgement"[55, p 2]. These three proposition have very different fregean structures. The first does predicate mortality of a person, Socrates, but neither of the other statements have a subject in the same way. The second statement is understood as the universal quantification of the incomplete statement *if x is human, x is mortal.* The statement is true if every object satisfies the incomplete statement. Frege changed what we mean by the word "predicate". Aristotlean predicates are terms, which are predicated of subjects in some manner, but can also be subjects. Fregean names and predicates are not of the same type. Names are complete expressions; fregean predicates are incomplete sentences.

Frege's *begrifftschrift* is far more powerful than Kant's logic. Indeed, at some points Frege's logic was inconsistent. Russell showed that Frege's infamous law five results in a contradiction. Inconsistency aside, Frege's *begrifftschrift* outstretches what is commonly used today. Frege makes use of second order quantification and no level of higher order quantification is ruled out. This was the result of his new approach to the pure logical forms of propositions.

Frege's *begrifftschrift* is the foundation on which modern logic is based. Its use of predicates, names, variables and quantifiers gives the structure that most logical systems use.

10. Russell [1872–1970]

At the beginning of the 20^{th} century, Russell was involved in a reductive project similar to Frege's. Russell employed the methods of Peano to give derivations of mathematical results based on logic alone. When he applied this approach to Cantor's proof that there is no greatest largest cardinal number he stumbled on paradox. The result appeared to conflict with Russell's assumption that there is a class which contains all other objects; this class would be the greatest cardinal. Running through Cantor's argument with this supposed universal class leads to Russell's class: the class of all classes which do not contain themselves. This class, were it to exist, would and would not contain itself. Russell showed that this conflicts with Frege's law five (roughly, that there is a class for any concept). Russell's paradox, along with others, became very important for modern logic. Unlike in the medieval era, paradoxes triggered serious doubts about core logical notions in many of the best logicians to follow Russell.

In *Principia Mathematica* [123], Russell and Alfred North Whitehead aimed to produce a paradox free reduction of mathematics to logic. In order to achieve this, they had to steer between the weak logic of Kant and Aristotle, and the inconsistent strength of Frege's logic. Russell and Whitehead certainly did not take a predefined notion of "logic" and reduce mathematics to it. In the preface to *Principia Mathematica*, Russell and Whitehead say:

> In constructing a deductive system such as that contained in the present work, there are two opposite tasks which have to be concurrently performed. On the one hand, we have to analyse existing mathematics, with a view to discovering what premisses are employed, whether these premisses are mutually consistent, and

> whether they are capable of reduction to more fundamental premisses. On the other hand, when we have decided upon our premisses, we have to build up again as much as may seem necessary of the data previously analysed, and as many other consequences of our premisses as are of sufficient general interest to deserve statement. The preliminary labour of analysis does not appear in the final presentation, which merely sets forth the outcome of the analysis in certain undefined ideas and undemonstrated propositions. It is not claimed that the analysis could not have been carried farther: we have no reason to suppose that it is impossible to find simpler ideas and axioms by means of which those with which we start could be defined and demonstrated. All that is affirmed is that the ideas and axioms with which we start are sufficient, not that they are necessary.

There is a sense in which the the reduction of mathematics is an uncovering of what logic must be. As a result, Russell's logic is much more intricate than Kant's. Nonetheless, russellian logic retained some of the *form of thought* characteristics it had for Kant and Frege. Russell says:

> The word symbolic designates the subject by an accidental characteristic, for the employment of mathematical symbols, here as elsewhere, is merely a theoretically irrelevant convenience. ...
>
> Symbolic logic is essentially concerned with inference in general, and is distinguished from various special branches of mathematics mainly by its generality. [99, section 11 and 12]

The (ramified) theory of types is central to Russell's approaches to logicism. We can think of the theories of types as theories of the forms propositions. The pure forms of propositions are more complex than in Aristotelian logic, but they are more restricted than in Frege's logic.

The ramified theory of types (we follow Church's [30] rather than [123] in our exposition) begins with *types*, or different domains of quantification. Domains of quantification are associated with variables which range over them; so, variables have specified types. Individual variables are of some basic type i. If $\beta_1, \ldots, \beta_m$ are types of variables, then there is a further type $(\beta_1, \ldots, \beta_m)/n$ of variables which contains m-place functional variables of level n. A type $(\alpha_1, \ldots, \alpha_m)/k$ is *directly lower* than the $(\beta_1, \ldots, \beta_m)/n$ if $\alpha_i = \beta_i$ and $k < n$.

0-place functional variables of level n are propositional variables of level n. Functional variables of 1 or more places are *propositional functions*. A 1-place functional variable of level n is a one place predicate of that level.

Formulas are restricted to the following forms. Propositional variables are well formed formulas. If f is a variable of a type $(\beta_1, \ldots, \beta_m)/n$ and x_i is a variable of, or directly lower than, type β_i, then $f(x_1, \ldots, x_m)$ is a well formed formula. On this base we add a recursive clause for negation, disjunction and universal quantification.

The forms of propositions are restricted by the requirement that variables are only applied to others of lower levels. The levels are cumulative in that the range of a variable of one type includes all the range of the variables of directly lower types.

Each variable type has an associated *order*. The order of a type is defined recursively. Individual variables (variables of type i) have order 0. A type $(\beta_1, \ldots, \beta_m)/n$

has order $N+n$ where N is the greatest order of the types $\beta_1, \ldots, \beta_m$. Orders of types are used to allow for controlled versions of Frege's law five.

- $(\exists p)(p \leftrightarrow P)$
- $(\exists f)(f(x_1, \ldots, x_m) \leftrightarrow P(x_1, \ldots, x_m))$

The crucial feature of these axioms is that the variables in the well formed formula P on the right hand side of the biconditionals are restricted by the order of the propositional variable or functional varible on the left hand side.

In order to achieve the results they wanted, Russell and Whitehead needed to introduce axioms of reducibility. The axioms ensure that every propositional function has a logically equivalent propositional function of level 1.

The logicist project of *Principia Mathematica* was intended reduce mathematics by providing an account of what is *said* by mathematical claims. The account had trouble because of what the "logical" base was committed to. The base required axioms like *the axiom of infinity* and *the axiom of reducibility* to be able to provide deductions of mathematics that Russell and Whitehead aimed at recapturing. Neither axiom seems *obviously* true. This leads to careful discussion of the epistemology of logic.

> In fact self-evidence is never more than a part of the reason for accepting an axiom, and is never indispensable. The reason for accepting an axiom, as for accepting any other proposition, is always largely inductive, namely that many propositions which are nearly indubitable can be produced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false, and nothing which is probably false can be deduced from it. [122, p 59]

Russell has some clear ideas about what logic is, but he is also clear that he has no adequate definition. In some places he rejects the axiom of infinity as logical, because it is not adequately *tautological*.

> It is clear that the definition of "logic" or "mathematics" must be sought by trying to give a new definition of the old notion of "analytic" propositions. ... They all have the characteristic which, a moment ago, we agreed to call "tautology". This, combined with the fact that they can be expressed wholly in terms of variables and logical constants (a logical constant being something which remains constant in a proposition even when all its constituents are changed) will give the definition of logic or pure mathematics. For the moment, I do not know how to define "tautology". It would be easy to offer a definition which might seem satisfactory for a while; but I know of none that I feel to be satisfactory, in spite of feeling thoroughly familiar with the characteristic of which a definition is wanted. At this point, therefore, for the moment, we reach the frontier of knowledge on our backward journey into the logical foundations of mathematics. [98, pp 204 – 205]

Later he says:

> It seems clear that there must be some way of defining logic otherwise than in relation to a particular logical language. The fundamental characteristic of logic, obviously, is that which is indicated

> when we say that logical propositions are true in virtue of their form. The question of demonstrability cannot enter in, since every proposition which, in one system, is deduced from the premises, might, in another system, be itself taken as a premise. If the proposition is complicated, this is inconvenient, but it cannot be impossible. All the propositions that are demonstrable in any admissible logical system must share with the premises the property of being true in virtue of their form; and all propositions which are true in virtue of their form ought to be included in any adequate logic.
>
> Some writers, for example Carnap in his "Logical Syntax of Language," treat the whole matter as being more a matter of linguistic choice than I can believe it to be. In the above mentioned work, Carnap has two logical languages, one of which admits the multiplicative axiom and the axiom of infinity, while the other does not. I cannot myself regard such a matter as one to be decided by our arbitrary choice. It seems to me that these axioms either do, or do not, have the characteristic of formal truth which characterises logic, and that in the former event every logic must include them, while in the latter every logic must exclude them. I confess, however, that I am unable to give any clear account of what is meant by saying that a proposition is "true in virtue of its form." But this phrase, inadequate as it is, points, I think, to the problem which must be solved if an adequate definition of logic is to be found.
>
> [99, Introduction]

11. Carnap [1891–1970]

Rudolf Carnap's intellectual development began within a dominantly Kantian tradition. He had the benefit of attending logic lectures by Frege in Jena (in the early 1910's), but this exposure to a father of modern logic only had significant philosophical impact on Carnap after a "conversion experience" through reading Bertrand Russell. Carnap was particularly struck by Russell's insistence that "[t]he study of logic becomes the central study in philosophy" [27, p 25]. Carnap was won over by the combination of rigour and philosophical applicability [31] of Russell's work.

Carnap began using logical methods in all of his work, following Russell but also heavily influenced by Frege's classes and also (unlike Russell and Frege) axiomatic theories in mathematics (especially Hilbert's program). Carnap was a great populariser of modern logic. He lead the way in producing textbooks and introductions to the area. He also did cutting edge work on the relations between completeness, categoricity and decidability. The core conjectures that he focussed on (formulated for the modern reader) are:

(1) An axiomatic system S is consistent (no contradiction is deducible from it) if and only if it is satisfiable, i.e., has a model.
(2) An axiomatic system S is semantically complete (non-forkable) if and only if it is categorical (monomorphic).
(3) An axiomatic system S is deductively complete if and only if it is semantically complete (non-forkable).

[92, p 187]

This work was closely connected to important results of Gödel and Tarski (see [92] for further details).

As we saw in an earlier quote from Russell, Carnap's philosophy and use of these tools were very different to Russell's. Carnap's works the *Der Logische Aufbau der Welt* [24, 26] and the *Logical Syntax of Language* [25] use techniques inspired by Russell and Frege, but the resulting philosophical picture is very different. Russell's theory of types provided one logical base for the reduction of mathematics but there were alternatives. In *Logical Syntax of Language*, Carnap aimed to show that there is no need to *justify* deviation from Russell's theory of types. His position was that no "new language-form must be proved to be 'correct' [nor] to constitute a faithful rendering of 'the true logic"' [25, p xiv]. Carnap's *Principle of Tolerence* gives us "complete liberty with regard to the forms of a language; that both the forms of construction for sentences and the rules of transformation ... may be chosen quite arbitrarily." [25, p xv] *Correctness* can only be determined within a system of rules; the adoption of a logical system (or language-form) is not done in this way. There may be reasons for or against adopting a particular system, but these are pragmatic choices — choices about which system will be more useful, rather than which system is correct.

Carnap argued for a viewpoint in which *philosophy was syntax*. The rules of formation and transformation of a language are conventional and we are at liberty to choose between systems. The rules of formation govern the forms of formulas (and propositions), and the transformation rules are the basis for logical consequence. The transformation rules determine a collection of legitimate transformations of propositions. If a proposition is the result of legitimate transformations of a collection of assumptions, then it is a consequence of these assumptions. This bases consequence on "syntactic" manipulation rather than semantic notions like truth or meaning. This was important for earlier followers of Carnap, like Quine [23], but later (inspired by the work of Tarski) Carnap came to change his approach. In the end, Carnap offered both transference and property style accounts of logical consequence.

Carnap was a logical empiricist (logical positivist). One facet of logical empiricism was the result of combining Russell's logicism with Wittgenstein's account of tautologies. The logical empiricists wanted to account for the necessity of mathematics without granting it any empirical substance. If *Principia Mathematica* [123] was successful in reducing mathematics to logic (and the logical empiricists thought this plausible) then it was only logic that needed accounting for. For this, they turned to Wittgenstein's *Tractatus* [124]. In the *Tractatus*, logical truths were true in virtue of language, not in virtue of how they represent the world; they were *empty* tautologies. The necessities of mathematics (via their reduction to logic) were necessary but this necessity only stems from the logical system one has adopted. How do we determine what logical system should be adopted? This is the point at which the Principle of Tolerance came in.

> In the foregoing we have discussed several examples of negative requirements (especially those of Brouwer, Kaufmann, and Wittgenstein) by which certain common forms of language — methods of expression and of inference — would be excluded. Our attitude to requirements of this kind is given a general formulation in the *Principle of Tolerance: It is not our business to set up prohibitions,*

> *but to arrive at conventions. . . . In logic, there are no morals.* Everyone is at liberty to build up his own logic, i.e. his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments.
>
> [25, pp 51 – 52, emphasis in original]

Carnap made suggestions regarding what choices may be better for certain purposes (in particular for unified empirical science) but this doesn't make these choices *correct*. Carnap developed two different systems in *Logical Syntax*. Logical consequence is an important feature of both. *Language I* is a weaker system which implements some constructivist constraints on consequence. This system is safer; it is less likely to produce inconsistency. *Language II* is a richer system based on the theory of types in Russell's work. If we prefer the safety of Brouwer's intuitionstism, we can adopt the former system. If we prefer the strength of the classical theory of types, we can adopt the latter. Neither give the *correct* consequence relation. Rightness or wrongness of arguments can be determined with respect to the logical system they are framed in, but a consequence relation is something we adopt for a purpose. The choice of logical system and consequence relation is not arbitrary; pragmatic considerations (like simplicity, fruitfullness, safety from inconsistency and the like) determine that some choices may be *better* than others.

12. Gentzen [1909–1945]

Gerhard Gentzen's work in logic, particularly his presentation of proof systems, has been highly influential. His natural deduction systems and sequent calculi were tremendous advances in proof theory. The ideas underlying these systems, and their connections to logical consequence, are important for understanding different ways of reasoning.

Gentzen's first publicly submitted work, in which he demonstrated significant technical ability, was a paper *On the Existence of Independent Axiom Systems for Infinite Sentence Systems* [56]. Importantly for us, he gave clear description of the connection between proof (and inference) in a logical system and an intuitive notion of logical consequence. With respect to the notion of prove used in the paper, he says:

> Our formal definition of provability and, more generally, our choice of the forms of inference will seem appropriate only if it is certain that a sentence $\mathfrak{q}$ is 'provable' from the sentences $\mathfrak{p}_1 \ldots \mathfrak{p}_v$ if and only if it represents informally a consequence of the $\mathfrak{p}$'s. We shall be able to show that this is indeed so as soon as we have fixed the meaning of the still somewhat vague notion of 'consequence'.
>
> [59, p 33]

Gentzen gave a regimented version of the "somewhat vague" notion and then proved the equivalence (in separate soundness and completeness stages) between proof and consequence for his system. A complication worth mentioning is that a "sentence" in this system is more like a sequent (or argument) than a formula.

Gentzen's natural deduction and sequent calculi, and the associated notions of normal proof and cut elimination, have become central to proof theory. In his *Investigations into Logical Deduction* [57, 58] Gentzen presents systems of proof which are intended to "[come] as close as possible to actual reasoning" [59, p 68].

In calculi of *natural deduction* one demonstrates that a formula follows from others by means of inference figures which license derivations. Each connective has both an introduction and an elimination rule.

The following example is a natural deduction style derivation of $(A \to C) \wedge (B \to C)$ from the premise $(A \vee B) \to C$. The derivation introduces disjunctions and conjunctions, and introduces and eliminates conditionals. Notice that the temporary premise A is discharged when the conditional $A \to C$ is introduced. (Some medieval discussions of inference and consequence come very close to natural deduction; one point of difference is that, even though the medievals performed hypothetical reasoning, it seems that they had no way to understand how one might discharge assumptions in order to introduce conditionals [67]. Consequence is always a local matter, and rules which allow 'action at a distance' are absent.)

$$\dfrac{\dfrac{\dfrac{(A \vee B) \to C \qquad \dfrac{[A]_a}{A \vee B}\vee I}{C}\to E}{(A \to C)}\to I, a \qquad \dfrac{\dfrac{(A \vee B) \to C \qquad \dfrac{[B]_b}{A \vee B}\vee I}{C}\to E}{(B \to C)}\to I, b}{(A \to C) \wedge (B \to C)}\wedge I$$

A natural deduction derivation proves that a conclusion *follows from* a collection of assumptions. Derivations show that formulas are consequences of premises. Inference rules show that consequences hold, if other consequences hold. This became clearer when Gentzen introduced his *sequent calculi.*

Sequent calculi make explicit the reasoning involved in inference steps. The inferences of sequent calculi operate explicitly on sequents (or argument forms), which have a sequence of antecedent formulas (or premises) and either a single succedent formula (or conclusion) in the case of intuitionistic logic or a sequence of succedent formulas in the case of classical logic. The inferences of the system can be divided into two kinds. The structural inferences *thinning*, *contraction*, *interchange* and *cut* are concerned with the structure of the antecedent and succedent formulas. The operational inferences introduce logical connectives into either the antecedent or the succeedent. A sequent calculus style proof of the argument $(A \vee B) \to C \mathrel{\therefore} (A \to C) \wedge (B \to C)$ is in the substructural logic section below. Note the correspondence between natural deduction introduction rules and sequent right hand side rules, elimination rules and sequent left rules.

A derivation of a sequent in in a sequent calculi proves that it corresponds to a valid argument. As classical sequents have sequences of conclusions, this broadens the category of arguments to include structures with multiple conclusions.

13. Tarski [1902–1983]

Alfred Tarski is a foundational figure for modern logic. Tarski's contributions to logic and mathematics guided much of the development of logic in the 20^{th} and early 21^{st} centuries. Tarski is particularly well known for the new degree of clarity that he brought to the study of logic. He advocated a *metamathematical* approach to logic. This has been very important for clarifying the central notions of logic. Tarski's metamathematical approach allowed him to study many semantic notions, like consequence and truth, at a time when they were thought troublesome at best and incoherent at worst.

Metamathematics is the branch of mathematics dedicated to the study of formalised deductive disciples: that is, "formalized deductive disciples form the field of research of metamathematics roughly in the same sense in which spactial entities form the field of research in geometry" [119, p 30]. A deductive discipline is composed of a collection of *meaningful sentences*. From a collection of sentences one can make various inferences; the resulting sentences are *consequences*. Every deductive science has associated rules of inference; "To establish these rules of inference, and with their help to define exactly the concept of consequence, is again a, task of special metadisciplines" [119, p 63]. Any sentence A which is derived from the collection of sentences Γ by means of these rules is a consequence of Γ. The consequences of Γ can be defined as the intersection of all the sets which contain the set A and are closed under the given rules of inference. This is a transference style approach to consequence; Tarski later gave a property based conception of consequence.

In *Fundamental Concepts of the Methodology of the Deductive Sciences* [119, chapter 5], Tarski studies deductive disciplines and their consequence relations at a high level of abstraction. On the basis of a small number of axioms regarding sentences and consequence, Tarski defines a number of important concepts for deductive disciplines. In the following axioms, S is a collection of sentences, $\subseteq$ is the standard sub-set or equals relation between sets, and Cn is an operation on sets of sentences. The set $Cn(X)$ is the set of consequences of the sentences in the set X. Sentences and consequence in a deductive discipline is subject to the following axioms:

Axiom 1: The collection of sentences, S, is denumerable.
Axiom 2: If $X \subseteq S$ (that is, X is a set of sentences), then $X \subseteq Cn(X) \subseteq S$.
Axiom 3: If $X \subseteq S$, then $Cn(Cn(X)) = Cn(X)$
Axiom 4: If $X \subseteq S$, then $Cn(X) = \bigcup\{Cn(A) : A \text{ is a finite subset of } X\}$
[119, pp 63 – 64]

On the basis of these axioms, Tarski proves a number of general theorems about sentences and consequence. He then focusses on *closed deductive systems* which contain all the consequences of their subsets. In *Fundamental Concepts of the Metamathematics* [119, chapter 3], Tarski uses a similar approach to investigate a narrowed field of deductive systems. In this approach, additional axioms are imposed on the sentences in the collection S. The first restriction is that S has at least one member which has every element of S as a consequence — an absurd sentence. The collection of sentences is required to contain the conjunction of any two of its members, and the negation of any of its members. The logical properties of these sentences are characterised in terms of the consequence operator. For example,

Axiom 9: If $x \in S$, then $Cn(\{x, n(x)\}) = S$ [119, p 32]

where $n(x)$ is the negation of x, ensures that every sentence follows from a set which contains a sentence and its negation.

It is now common to refer to relations, R, between sets of sentences (X and Y in the properties below) and sentences (a and c) with the properties:

Reflexivity: If $a \in X$ then RaX

Transitivity: If RcY and $Ra(X \cup \{c\})$, then $Ra(X \cup Y)$
Monotonicity: If RaX and $X \subseteq Y$ then RaY

as a *Tarski consequence relation.* If the relation is between sets of sentences, that is a set of sentences can have a set of sentences as a consequence, and if the properties are appropriately altered, it is a *Tarski-Scott consequence relation.*

Tarski is famous for his definition of truth in formal languages, but his analysis of logical consequence for predicate logic has become a central part of orthodox logic. In *On the concept of logical consequence* (other references???), Tarski used *models* and *satisfaction* to give a theory of logical consequence for predicate logic. Sentences of predicate logic have a recursive structure. They are either atomic or are formed using clauses for conjunction, disjunction, negation, universal quantification, existential quantification, etc. Tarski shows how the truth of each type of sentence is to be determined relative to a model. The propositional part of this is relatively straight forward. The quantifiers, however, required particular care. The truth of the formula $(\forall x)(Fx \supset Gx)$ depends on the properties of $Fx \supset Gx$. It cannot depend on the *truth* of $Fx \supset Gx$ as it contains the unbound variable x. Tarski used *satisfaction in a model relative to an assignment of values to variables* to define truth in a model. The formula $Fx \supset Gx$ may be satisfied in a model relative to some assignments but not others. For $(\forall x)(Fx \supset Gx)$ to be satisfied relative to an assignment requires that $Fx \supset Gx$ is satisfied in the model relative to *all* assignments of values to variables. A model is a *model of a sentence* if the sentence is satisfied by an assignment of variables in the model. This leads to the definition of consequence:

> The sentence X *follows logically* from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.
>
> [120, p 417, emphasis in original]

This gives a very restricted version of the intuitive notion of logical consequence. Tarski was well aware that some semantic notions cannot be defined in full generality. As another example, his definition of truth is for a carefully regimented language. He says that the intuitive concept as found in natural languages is inconsistent, and he demonstrates that there are a number of contexts in which truth cannot be defined. He is similarly aware that there are restrictions on definitions of logical consequence.

> Any attempt to bring into harmony all possible vague, sometimes contradictory, tendencies which connected with the use of this concept, is certainly doomed to failure. We must reconcile ourselves from the start to the fact that every precise definition of this concept will show arbitrary features to a greater or less degree
>
> [119, p 411]

He steered the investigation in a meta-logical or metamathematical direction in recognition that no attempt to supplement rules of inference in deductive systems can capture the intuitive notion of logical consequence.

> By making use of the results of K. Gödel we can show that this conjecture is untenable. In every deductive theory (apart from certain theories of a particularly elementary nature), however much

> we supplement the ordinary rules of inference by new purely structural rules, it is possible to construct sentences which follow, in the usual sense, from the theorems of this theory, but which nevertheless cannot be proved in this theory on the basis of the accepted rules of inference.
>
> [119, p 413]

14. Gödel [1906–1978]

Kurt Gödel and Alfred Tarski were, arguably, the greatest logicians of the 20^{th} Century. Gödel's results that are most relevant to our discussion are the completeness of predicate logic and the incompleteness of arithmetic.

In his thesis [61, see 1929], Gödel proved the *completeness* of predicate logic. He proved that a proof calculus of Hilbert and Ackerman proves all the formulas which are correct for any domain of individuals. Gödel proved that any formula is either refutable in the proof system (meaning that its negation can be proved from the axioms and proof rules) or is satisfiable (meaning that it is true in some domain of individuals). This showed that the proof system is complete: any unsatisfiable formula can be refuted by the proof system. In conjunction with Tarski's model theoretic analysis of consequence, this gives the result that logical consequence for predicate (classical) logic is captured by Hilbert and Ackerman's proof system. This shows that, in at least this case, there are transference and property style approaches to consequence that agree.

Gödel's completeness results were quickly followed by his *incompleteness* results. These were the results that Tarski referred to (in the above quote) in dismissing a derivation based approach to defining logical consequence. Gödel's incompleteness theorems were two of the most unexpected and outstanding results of the last century. The first theorem is given by the following quote:

> In the proof of Theorem VI no properties of the system P were used besides the following:
>
> (1) The class of axioms and the rules of inference (that is, the relation "immediate consequence") are recursively definable (as soon as we replace the signs in some way by natural numbers).
> (2) Every recursive relation is definable . . . in the system P.
>
> Therefore, in every formal system that satisfies the assumptions 1 and 2 and is ω-consistent, there are undecidable propositions of the form $(x)Fx$, where F is a recursively defined property of natural numbers, and likewise in every extension of such a system by a recursively definable ω-consistent class of axioms. As can easily be verified, included among the systems satisfying the assumptions 1 and 2 are the Zermelo-Fraenkel and the von Neumann axiom systems of set theory, as well as the axiom systems of number theory consisting of the Peano axioms, recursive definition, and the rules of logic. [61, p 181]

The second incompleteness theorem is that in every *consistent* (rather than the stronger ω-consistent) formal system that satisfies assumptions 1 and 2 there are undecidable propositions. In particular, a system of this type cannot prove the coded sentence which expresses the systems consistency.

> [Gödel's theorems] sabotage, once and for all, any project of trying to show that all basic arithmetical truths can be thought of as deducible, in some standard deductive system from one unified set of fundamental axioms which can be specified in a tidy, [primitively recursive], way. In short, *arithmetical truth isn't provability in some single axiomatizable system.* [108, p 161]

Gödel's incompleteness theorems show that certain types of transference/deductive accounts of consequence and property/truth accounts come apart for languages with strong enough expressive resources. Any formal system that is as strong as Peano arithmetic (for example, second order logic) cannot have a complete primitive recursive proof theory.

It doesn't follow that transference style approaches to consequence must fail. Carnap provided transference style approaches for arithmetic consequence while fully aware of Gödel's results (see [97, section 3] for a good discussion). Gentzen gave proof theoretic proofs of the consistency of arithmetic [59, # 4]. In both cases, infinitary (in some sense) techniques must be admitted. Nonetheless, Gödel's results show that the class of primitive recursive transference approaches to consequence cannot capture all of classical arithmetic (while remaining consistent). This leaves the consequence theorist with a choice: abandon primitive recursive transformations (whether by adopting an appropriate property based account or by incorperating some infinitary transformations) or abandon the consequence relation of classical arithmetic.

15. Modal Logics

The history of modal logic is woven throughout the history of logic and logical consequence. Modal logic began with Aristotle's syllogisms and was further developed by the Stoics and medievals. In the medieval period, necessity was connected via modal logic to logical consequence. MacColl investigated modal logic in the algebraic tradition of Boole. Many of the logicians focussed on by this entry worked on modal logic (for example: Carnap, Gödel, and Tarski). Modern modal logic began with C. I. Lewis' response to Whitehead and Russell's *Principia Mathematica*. The controversy was over the material conditional of the *Principia*. Lewis argued that Whitehead and Russell's conditional doesn't adequately capture implication. There was still confusion over to the connections between conditionals, implication, entailment and logical consequence (Peano used "C" for consequence and a backwards "C" for . . . *is deducible from* . . . , the backwards 'C' became our hook symbol $\subset$ for the material conditional [65, p 84]). Lewis drew attention to the *paradoxes of material implication*. His various attempted improvements make use of modal notions, for example: necessity, possibility and compatibility.

The modern semantic period of modal logic started with the work of Krikpe, Prior, Hintikka, Montague and others (see [62, section 4] for some discussion of the controversy involved). The model theoretic approach to modal logic draws on the ideas of Leibniz: truth is relativised to possible worlds. In a modal propositional language with the operators $\Diamond$ (for possibility) and $\Box$ (for necessity) the formulas $\Diamond A$ and $\Box A$ are assigned truth values relative to each world in the model (which is composed of a collection of worlds, a binary *accessibility* relation, and a valuation):

Diamond: $\Diamond A$ is true at world w in the model $\mathfrak{M} = \langle W, R, v\rangle$ (in symbols: $\mathfrak{M}, w \Vdash \Diamond A$) if and only if there is some world u related to w by R (that is, Rwu) and A is true at u in $\mathfrak{M}$.

Box: $\mathfrak{M}, w \Vdash \Box A$ if and only if, for all u such that Rwu, $\mathfrak{M}, u \Vdash A$

There are two notions of logical consequence which can be defined in terms of truth in modal, or Krikpe, models. *Local* consequence is based on preservation of truth at each individual world in a model, while *global* consequence is based on preservation of truth-at-all-worlds in each model. Here is the formal characterisation of these distinct notions

- The formula C is a *local consequence* of the formulas Γ relative to the models $\mathbf{M}$ if and only if, for any model $\mathfrak{M}$ in $\mathbf{M}$, and any world w in $\mathfrak{M}$ if, for each $A \in \Gamma$, $\mathfrak{M}, w \Vdash A$, then $\mathfrak{M}, w \Vdash C$.
- The formula C is a *global consequence* of the formulas Γ relative to the models $\mathbf{M}$ if and only if, for any model $\mathfrak{M}$ in $\mathbf{M}$, if for any $A \in \Gamma$ and any world w in $\mathfrak{M}$, $\mathfrak{M}, w \Vdash A$, then for any world w in $\mathfrak{M}$, we have $\mathfrak{M}, w \Vdash C$.

The two notions are both consequence relations in at least the sense that they are Tarski consequence relations (both relations are reflexive, transitive and monotonic). They also differ in important respects. The $\Box p$ is a global consequence of p in all classes of Kripke models (if p is true at every world in a model, then so is $\Box p$) but $\Box p$ is rarely a local consequence of p. There are many models in which we have worlds w at which p holds but $\Box p$ does not.

Which of these two notions of consequence is the *correct* notion if we are dealing with modal notions?

Various logicians and philosophers of logic have thought that consequence should be a necessary, *a priori* relation based on the meaning of logical vocabulary. The categories of the necessary, the *a priori* and the analytic have all received criticism during the history we have discussed. The *a priori* and necessity were attacked by empiricists, including Carnap and the logical empiricists. The logical empiricists allowed a remnant of necessity to remain under the guise of the analytic but insisted, inspired by Wittgenstein, that analytical truths said nothing. This was not far enough for Quine, who continued the attack on necessity and analyticisty. The recent approach of two dimensional semantics attempts to capture all three notions in a single framework.

> First, Kant linked reason and modality, by suggesting that what is necessary is knowable a priori, and vice versa. Second, Frege linked reason and meaning, by proposing an aspect of meaning (sense) that is constitutively tied to cognitive significance. Third, Carnap linked meaning and modality, by proposing an aspect of meaning (intension) that is constitutively tied to possibility and necessity. ... The result was a golden triangle of constitutive connections between meaning, reason, and modality.
>
> Some years later, Kripke severed the Kantian link between apriority and necessity, thus severing the link between reason and modality. Carnaps link between meaning and modality was left intact, but it no longer grounded a Fregean link between meaning and reason. In this way the golden triangle was broken: meaning and

> modality were dissociated from reason. Two-dimensional semantics promises to restore the golden triangle. [28, p 55]

We will focus on necessity and *a priori* knowable and a two dimensional system based on Davies and Humberstone's [33].

In a two dimensional model, truth is relativised to pairs of worlds, rather than single worlds. A proposition A is true at the pair $\langle u, v\rangle$ if and only if, were u the actual world, then A would have been true in the possible world v. There are typically three important modal operator:

Box: $\mathfrak{M}, \langle u, v\rangle \Vdash \Box A$ if and only if, for all w, $\mathfrak{M}, \langle u, w\rangle \Vdash A$
Fixedly: $\mathfrak{M}, \langle u, v\rangle \Vdash FA$ if and only if, for all w, $\mathfrak{M}, \langle w, v\rangle \Vdash A$
At: $\mathfrak{M}, \langle u, v\rangle \Vdash @A$ if and only if $\mathfrak{M}, \langle u, u\rangle \Vdash A$

Intuitively: $\Box A$ is true if A is true in all the ways the world could be; FA is true if A is true in all the epistemic alternatives (the ways the world could, for all we know, turn out to be); and $@A$ is true if A is true the actual world as it actually will turn out to be. In this two dimensional analysis $\Box$ plays the role of necessity and $F@$ plays the role of knowable *a priori*.

The logical and philosophical details of two dimensional approaches are varied and often far more complex than this short exposition. There are two points which we want to draw out of this short description. First, the different properties that consequence is sometimes thought to have are brought together in this single framework. Secondly, as with other modal logics, there are a number of different consequence relations for two dimensional modal logic, and they connect to necessity and *a priori* knowledge in different ways. Are the arguments "A therefore $@A$" and "$@A$ therefore A" logically good ones? Are the conclusions consequences of the premises? If consequence is defined as truth preservation at *diagonal* pairs (where the first and second element are the same world), these are valid arguments. This notion of consequence (according to this two dimensional analysis), is one where the premises are *a priori* reasons for the conclusion. The premises, however, do not *necessitate* (according to this two dimensional analysis) the conclusion. If the premises necessitate the conclusion, then the truth of the premise is sufficent for the truth of the conclusion in *any* hypothetical situation. Neither the truth of $@A$ nor of A is sufficient for the other in arbitrary hypotheticals. For this consequence relation we require truth preservation over all the pairs in the model.

16. Nonmonotonic Options

Tarski's conditions of reflexivity, transitivity and monotonicity have been central in the study of all sorts of consequence relations. However, they are not necessarily all that we want in any notion of consequence, broadly understood. Let's remind ourselves of Tarski's three conditions, for a consequence relation $\vdash$.

Reflexivity: If $A \in X$ then $X \vdash A$
Transitivity: If $Y \vdash C$ and $X \cup \{C\} \vdash A$, then $X \cup Y \vdash C$
Monotonicity: If $X \vdash A$ and $X \subseteq Y$ then $Y \vdash A$.

Not everything that we might want to call a consequence relation satisfies these three conditions. In particular, the monotonicity condition rules out a lot of what we might broadly consider 'consequence.' Sometimes we wish to conclude something beyond what is deductively entailed. If I learn that someone is a Quaker, I might quite reasonably conclude that they are a pacifist. To be sure, this conclusion may be tentative, and it may be be undercut by later evidence, but it may still be a genuine *conclusion* rather than an hypothesis or a conjecture. Work in the second half of the 20th Century and beyond has drawn out some of the structure of these more general kinds of consequence relations. There is a structure here, to be examined, and to be considered under the broader banner of 'logic.'

The first thing to notice is that these relations do not satisfy monotonicity. Using '$\mid\!\sim$' to symbolise this sort of consequence relation, we may agree that

$$x \text{ is a Quaker} \mid\!\sim x \text{ is a Pacifist}$$

while we deny that

$$x \text{ is a Quaker}, x \text{ is a Republican} \mid\!\sim x \text{ is a Pacifist}$$

We may have $Q \mid\!\sim P$ and $Q, R \not\mid\!\sim P$, and so, the monotonicity condition fails for $\mid\!\sim$. However, work on such 'nonmonotonic' consequence relations has revealed a stricter monotonicity condition which plausibly does hold.

Cautious Monotonicity: If $X \mid\!\sim A$ and $X \mid\!\sim B$ then $X, B \mid\!\sim A$.

Adding *arbitrary* extra premises may defeat a conclusion, but we add as a premise something which may be concluded from premises this does not undercut the original conclusion [49]. This seems to be a principle that is satisfied by a number of different ways to understand nonmonotonic consequence relation and to distinguish it from broader notions, such as compatibility.[7]

Much work has been done since the 1970s on nonmonotonic consequence relations and ways that they might come about. John McCarthy's work on circumscription analyses nonmonotonic consequence relations in terms of the minimality of the extensions of certain predicates. The core idea is that, *ceteris paribus*, some predicates are to have as small extensions as possible. If we take all *normal* or *typical* Quakers to be Pacifists, then the idea is that we want there to be as few *abnormal* or *atypical* items as possible, unless we have positive information to the effec that there are abnormal or atypical items. McCarthy represents this constraint formally, using second order quantification to express the condition [85].

Other accounts of nonmonotonic logic—such as Reiter's *Default Logic* [93] characterise the relation $\mid\!\sim$ in terms of primitive default rules. We can specify models in terms of default rules (stating that Quakers are generally Pacifists and Republicans are generally not Pacifists). Networks of default rules like these can be used to define a nonmonotonic consequence relation over a whole language, interpreting default rules as defeasible *inheritance* rules, where properties flow through the network as far as possible. Different ways to treat conflicts in inheritance graphs (e.g. the conflict in the case of Republican Quakers) have been characterised as *skeptical* and *credulous* approaches to conflict. A useful and straightforward introduction to this and many other issues in the interpretation and application of nonmonotonic consequence relations is given by Antonelli [4].

17. The Substructural Landscape

Tarski-Scott consequence relations are relations between sets. Gentzen's sequent calculi are proof systems with sequents of sequences but, with their structural

[7] If we think of $X \mid\!\approx A$ as 'A is compatible with X' then cautious monotonicity fails for $\mid\!\approx$. Here $p \mid\!\approx q$ and $p \mid\!\approx \neg q$ but we do not have $p, \neg q \mid\!\approx q$. Whatever satisfies cautious monotonicity, it is a more discriminating relation than mere compatibility.

rules, they function very much like the sets of Tarski-Scott consequence relations. There are, however, consequence relations between other structures. There are numerous ways of refining the structure of premises and conclusions in arguments and of restricting the available of structural rules. This tends to result in weaker consequence relations. A consequence relation where the structure of premises and conclusions is increased and structural rules are restricted is a *substructural* consequence relation.

The following example shows a Gentzen style sequent calculi derivation (it is a proof for the same argument as in the natural deduction from the Gentzen section). The structures on the left and right of the symbol $\Rightarrow$ are *multi-sets* (the order of presentation is irrelevant but the number of appearances of a formula in the presentation is). Disjunctions and a conjunction are introduced on the right of the double arrow. Conditionals are introduced on both the right and the left. Note that the disjunctions are introduced from multiple conclusions. This proof looks much bigger than the natural deduction above. The reason is that more attention has been paid to the structure of the premises and conclusions. The natural deduction example took no notice that there were two instances of the premise at the end of the deduction (one on each branch of the tree), this deduction uses the structural rule of contraction on the left (LW) to remove the duplication. The structural rule of weakening on the right (RK) is used to introduce the additional disjunct before the disjunction is introduced.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, B}\,RK}{A \Rightarrow A \vee B}\,R\vee \quad C \Rightarrow C}{(A \vee B) \to C, A \Rightarrow C}\,R\to}{(A \vee B) \to C \Rightarrow (A \to C)}\,L\to \quad \dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow A, B}\,RK}{B \Rightarrow A \vee B}\,R\vee \quad C \Rightarrow C}{(A \vee B) \to C, B \Rightarrow C}\,R\to}{(A \vee B) \to C \Rightarrow (B \to C)}\,L\to}{(A \vee B) \to C, (A \vee B) \to C \Rightarrow (A \to C) \wedge (B \to C)}\,R\wedge}{(A \vee B) \to C \Rightarrow (A \to C) \wedge (B \to C)}\,LW$$

As can be seen from the above history of consequence, there are close connections between logical consequence and conditionals. The distinctions between these two have not always been very clear. The *Residuation condition* is one way of expressing the connection.

$$A, B \Rightarrow C \text{ if and only if } A \Rightarrow B \to C$$

The left to right direction of this law can be seen in the above example derivation in the application of the conditional on the right rule. The law of risiduation connects three notions together: logical consequence (indicated by $\Rightarrow$), conditionals (by $\to$) and *premise combination* (indicated by the comma). Structural rules (excluding identity and cut) operate on the comma in premise combination and, thus by the law of residuation, impact the properties of the conditional.

In the example above, conclusions are *weakened* into the proof. Logics which permit weakening on both the left and right are monotonic. The nonmonotoic logics from the previous section are all substructural in the sense that they do not respect the substructural rule of weakening. It is particularly important for relevant/relevance logics that weakening is abandoned. It doesn't follow from the consequence $X \Rightarrow A$ where the conclusion follows relevantly from the premises (each premise is important in the *following from* relation), that in the argument $X, B \Rightarrow A$ the conclusion *follows from* the premises in the same relevant way. Relevance in deduction and consequence was studied by Moh (1950), Church (1951) and Ackermann (1956). The canonical text on later developments is Anderson and Belnap's *Entailment* [1, 2].

While weakening allows for the addition of premises and conclusions, contraction removes duplicated premises and conclusions. In resource conscious logics (like Girard's *linear logic* [60]) contraction is dropped. Resource conscious logics pay attention to the number of times a formula is required in deriving a conclusion. So, a conclusion may follow from n uses of A, but not from any fewer. A number of contraction free logics are also weakening free.

Where the weakening rule has been connected to the paradoxes of material implication, the contraction rule is connected to Curry's paradox. Logics without contraction, like Łukasiewicz many-valued logics, have sustained interest because of this connection.

The example above uses multi-sets in the left and right of the sequents. Gentzen's original formulation used sequence, rather than multi-sets. In this case rules like *exchange*

$$\frac{\Gamma, A, B, \Gamma' \Rightarrow \Delta}{\Gamma, B, A, \Gamma' \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Delta'}{\Gamma \Rightarrow \Delta, B, A, \Delta'}$$

need to be included for classical and intuitionistic logic. Substructural logics like Lambek's calculus [78, 79] drop these rules. Lambek used mathematical techniques to model language and syntax. "Premise" combination is used to model composition of linguistic units. In these cases, it is inappropriate to change the order of the "premises".

In the sequent derivation above, the sequents have finite sets of formulas in the premise positions and in the conclusion positions. By restricting the maximum number of formulas which can appear in the antecedent or consequent sequence, you restrict the consequence relation. Sequent system for classical logic have multi-sets or sequences of any (finite) number of formulas on either consequent or antecedent position. Gentzen's system for intuitionist logic, however, is restricted to no more than one formula on the right hand side. This restriction on the structure on the right hand side makes the difference between intuitionist and classical logic. This restriction is a restriction on the structural rules of Intuitionist logic, the identity and cut rules are suitably restricted to single formulas on the right of sequents and weakening on the right is restricted.

These structural rules can be included, dropped entirely, or restricted in some way. The structural rule *mingle*:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A, A \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A, A}$$

is a restricted version of weakening. The rule doesn't allow for arbitrary additions of premises, but if a premise has already been included in a relevant deduction of the conclusion, it can be weakened in any number of times.

We began this entry by claiming that theorists of logical consequence have to answer the question "In what ways can premises combine in an argument?". The substructural landscape shows that this is a genuine requirement on theorists. Different answers, in combination with different structural rules, produce very different consequence relations.

18. Monism or Pluralism

Given such a variety of different accounts of logical consequence, what are we to say about them? Are some of these accounts correct and others incorrect? Is there One True Logic, which gives the single correct answer to the question "is this valid?" when presented with an argument, or is there no such logic but rather, a Plurality of consequence relations? This is the issue between *Monists* [89, 91] and *Pluralists* [9, 10] about logical consequence.

In this debate we can at first set aside some consequence relations. Some formal consequence relations we call 'logics' are clearly logical consequence relations by courtesy only. They are not intended as giving an account of a consequence relations between statements or propositions. They are logical consequence relations as abstract formal structures designed to model something *quite like* a traditional consequence relation. The substructural logic of the Lambek calculus, when interpreted as giving us information about syntactic types is a good example of this. It is a 'logic' because it has a formal structure like other logics, but we do not think of the consequence relation between statements as anything like *truth* preservation. To know that A follows from B in the Lambek calculus interpreted in this way means that any syntactic string of type A is also a string of type B.[8] So, we may set these logics (or logical consequence relations interpreted in these ways) aside from our consideration, and consider consequence relations where the verdict that A logically entails B is to tell us something about the relationship between the *statements* A and B.

Consider four examples where there is thought to be a genuine disagreement about consequence between given statements, in debates over second order consequence; intuitionist logic; classical logic; and indexicality.

SECOND ORDER LOGIC: Does $a \neq b$ entail $\exists X(Xa \wedge \neg Xb)$? According to standard accounts of second order logic, it does.[9] If a and b are distinct, there is some way to assign a value to the second order variable X to ensure that (relative to this assignment), $Xa \wedge \neg Xb$ is satisfied.[10] If second order quantification is not properly logical, then perhaps the entailment fails. What are we to say? Perhaps there is to be addressed by singling out some class of vocabulary as the *logical constants*, and

[8] So, we could interpret the consequence relation between A and B as telling us that the typing judgement 'x is of type B' somehow follows from the typing judgement 'x is of type A,' but these typing judgements are not the relata of the 'consequence relation' in question. The types A and B are those relata.

[9] This is a relatively uncontroversial example. For a more difficult case, consider the statement $\forall R(\forall x \exists y Rxy \rightarrow \exists f \forall x Rxfx)$, which is a statement of the axiom of choice in the vocabulary of second order logic. Is this a tautology (a consequence of the empty set of premises)?

[10] Perhaps we can assign X the extension $\{d\}$ where d is the object in the domain assigned as the denotation of the name a.

then we are to decide which side of the boundary the second order quantifiers are to fall. Perhaps, on the other hand, we are to address the question of the validity of this argument by other means. Perhaps we are to ask what are the admissible interpretatons of the statment $(\exists X)(Xa \wedge \neg Xb)$, or the circumstances in which its truth or falsity may be determined.

Monists take there to be a single answer to this question. If they accept the distinction between logical constants and non-logical vocabulary, then second order logic must fall on one side or other of that boundary. If they do not accept such a distinction, then if consequence is understood in terms of truth preservation across a range of cases, there is a definitive class of cases in which one is to interpret the second order vocabulary.

Pluralists, on the other hand, can either say that the distinction between logical and non-logical vocabulary admits of more than one good (and equally correct) answer, or that if we do not accept such a distinction, we can say that there is a range of circumstances appropriate for interpreting second order vocabulary. For example, in this case we could say that in standard models of the second order vocabulary, $a \neq b$ does entail $\exists X(Xa \wedge \neg Xb)$, but in the more generous class of Henkin models, we can find interpretations in which $a \neq b$ holds yet $\exists X(Xa \wedge \neg Xb)$ fails, because we have fewer possible assignments for the second order variable X. Pluralists in this case go on to say that the choice between the wider and narrower class of models need not be an all-or-nothing decision. We can have two consequence relations: according to one, the argument is valid (and has no—standard model—counterexamples), and according to the other it is invalid (and has a—Henkin model—counterexample).

Is this sort of answer enough to satisfy? For the pluralist, it may well. For the monist, it does not. The monist would like to know whether the putative counterexample to the argument is a *genuine* counterexample or not, for then and only then will we know whether the argument is valid or not.

Not all disagreements between pluralists and monists emerge on the ground where we may dispute over whether or not we have a logical constant. The next two disagreements over logical consequence are on the battleground of the standard propositional connectives, and the consensus is that these are logical constants if anything deserves the name.

CONSTRUCTIVE LOGIC: Is $p \vee \neg p$ a tautology? Is the argument from $\neg\neg p$ to p valid? The intuitionist says that they are not—that we may have a construction that shows $\neg\neg p$ (by reducing $\neg p$ to absurdity) while not also constructing the truth of p. Constructions may be incomplete. Similarly, not all constructions will verify $p \vee \neg p$—not through *refuting* it by verifying $\neg p \wedge \neg\neg p$ but by neither verifying p nor $\neg p$. In these cases, the genuine intuitionists claim to have counterexamples to these classically valid arguments, while the orthodox classical logicians take these arguments to have no counterexamples. The debate here is *not* over whether negation or disjunction are logical constants. The monist (whether accepting classical or intiuitionist logic) holds there to be a definitive answer to the question of whether these constructive counterexamples are worth the name. If they are, then the arguments are *invalid* (and classical logic is not correct) and if they are not, then classical logic judges some invalid arguments to be invalid, and is hence to be rejected. This is the standard intuitioinst response to classical logic.[11]

The pluralist, on the other hand, cannot say this. The pluralist can say, on the other hand, that the argument from $\neg\neg p$ to p is valid in one sense (it is classically valid: it has no counterexample in consistent and complete *worlds*) and invalid in another (it is constructively invalid: there are *constructions* for $\neg\neg p$ that are not constructions for p). The pluralist goes on to say that these two verdicts do not conflict with one another. The one person can agree with both verdicts. It is just that the notion of a counterexample expands from the traditional view according to which an argument either has counterexamples or it does not, and that is the end of the matter. Instead, we have narrower (classical, worlds) and wider (intuitionistic, constructions) classes of circumstances, and for some theoretical purposes we want the narrower class, and for others the wider. There are two consequence relations here, not one [94].

RELEVANCE: The same sort of consideration holds over debates of relevance. For relevantists, the argument form of explosion, from $p \wedge \neg p$ to q and disjunctive syllogism, from $p \wedge (\neg p \vee q)$ to q are invalid, while for classical and constructive accounts of validity, they are valid. The monist holds either that classical logic is correct in this case, or that it is incorrect and a relevant account of consequence is to be preferred. This has proved to be a difficult position to take for the relevantist: for there seems to be a clear sense in which disjunctive syllogism is *valid*—there are no consistent circumstances in which $p \wedge (\neg p \vee q)$ is true and q is not. There are, for the relevantist, counterexamples to this argument, but they involve inconsistent circumstances, which make p and $\neg p$ both true. Now, it may be the case that inconsistent circumstances like these are suitable for individuating content in a finely-grained way—a situation inconsistent about p need not be inconsistent about q so we can distinguish them as having different subject matter, even in the case where the premise $p \wedge \neg p$ is inconsistent. However, for many purposes, we would like to ignore these inconsistent circumstances. In many practical reasoning situations, we would like very much to deduce q from $p \wedge (\neg p \vee q)$, and inconsistent situations where $p \wedge (\neg p \vee q)$ holds and q doesn't seem neither here nor there.

In this case the pluralist is able to say that there is more than one consequence relation at work, and that we need not be forced to choose. If we wish to include incosistent situations as 'counterexamples' then we have a strong—relevant—consequence relation. Without them, we do not. The monist has no such response, and this has caused some consternation for the monist who wishes to find a place for relevant consequence [11].

INDEXICALITY: Is the argument from 'It is raining' to 'It is raining *here*' a valid one? Is the argument form from p to $@p$ valid? Again, it depends on the admisslbe ways to interpret the actuality operator @ or the indexical 'here.' In one sense it is clearly valid, and in another sense, it is cleary not valid. If we take the constituents of arguments here to be *sentences*, then there is a sense in which the argument from 'It is raining' to 'It is raining *here*' is valid, for any circumstance in which the sentence 'It is raining' is expressed to state something true, the sentence

[11]It may be augmented, of course, by saying that classical logic has its place as a logic of decidable situations, which is to say that in *some* circumstances, $p \vee \neg p$ is true, for some statements p. It is not necessarily to say that classical logic is the right account of *validity* for any class of arguments.

'It is raining here'expressed in that circumstance is also true. If, on the other hand, we take the constituents of the argument to be the *contents* of the sentences, then the argument can be seen to be invalid. For the claim that it is raining can be true without it being true that it is raining *here*. Had it been raining over there, then relative to that circumstance, it would be raining, but it wouldn't be raining *here*.[12] Another way to be a pluralist, then, is to say that sometimes it is appropriate to evaluate consequence as a relation between sentences, and sometimes it is appropriate to think of it as a relation between contents of those sentences. In any case, we must pay attention to the kinds of items *related* by our consequence relations, as our choices here will also play a part in determining what kind of relation we have [100], and perhaps, how many of those relations there might be.

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[12] This is equivalent, formally, to evaluating the sentences 'off the diagonal' in a two-dimensional modal model.

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