Reflections on Brady's Logic of Meaning Containment

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Abstract: This paper is a series of reflections on Ross Brady's favourite substructural logic, the logic MC of *meaning containment*. In the first section, I describe some of the distinctive features of MC, including depth relevance, and its principled rejection of some concepts that have been found useful in many substructural logics, namely intensional or multiplicative conjunction (sometimes known as 'fusion'), the Church constants ($\top$ and $\bot$), and the Ackermann constants (t and f). A further distinctive feature of the axiomatic formulation of MC is its *meta-rule*, which is a unique feature of MC Hilbert proofs. This meta-rule gives rise to one further special property of MC, in that the logic is *distributive* in one sense, and *non-distributive* in another. The distribution of additive conjunction over disjunction (the step from $p \wedge (q \vee r)$ to $(p \wedge q) \vee (p \wedge r)$) holds in MC as a *rule*, but not as a provable conditional, and in this way, MC is distinctive among popular substructural logics. (Anderson and Belnap's favourite logics R and E are distributive in *both* senses, while Girard's linear logic is distributive in neither.)

In this paper, I aim to increase our understanding of each of these distinctive features of MC, giving an account of what it might take for a propositional logic to meet these constraints. I will start with a presentation of Hilbert proofs for MC, and then showing how *Brady Lattices* (a natural class of algebraic models for MC) can help us understand each of these special features of Brady's logic of meaning containment.

1 Introducing MC

Ross Brady's MC is an important substructural logic, with a number of features that set it apart from other logics in the wider substructural family. Alongside R, it is a *relevant* logic. If a conditional $A \to B$ is provable, then the antecedent

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A and the consequent B must share a propositional atom. But MC goes further than R in that this shared content must occur at the same *depth* in the antecedent and in the consequent. The so-called *modus ponens axiom*[1] $p \mathbin{\&} (p \to q) \to q$ fails to be depth relevant since the content shared between antecedent $p \mathbin{\&} (p \to q)$ and consequent q (in this case, the atom q) occurs at different depth in the antecedent than the consequent. In the antecedent, it has depth 1 because it is inside a further conditional, while in the consequent it has depth 0 [2].

Depth relevance provides a powerful filter on formulas. Consider these two candidates for a formula expressing the transitivity of the conditional. This first formula is the *conjunctive syllogism* axiom:

$$(A \to B) \mathbin{\&} (B \to C) \to. A \to C$$

The second formula, sometimes known as the *suffixing axiom*, also expresses the transitivity of the conditional:

$$A \to B \to. B \to C \to. A \to C$$

The *conjunctive syllogism* axiom is depth relevant, since A and C occur at the same depth in the antecedent $(A \to B) \mathbin{\&} (B \to C)$ and in the consequent $A \to C$ (both occuring under just one conditional). However, *suffixing* fails to be depth relevant, since A and B occur under one conditional in its antecedent $A \to B$, while they occur under two conditionals in the consequent $B \to C \to. A \to C$. This is why Brady's logic of meaning containment includes conjunctive syllogism, but rejects suffixing. From the perspective of other familiar substructural logics (say, linear logic, or the contraction-free variant RW of Anderson and Belnap's logic R), the conjunctive syllogism axiom is a close cousin of the *contraction* axiom

$$A \to (A \to B) \to. A \to B$$

in that, from the point of view of RW or linear logic, adding one of conjunctive syllogism and contraction as an axiom brings the other in its wake. For those familiar with natural deduction proofs of the Gentzen/Prawitz style [8,10], it seems that conjunctive syllogism, contraction, and the *modus ponens* axiom each arise

[1] Here, and throughout the paper, I follow Brady's notational conventions. The ampersand (&) is used for extensional (additive) conjunction, and the wedge ($\vee$) for extensional disjunction. Parenthesis use is minimised: conjunction and disjunction bind more tightly than the conditional ($\to$), so $p \mathbin{\&} q \to r \vee s$ is a conditional with antecedent $p \mathbin{\&} q$ and consequent $r \vee s$. The conditional associates to the left by default, so $p \to q \to r$ is the conditional with antecedent $p \to q$ and consequent r. Finally, a period after a connective is treated as a left parenthesis whose mate is as far to the right in the expression as grammatically possible. So, $p \to. p \to p \to p$ would be written, with parentheses, as $p \to ((p \to p) \to p)$.

from a common source: their natural deduction proofs require that more than one occurrence of a single assumption formula be discharged. Consider this proof of contraction:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \to. A \to B]^2 \qquad [A]^1}{A \to B}\,{\to}E \qquad [A]^1}{B}\,{\to}E}{A \to B}\,{\to}I^1\,\textbf{!}}{A \to (A \to B) \to. A \to B}\,{\to}I^2$$

Here, the two occurrences of A are discharged at the marked $\to$I step. In the following proof of conjunctive syllogism

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \to B) \,\&\, (B \to C)]^2}{B \to C}\,\&E \qquad \dfrac{\dfrac{[(A \to B) \,\&\, (B \to C)]^2}{A \to B}\,\&E \qquad [A]^1}{B}\,{\to}E}{C}\,{\to}E}{A \to C}\,{\to}I^1}{(A \to B) \,\&\, (B \to C) \to. A \to C}\,{\to}I^2\,\textbf{!}$$

the two instances of $(A \to B) \,\&\, (B \to C)$ are discharged at once, at the marked $\to$I step. The same goes for the *modus ponens* axiom.

$$\dfrac{\dfrac{\dfrac{[A \,\&\, (A \to B)]^1}{A \to B}\,\&E \qquad \dfrac{[A \,\&\, (A \to B)]^1}{A}\,\&E}{B}\,{\to}E}{A \,\&\, (A \to B) \to B}\,{\to}I^1\,\textbf{!}$$

So, for those of us who are steeped in natural deduction, it at least seems natural to classify conjunctive syllogism alongside contraction as a principle involving duplicate discharge, and to prefer logics that contain *neither* of these principles (say, linear logic, the Lambek calculus, RW and their many relatives) or those which contain *both* (say, R or E and their relatives). MC stands apart in that it includes one of these principles and not the other.

A strict adherence to depth relevance brings with it other commitments concerning the vocabulary of MC. On some formulations of substructural logics, it is natural to include in the vocabulary the so-called 'Church' constants $\top$ and $\bot$ for a maximal truth and minimal falsity, respectively. These are rejected in MC on depth relevance grounds. As logical constants, there are no atoms present in $\top$ or in $\bot$, so the candidate axioms $\bot \to A$ and $A \to \top$ violate depth relevance.

Not only are the extremal propositional constants rejected, but so is the 'Ackermann' constant t, defined by way of the rules $A \Rightarrow t \to A$ and $t \to A \Rightarrow A$. If we treat t as a logical constant with no distinctive content of its own, it violates depth relevance in just the same way as the Church constants, and so, is rejected.

MC also exchews fusion ($\circ$), the intensional conjunction central to the formulation of many substructural logics.[2] Introduced by the rules $A \to . B \to C \Rightarrow A \circ B \to C$ and $A \circ B \to C \Rightarrow A \to . B \to C$, it is not surprising that such a connective raise suspicions on the grounds of depth relevance, since the nesting of formulas under arrows shifts between $A \circ B \to C$ and $A \to . B \to C$.

However, Brady does not reject fusion on the grounds of depth relevance alone. The logic MC exchews the use of fusion because adding fusion to the vocabulary (in the presence of conjunctive syllogism at least) allows for a derivation of inconsistency from a the naïve class comprehension axiom.[3] Rather than listing all of the different inclusions and exclusions from MC and discussing the grounds for each judgement, let's skip ahead and consider MC as a whole. Here are the axioms and rules of MC:

A1. $A \to A$
A2. $A \mathbin{\&} B \to A$
A3. $A \mathbin{\&} B \to B$
A4. $(A \to B) \mathbin{\&} (A \to C) \to . A \to B \mathbin{\&} C$
A5. $A \to A \vee B$
A6. $B \to A \vee B$
A7. $(A \to C) \mathbin{\&} (B \to C) \to . A \vee B \to C$
A8. $\sim\sim A \to A$
A9. $A \to \sim B \to . B \to \sim A$
A10. $(A \to B) \mathbin{\&} (B \to C) \to . A \to C$

R1. $A, A \to B \Rightarrow B$
R2. $A, B \Rightarrow A \mathbin{\&} B$
R3. $A \to B, C \to D \Rightarrow B \to C \to . A \to D$

However, the axioms and rules are not enough to define MC. It contains a *meta-rule* of the following form:

[2] In the linear logic literature, $\circ$ is written '$\otimes$' and is called the *multiplicative* conjunction, in contrast to the *additive* conjunction and disjunction.

[3] See Brady's *Universal Logic* [4, p. 241] for the explanation of exactly *how* fusion brings triviality in naïve comprehension. From conjunctive syllogism we can prove $A \circ B \to A \circ (A \circ B)$, and when we take the set C to be $\{x : x \in x \circ (p \to p) \to q\}$ we can prove $C \in C \leftrightarrow C \in C \circ (p \to p) \to q$, which, by the definition of fusion, gives $C \in C \circ (C \in C \circ (p \to p) \to q)$ which, by $A \circ B \to A \circ (A \circ B)$, gives $C \in C \circ (p \to p) \to q$, which is enough to prove $C \in C$ (applying the biconditional in reverse), and then since we can prove $p \to p$ we have $C \in C \circ (p \to p)$, which implies q, which was arbitrary.

MR1. If $A, B \Rightarrow C$ then $D \vee A, D \vee B \Rightarrow D \vee C$.

Ross Brady introduced *meta*-rules in a 1984 paper on proof systems for quantified relevant logics [3]. If we think of the axioms and rules as merely providing a presentation of a Tarskian consequence relation in the natural way—as the smallest such relation in which each *axiom* is a consequence of the empty set, and such that the relation is closed under each of the *rules*, then we can think of the meta-rule is a higher-level closure condition on this consequence relation. Our target is the minimal consequence relation closed under the rules and satisfying the meta-rule.

However, a Hilbert system is not only a recursive characterisation of a consequence relation: it is also a way of defining *proofs*. Brady did give some brief remarks concerning how one might understand proofs using a meta-rule (in which you enter the antecedent proof as a sub-proof of the main proof [3, p. 358]), but as far as I am aware, the literature contains neither an explicit definition of what counts as a proof in a Hilbert system with meta-rules, nor any concrete example of such a thing. So, as a small contribution to the literature, I will expand on Brady's comments in the next section with a formal definition of an extended Hilbert proof, followed by an example.

2 MC PROOFS

DEFINITION 1 [EXTENDED HILBERT PROOF] Given some family of *axioms*, *rules* and *meta-rules*, an *extended Hilbert proof* from the premises X to the conclusion C (a *proof* for $X \Rightarrow C$, for short) is a structured list of formulas and proofs, ending in the conclusion formula C, such that each item in the proof is either (a) an *axiom*, or (b) a member of the set P of premises, or (c) the concluding formula formula of some *rule*, where each of the premises of that rule have appeared earlier in the proof, or (d) another *proof* (which we call a *sub-proof* of this proof), or (e) the concluding formula D' of the *meta-rule* (if $Y_1 \Rightarrow D_1, \ldots, Y_n \Rightarrow D_n$, then $Y' \Rightarrow D'$), where a sub-proof for each $Y_i \Rightarrow D_i$, and the members of Y' each appear earlier in the proof.

In this definition, it is important to understand that any premises used in a sub-proof of a proof are not themselves premises for the proof as a whole. Sub-proofs are included in the main proof as a means to underwrite the application of meta-rules. To illustrate, Figure 1 displays a proof from the premise $p \mathbin{\&} (q \vee r)$ to the conclusion $(p \mathbin{\&} q) \vee r$. The entries of the proof are given line numbers to the left, and annotations to the right, indicating the provenance of each entry. The sub-proof on line 14 is presented in a box of its own, and it underwrites the application

1.	$p \mathbin{\&} (q \vee r)$	PREMISE
2.	$p \mathbin{\&} (q \vee r) \to p$	A2
3.	$p \mathbin{\&} (q \vee r) \to q \vee r$	A3
4.	p	1, 2 R1
5.	$p \to r \vee p$	A6
6.	$r \vee p$	4, 5 R1
7.	$q \vee r$	1, 3 R1
8.	$q \to r \vee q$	A6
9.	$r \to r \vee q$	A5
10.	$(q \to r \vee q) \mathbin{\&} (r \to r \vee q)$	8, 9 R2
11.	$(q \to r \vee q) \mathbin{\&} (r \to r \vee q) \to .\ q \vee r \to r \vee q$	A7
12.	$q \vee r \to r \vee q$	10, 11 R1
13.	$r \vee q$	7, 12 R1
14.	1. p PREMISE 2. q PREMISE 3. $p \mathbin{\&} q$ 1, 2 R2	SUB-PROOF
15.	$r \vee (p \mathbin{\&} q)$	6, 13, 14 MR1
16.	$r \to (p \mathbin{\&} q) \vee r$	A6
17.	$p \mathbin{\&} q \to (p \mathbin{\&} q) \vee r$	A5
18.	$(r \to (p \mathbin{\&} q) \vee r) \mathbin{\&} (p \mathbin{\&} q \to (p \mathbin{\&} q \vee r))$	16, 17, R2
19.	$(r \to (p \mathbin{\&} q) \vee r) \mathbin{\&} (p \mathbin{\&} q \to (p \mathbin{\&} q) \vee r) \to .$ $(r \vee (p \mathbin{\&} q) \to (p \mathbin{\&} q) \vee r)$	A7
20.	$r \vee (p \mathbin{\&} q) \to (p \mathbin{\&} q) \vee r$	18, 19 R1
21.	$(p \mathbin{\&} q) \vee r$	15, 20 R1

Figure 1: An MC proof for $p \mathbin{\&} (q \vee r) \Rightarrow (p \mathbin{\&} q) \vee r$

of the meta-rule, allowing for the inference of $r \vee (p \mathbin{\&} q)$ from $r \vee p$ and $r \vee q$, on line 15.

The meta-rule MR1 allows for the derivation of the rule form of distribution $p \mathbin{\&} (q \vee r) \Rightarrow (p \mathbin{\&} q) \vee r$, where the *axiom* form $p \mathbin{\&} (q \vee r) \to (p \mathbin{\&} q) \vee r$ is absent from MC. It is not too hard to show that distribution in the form $p \mathbin{\&} (q \vee r) \Rightarrow (p \mathbin{\&} q) \vee (p \mathbin{\&} r)$ follows from distribution in this simpler form,[4] while again, the axiom form of distribution, $p \mathbin{\&} (q \vee r) \to (p \mathbin{\&} q) \vee (p \mathbin{\&} r)$ is absent from MC. So, MC is both *distributive* and *non*-distributive in ther sense that distribution holds at the level of Hilbert validity, but not at the level of

[4]We have $p \mathbin{\&} (q \vee r) \Rightarrow (p \mathbin{\&} q) \vee r$, from which it is not hard to get $p \mathbin{\&} (q \vee r) \Rightarrow r \vee (p \mathbin{\&} q)$, commuting a disjunction, and we also have $p \mathbin{\&} (q \vee r) \Rightarrow p$. Together these give $p \mathbin{\&} (q \vee r) \Rightarrow p \mathbin{\&} (r \vee (p \mathbin{\&} q))$, and $p \mathbin{\&} (r \vee (p \mathbin{\&} q)) \Rightarrow (p \mathbin{\&} r) \vee (p \mathbin{\&} q)$ follows from a second application of the form of distribution we already have. Chaining these together, and commuting the final disjunction, we have $p \mathbin{\&} (q \vee r) \Rightarrow (p \mathbin{\&} q) \vee (p \mathbin{\&} r)$ as desired.

provable conditionals.[5]

With our short exploration of Hilbert proofs complete, we can return to MC and its distinctives. In the remainder of this paper, I will focus on four distinctive features of MC that make it stand out among substructural logics.

1. MC takes the distribution of lattice connectives to hold as a *rule*, but not as a provable *conditional*.
2. MC has conjunctive syllogism, but no contraction.
3. MC excludes the Church and Ackermann constants.
4. MC excludes fusion.

These features are not *unique* to MC (we can formulate the paradigm relevant logic R without the Ackermann or Church constants, and without fusion if we wish), but these additions are very *natural* when we formulate the logic by way of natural deduction [13], sequent calculus [1,7], algebraic semantics [6], or Routley–Meyer ternary relational models [12]. In each different formulation of a logic in this broad family, fusion and the Ackermann and Church constants are either easily and simply definable out of the material at hand, or they are *essential* for the enterprise to get off the ground in the first place. In sequent systems and in natural deduction, the conditional is governed by rules looking rather like the traditional rules of *modus ponens* and conditional proof, which involve the addition of premises and the discharging of premises previously added. The notion of addition at play in those rules is not the extensional conjunction &, but is naturally made explicit in the language by way of the fusion connective. In the ternary relational semantics the conditional is modelled by a universal forward-facing clause concerning the ternary relation R, and fusion is naturally modelled by the corresponding existential backward-facing clause.[6] In such formulations it is *possible* to have conjunctive syllogism present in a logic without fusion, but this possibility does not stand out as particularly natural. Contraction corresponds to the collapse of multiple repetitions of assumuptions in one discharge. Conjunctive syllogism is a special case of contraction, when viewed in this light, and viewed

[5]Here, and elsewhere, I will write $A, B, C \Rightarrow D$ to state that there is an MC-proof from premises A, B, C to conclusion D.

[6]This is a binary connective analogue to the duality in temporal logic between a necessity-like 'at every time in the future' operator $\Box^+$ and a possibility-like 'at some time in the past' operator $\Diamond^-$, for which we have the equivalence between the entailments $A \Rightarrow \Box^+ B$ and $\Diamond^- A \Rightarrow B$. Perhaps the fact that some modal logics, such as temporal logics, provide interpretations of these connectives in which both make sense, while *other* interpretations of a modality, plausibly, define notions of necessity where there is *no* natural sense to a backward-looking modality, can help us think about why it might be that the formal definability of a fusion operator in some model theory may not be enough to underwrite the claim that it makes sense.

from the point of view of fusion this contraction does not seem to have anything in particular to recommend it. If I can indulge in a moment of autobiography, my own introduction to substructural logics [11] took its lead from the traditions I have enumerated above, and so, it was difficult for me to bring Brady's work on logics in the vicinity of MC into proper focus when viewed from that particular orientation. In this note, I will attempt to remedy this shortcoming by adapting *one* of the tools used in that book—algebraic semantics—to better model MC, so I can better understand some of the distinctive features of this logic.

3 MC ALGEBRAS

One way to conceive of an algebraic model of a logic is to think of the items in the algebra as providing a range of semantic values for the formulas in the language to be assigned. For each formula A, its semantic value $[\![A]\!]$ is a member of the algebra. The cruical constraint in interpretations of this kind is that the value of a complex formula (say, the value $[\![A \rightarrow B]\!]$) depends on the values of its subformulas (in this case, the values $[\![A]\!]$ and $[\![B]\!]$). If this compositionality constraint is satisfied by some connective (here, the conditional) then the syntactic connective corresponds to an *operator* on the algebra in the familiar way.

An algebra provides a different perspective on the logic by way of its identifications: two different formulas A and B might have the same semantic values. In the extreme case of the two-valued Boolean algebra on $\{0, 1\}$, there are only two values to choose from, and there are very many identifications. In other algebras with many more values, fewer formulas are identified in any valuation. However, the upper limit for our algebras is fixed: if A and B are logically equivalent, then $[\![A]\!] = [\![B]\!]$.

In logics like MC, the salient notion of logical equivalence cannot be given by the Tarskian consequence relation $\Rightarrow$ alone. We have $p \rightarrow p \Rightarrow q \rightarrow q$ and vice versa, but $p \rightarrow p$ and $q \rightarrow q$ cannot be substituted one for the other in every context, if we wish to keep track of validity. For example, $(p \rightarrow p) \rightarrow (p \rightarrow p)$ is an MC-theorem (an instance of A1), while $(p \rightarrow p) \rightarrow (q \rightarrow q)$ is not an MC-theorem. So, the salient notion of equivalence for MC is stricter: A and B are logically equivalent if and only if MC proves $\Rightarrow A \rightarrow B$ and $\Rightarrow B \rightarrow A$. This notion of equivalence is well-suited for our task, since we can prove the following theorem:

THEOREM 1 [SUBSTITUTIVITY OF EQUIVALENTS] *If A and A' are logically equivalent in MC then so are $C(A)$ and $C(A')$.*

Proof: We prove this result by induction on the formula context $C(-)$ in which A and A' are placed. The result is immediate when $C(-)$ is $-$ alone. For the induction steps, we need to show that if A and A' are logically equivalent, so are $\sim A$

and $\sim A'$; so are $A \mathbin{\&} B$ and $A' \mathbin{\&} B$ (and $B \mathbin{\&} A$ and $B \mathbin{\&} A'$); so are $A \vee B$ and $A' \vee B$ (and $B \vee A$ and $B \vee A'$); so are $A \to B$ and $A' \to B$ (and $B \to A$ and $B \to A'$). These facts follow from stronger principles which may each be derived in MC.

- $A \to A' \Rightarrow {\sim}A' \to {\sim}A$
- $A \to A' \Rightarrow A \mathbin{\&} B \to A' \mathbin{\&} B$
- $A \to A' \Rightarrow A \vee B \to A' \vee B$
- $A \to A' \Rightarrow B \to A \to. B \to A'$
- $A \to A' \Rightarrow A' \to B \to. A \to B$

Finding a proof for the first requires a little creative fiddling with the negation axioms of MC, while the other proofs are quite straightforward. I complete the first two, leaving the others to the reader.

1.	$A \to A'$	PREMISE
2.	${\sim}A' \to {\sim}A' \to. A' \to {\sim}{\sim}A'$	A9
3.	${\sim}A' \to {\sim}A'$	A1
4.	$A' \to {\sim}{\sim}A'$	2, 3 R1
5.	$A' \to A' \to. A \to {\sim}{\sim}A'$	1, 4 R3
6.	$A' \to A'$	A1
7.	$A \to {\sim}{\sim}A'$	5, 6 R1
8.	$A \to {\sim}{\sim}A' \to. {\sim}A' \to {\sim}A$	A9
9.	${\sim}A' \to {\sim}A$	7, 8 R1

1.	$A \to A'$	PREMISE
2.	$A \mathbin{\&} B \to A$	A2
3.	$A \to A \to. A \mathbin{\&} B \to A'$	1, 2 R3
4.	$A \to A$	A1
5.	$A \mathbin{\&} B \to A'$	1, 4 R3
6.	$A \mathbin{\&} B \to B$	A3
7.	$(A \mathbin{\&} B \to A') \mathbin{\&} (A \mathbin{\&} B \to B)$	5, 6 R2
8.	$(A \mathbin{\&} B \to A') \mathbin{\&} (A \mathbin{\&} B \to B) \to. A \mathbin{\&} B \to A' \mathbin{\&} B$	A4
9.	$A \mathbin{\&} B \to A' \mathbin{\&} B$	7, 8 R1

With this family of principles, we can derive the replacement of equivalents by a simple induction on the structure of the formula context $C(-)$. ∎

This substitutivity result means that we can define the *Lindenbaum Algebra* for MC.

DEFINITION 2 [LINDENBAUM ALGEBRA] The Lindenbaum Algebra for MC (given a language $\mathcal{L}$) is the algebra on the set of equivalence classes of logically equivalent formulas in $\mathcal{L}$:

$$[A] = \{A' : \Rightarrow A \to A' \text{ and } \Rightarrow A' \to A\}$$

Each connective lifts to an operator on equivalence classes in the natural way

$$\begin{array}{rclrcl} [A] \,\&\, [B] & = & [A \,\&\, B] & [A] \to [B] & = & [A \to B] \\ [A] \vee [B] & = & [A \vee B] & {\sim}[A] & = & [{\sim}A] \end{array}$$

since the substitutivity of equivalents means that the choice of representative for an equivalence class is irrelevant. Our equivalence classes are naturally ordered by provable implication:

$$[A] \leq [B] \text{ iff } \Rightarrow A \to B$$

Since the equivalence classes are defined by logical equivalence, this relation is indeed a partial order. If $[A] \leq [B]$ and $[B] \leq [A]$ then $[A] = [B]$, since A and B are logically equivalent. The Lindenbaum algebra is a lattice under the ordering $\leq$, for which conjunction and disjunction are greatest lower and least upper bounds, respectively.

Finally, it is useful to have a record in the algebra of the theorems of the logic, so we isolate a subset of elements like so: $T = \{[A] : \ \Rightarrow A\}$.[7]

This set T is a *filter* in MC's Lindenbaum algebra: It is closed upward under the order (if $x \in T$ and $x \leq x'$ then $x' \in T$ too), since anything that follows from a theorem is itself a theorem, and it is closed under conjunction (if $x, y \in T$ then $x \,\&\, y \in T$). In fact, T is rather special in Lindenbaum algebra for MC: T is *prime*, in the following sense: If $x \vee y \in T$ then either $x \in T$ or $y \in T$, since in MC, we have $\Rightarrow A \vee B$ if and only if either $\Rightarrow A$ or $\Rightarrow B$ [5, p. 366–7].[8]

The Lindenbaum algebra is but one example of an algebraic model for MC. It is an algebra in which we perform the fewest identifications possible. There are many more algebras in which many more formulas are identified. (A Lindenbaum algebra for classical propositional logic is a free Boolean algebra generated by the set of propositional atoms. There are many other Boolean algebras, right down to the two-element algebra familiar from truth tables.) In the following definition I characterise a class of algebras which model MC, of which the Lindenbaum algebra is an extreme case:

[7] As we have seen, in MC, as with other relevant logics, there are many different non-equivalent theorems, so T is not a singleton. $[p \to p]$ and $[q \to q]$ are distinct members of T.

[8] Brady proves that the theorems of MC form a prime filter by a metavaluation argument [5, p. 366–7].

DEFINITION 3 [BRADY ALGEBRAS] A *Brady algebra* is a *lattice* $\langle B, \leq, \&, \vee\rangle$ (in which $\&$ and $\vee$ are greatest lower bound and least upper bound for the order $\leq$ respectively) equipped with one unary operator $\sim$, one binary operator $\rightarrow$, and a distinguished set T of elements, where

- T is a *filter* (if $x \in T$ and $x \leq x'$ then $x' \in T$; and if $x \in T$ and $y \in T$ then $x \,\&\, y \in T$), which is also *prime* (if $x \vee y \in T$ then $x \in T$ or $y \in T$).
- $\sim$ is *order inverting* (if $x \leq y$ then $\sim y \leq \sim x$) and *period two* ($\sim\sim x = x$).
- $\rightarrow$ *reflects the lattice order inside* T ($x \leq y$ iff $x \rightarrow y \in T$).

In addition, the following conditions are satisfied:

* $(x \rightarrow y) \,\&\, (x \rightarrow z) \leq x \rightarrow (y \,\&\, z)$

* $(x \rightarrow z) \,\&\, (y \rightarrow z) \leq (x \vee y) \rightarrow z$

* $(x \rightarrow y) \,\&\, (y \rightarrow z) \leq x \rightarrow z$

* If $x \leq y$ and $v \leq w$ then $y \rightarrow v \leq x \rightarrow w$

* $x \rightarrow \sim y \leq y \rightarrow \sim x$

Once we have a Brady algebra $\mathcal{B}$, it is straightforward to define an *evaluation* map $[\![\cdot]\!]$ from formulas into values in $\mathcal{B}$. An evaluation is a map that respects each of MC's connectives: $[\![A \,\&\, B]\!] = [\![A]\!] \,\&\, [\![B]\!]$, $[\![A \rightarrow B]\!] = [\![A]\!] \rightarrow [\![B]\!]$, and so on, for the four connectives of propositional MC. With evaluations defined, we can prove the following theorem.

THEOREM 2 [BRADY ALGEBRAS MODEL MC] *There is an* MC*-Hilbert proof for* $X \Rightarrow A$ *if and only if, for each Brady algebra* $\mathcal{B}$ *and any valuation* $[\![\cdot]\!]$ *into* $\mathcal{B}$*, if* $[\![B]\!] \in T$ *for each member* B *of* X*, then* $[\![A]\!] \in T$.

Proof: The left-to-right direction (soundness) is an induction on the structure of the Hilbert proof for $\Rightarrow A$, while the right-to-left direction (completeness) appeals to a generalisation of the Lindenbaum algebra construction. The techniques are utterly standard, so I will cover this ground rather briskly.

For soundness, it suffices to check that every MC axiom lands in the truth filter for every Brady algebra, and that inference steps using the rules, or the *meta*-rule never lead us away from T, in a standard inductive argument. That the axioms land inside T is straightforward to verify, using the fact that the arrow in the algebra reflects the order inside the truth filter, and the particular features of Brady algebras in the definition. For example, axiom A4 lands in the truth filter, since in Brady algebras we have imposed the condition $([\![A]\!] \rightarrow [\![B]\!]) \,\&\, ([\![A]\!] \rightarrow [\![C]\!]) \leq [\![A]\!] \rightarrow ([\![B]\!] \,\&\, [\![C]\!])$. This gives $[\![(A \rightarrow B) \,\&\, (A \rightarrow C)]\!] \leq [\![A \rightarrow (B \,\&\, C)]\!]$, which means that $[\![(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow. A \rightarrow (B \,\&\, C)]\!] \in T$, as desired.

To verify that applications of the rules never lead us outside T, it suffices to note that the fact that T is a filter (so, if $[\![A]\!] \in T$ and $[\![B]\!] \in T$ then $[\![A \,\&\, B]\!] \in T$), that $\rightarrow$ reflects the order inside T (so, if $[\![A \rightarrow B]\!] \in T$ then $[\![A]\!] \leq [\![B]\!]$ and so, if $[\![A]\!] \in T$ we must have $[\![B]\!] \in T$ too), and Brady algebras satisfy the constraint that if $[\![A]\!] \leq [\![B]\!]$ and $[\![C]\!] \leq [\![D]\!]$ then $[\![B \rightarrow C]\!] \leq [\![A \rightarrow D]\!]$, which is exactly what we need to verify rule R3.

Finally, for the meta-rule, if our proof contains a sub-proof for $A, B \Rightarrow C$ and also contains $D \vee A$ and $D \vee B$, and we wish to infer $D \vee C$, then, by induction, we can assume not only that $[\![D \vee A]\!] \in T$ and $[\![D \vee B]\!] \in T$ but also that if $[\![A]\!]$ and $[\![B]\!]$ are in T, then so is $[\![C]\!]$, since we have already verified soundness for the sub-proofs of our proof. To verify that $[\![D \vee C]\!]$ is in T, we note that T is prime, so at least one of $[\![D]\!]$ and $[\![A]\!]$ are in T, and at least one of $[\![D]\!]$ and $[\![B]\!]$ are in T, too. This means that either $[\![D]\!]$ is in T (giving us $[\![D \vee C]\!] \in T$ as desired), or bothboth $[\![A]\!]$ and $[\![B]\!]$ are in T which (by the validity of the sub-proof for $A, B \Rightarrow C$) gives $[\![C]\!] \in T$, which again means that $[\![D \vee C]\!] \in T$, as we wished to prove. This completes the proof of soundness.

For completeness, we wish to show that if there is no proof for $X \Rightarrow A$ then there is some Brady algebra in which each member of X is in the truth filter but A is not. This is a *strong* completeness theorem, as we allow X to be non-empty. Showing that if $\not\Rightarrow A$ then there is some map into a Brady algebra where A lands outside the truth filter is a matter of showing that the Lindenbaum algebra is a Brady algebra. Proving strong completeness is not much more difficult, as the technique is a relatively standard construction. Instead of appealing to the Lindenbaum algebra, we construct *relativised* Lindenbaum algebra, in which we expand the truth filter to include each member of X, while A remains evaluated outside it. The main constraint on such a construction is ensuring that the expanded truth filter be *prime*. To do this, we start by noting that the pair $\langle X, \{A\}\rangle$, is *safe* in the sense that there is no MC-proof from the members of the left component to any disjunction of members of the right component. And we appeal to a standard result to the effect that (if the underlying logic satisfies certain conditions) any safe pair can be extended into a partition of the language $\langle L, R\rangle$, which remains safe.[9] Using this such a safe partition $\langle L, R\rangle$ in which X is a subset of L

[9]This is the Pair Extension Theorem (Theorem 5.17 in *An Introduction to Substructural Logics* [11, Ch. 5]). The key feature of MC is that the Hilbert consequence relation satisfies the distribution law, which we have proved above. It is proved by showing that if $\langle L, R\rangle$ is safe then for any formula B, either $\langle L \cup \{B\}, R\rangle$ or $\langle L, \{B\} \cup R\rangle$ must be safe, and for this we appeal to distribution.

and $A \in R$, then, we can construct a relativised Lindenbaum algebra, in which whose members are classes of formulas that are equivalent *given* L.

$$[B]_L = \{B' : L \Rightarrow B \to B' \text{ and } L \Rightarrow B' \to B\}$$

The ordering on the elements of the algebra is also L-relativised.

$$[B]_L \le [C]_L \text{ iff } L \Rightarrow B \to C$$

The natural truth filter in this algebra is determined by L:

$$T_L = \{[B]_L : L \Rightarrow B\}$$

We can define the operations $\&$, $\vee$, $\sim$ and $\to$ on the elements of this algebra just as we did in the Lindenbaum algebra, since the substitutivity of equivalents generalises: our proof of Theorem 1 straightforwardly generalises to show that if $L \Rightarrow B \to B'$ and $L \Rightarrow B' \to B$ then $L \Rightarrow C(B) \to C(B')$ and $L \Rightarrow C(B') \to C(B)$ for any context $C(-)$.

The properties of implication immediately ensure that the L-relativised Lindenbaum algebra is a lattice under $\&$ and $\vee$, and that T_L is a filter. The fact that $\langle L, R\rangle$ is a safe partition of the language ensures that T_L is *prime*. If $[B \vee C]_L \in T_L$ then $B \vee C \in L$ (L must contain all of its MC-consequences, since if one of its consequences were in R and not L, the partition would not be safe.) If neither B nor C were in L, then they would both be in R, which again would ensure that $\langle L, R\rangle$ is not safe, since an element of L (namely $B \vee C$) would MC-entail a disjunction of members of R (namely, $B \vee C$). So, one of B and C are in L, and hence, one of $[B]_L$ and $[C]_L$ are in T_L.

The negation operator on our algebra is order inverting (since if $L \Rightarrow B \to C$ it folows that $L \Rightarrow {\sim}C \to {\sim}B$) and period two, since MC-theorems (here, ${\sim}{\sim}B \to B$ and $B \to {\sim}{\sim}B$) can be proved, vacuously, from L.

The conditional operator reflects the lattice order inside T_L in this algebra, since this is how the ordering was defined: $[B]_L \le [C]_L$ iff $L \Rightarrow B \to C$ which holds if and only if $[B]_L \to [C]_L \in T_L$.

For the other conditions on Brady algebras, we reason as we did for the negation conditions: these correspond to MC-theorems, so they hold in the L-relativised Lindenbaum algebra, too. That is, all except the arrow ordering condition to the effect that if $[B]_L \le [B']_L$ and $[C]_L \le [C']_L$ then $[B']_L \to [C]_L \le [B]_L \to [C']_L$. This corresponds, not to an axiom, but to rule R3. To verify that it holds in the L-relativised algebra, we reason as follows. If $[B]_L \le [B']_L$ and $[C]_L \le [C']_L$ then we have $L \Rightarrow B \to B'$ and $L \to C \to C'$. So, we can extend a proof from L, which reaches to the conclusions $B \to B'$ and $C \to C'$, by appealing to R3, the conclusion $(B' \to C) \to (B \to C')$ which ensures that $L \Rightarrow (B' \to C) \to (B \to C')$, and hence, that $[B']_L \to [C]_L \le [B]_L \to [C']_L$, as desired.

This completes the verification that the L-relativised Lindenbaum algebra is a Brady algebra. The completeness proof is then immediate, if there is no proof for $X \Rightarrow A$ then we find a safe partition $\langle L, R\rangle$ extending $\langle X, \{A\}\rangle$, and the L-relativised Lindenbaum algebra will provide a counterexample, given the homophonic valuation, where we set $[\![B]\!] = [B]_L$. This is indeed a valuation function (since the operators reflect the connectives, by design), and it sends every member of X into the truth filter T_L, but it leaves A outside, so it is also a counterexample to the validity of the argument. The strong completeness theorem, then, is proved. ■

It follows that Brady lattices model MC. So, we can, therefore, use them to shed light on MC's distinctive features. That is my plan for the rest of this paper.

4 ILLUSTRATING THE DISTINCTIVES OF MC

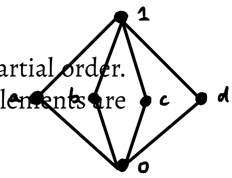
Recall, the features of MC that I would like to better understand are these:

1. MC takes the distribution of lattice connectives to hold as a *rule*, but not as a provable *conditional*.
2. MC has conjunctive syllogism, but no contraction.
3. MC excludes the Church and Ackermann constants.
4. MC excludes fusion.

We will start with the first feature, since it is relatively straightforward. There are many examples of lattices that fail to be distributive. In any Brady algebra in which $x \mathbin{\&} (y \vee z) \not\leq (x \mathbin{\&} y) \vee (x \mathbin{\&} z)$ (that is, any such algebra in which the underlying lattice is not distributive) then the corresponding conditional $p \mathbin{\&} (q \vee r) \to (p \mathbin{\&} q) \vee (p \mathbin{\&} r)$ will also fail, since the conditional reflects the order inside the truth filter. We then need to show how some non-distributive lattices can be equipped with a conditional operator, a negation, and a prime truth filter satisfying all the conditions of a Brady algebra. We will do just this in the next section, shedding light on how, exactly, distribution may be present in one sense and absent in another.

4.1 HOW DISTRIBUTION IS BOTH PRESENT AND ABSENT

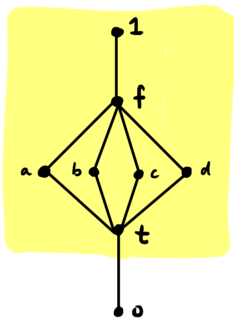
Non-distributive lattices are familiar structures. The Hasse diagram[10] to the right depicts a six-element lattice, O_6, which fails to



[10] You read a Hasse diagram from bottom-to-top, with the lines indicating the partial order. Here, 0 is under a, b, c and d, which are all incomparable with one another, but all elements are under 1.

be distributive. In this lattice, $a \ \& \ (b \vee c) = a \ \& \ 1 = a$, while $(a \ \& \ b) \vee (a \ \& \ c) = 0 \vee 0 = 0$.

Can we make O_6 into a Brady lattice? If we define the operator $\sim$ on the algebra by setting $\sim 1 = 0$, $\sim a = b$ and $\sim c = d$, and imposing the condition that $\sim\sim x = x$ for every x, then this operation is indeed order inverting and of period two.[11] This satisfies the negation-only conditions for a Brady lattice. However, O_6 cannot be made into a Brady lattice, since there is no non-trivial prime filter on O_6. Any truth filter on a Brady algebra of size ≥ 2 must exclude *some* member of the algebra. (Pick two elements x, y where $x \not\leq y$. We have $x \rightarrow y \notin T$.) The only non-trivial filters in O_6 are $\{1\}, \{1, a\}, \{1, b\}, \{1, c\}$ and $\{1, d\}$, since if F is a filter and (for example), $a, b \in F$ then $a \ \& \ b = 0 \in F$, and hence, F excludes no element of O_6 (since once a filter contains the bottom element 0, it excludes nothing.). However, none of these filters is prime. For any filter on the list, choose two elements from among a, b, c, d that are *not* in the filter. The disjunction of those two elements is 1, which *is* in the filter, and so, we have a counterexample to primality. O_6, therefore, is a non-starter for constructing a Brady lattice.



Here is the structure in which we *do* have a prime filter in a non-distributive lattice. L_8, to the right, an eight-element lattice, is found by extending O_6 with a new top element and new bottom element. In this algebra, we can assign negation in a natural way, setting $\sim f = t$. Now it is no longer an ortholattice, but remains a lattice with an order inverting involution. There is a trivial way to assign values for implication: $x \rightarrow y = 1$ if $x \leq y$ and $x \rightarrow y = 0$ otherwise. The filter, set in yellow, consisting of all of the elements other than 0, is prime, since if $x \vee y \in T$ then $t \leq x \vee y$, and so $t \leq x$ or $t \leq y$, which means $x \in T$ or $y \in T$.[12] The implication, as evaluated with our crude all-or-nothing values, clearly reflects the order in T, since $x \rightarrow y \in T$ iff $x \rightarrow y = 1$ (since $0 \notin T$, and 0 and 1 are the only values $x \rightarrow y$ can take), iff $x \leq y$.

It is not too hard to show that, in fact, this choice of T and the definition of the implication operation on L_8 makes the structure a Brady lattice. Instead of working through the verification in this particular case, I will pause to note that the construction turns out to be quite general. Let's construct many Brady algebras in one go:

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[11] In fact, it is an *orthonegation*, and O_6 with this negation operator is an ortholattice, which is a bounded lattice with a negation operator $\sim$ where $x \ \& \ \sim x$ is the least element 0, and $x \vee \sim x$ is the greatest element 1; $\sim\sim x = x$, and whenever $x \leq y$ then $\sim y \leq \sim x$.

[12] There is one other prime filter on the lattice L_8, the singleton $\{1\}$. The choice of the more generous truth filter in my example is purely a matter of taste.

THEOREM 3 *If L is a bounded lattice with an order inverting negation operation of period 2, with a non-trivial prime filter, T, then if we define the extra operation $\rightarrow$ on L by setting $x \rightarrow y = 1$ iff $x \leq y$ and $x \rightarrow y = 0$ otherwise, then the result is a Brady lattice.*

Proof: The structure is a lattice with bounds 0 and 1, with an order inverting, period-two negation, and by definition, $\rightarrow$ reflects the lattice order inside the non-trivial prime filter T. It remains to show that the five remaining Brady lattice conditions on $\rightarrow$ are satisfied. I take them in turn.

To show that $(x \rightarrow y) \,\&\, (x \rightarrow z) \leq x \rightarrow (y \,\&\, z)$, it suffices to rule out the case where $x \rightarrow y$ and $x \rightarrow z$ are 1 while $x \rightarrow (y \,\&\, z)$ is 0. But that would mean that $x \leq y$ and $x \leq z$ but $x \not\leq y \,\&\, z$, which violates the lattice conditions on L. The same reasoning applies for the second condition, concerning disjunction.

To show that $(x \rightarrow y) \,\&\, (y \rightarrow z) \leq x \rightarrow z$, it suffices to show that we cannot have $x \rightarrow y = 1$ and $y \rightarrow z = 1$ and $x \rightarrow z = 0$, but this is immediate, given the transitivity of $\leq$.

For the fourth condition, if $x \leq y$ and $v \leq w$ then if $y \leq v$ then $x \leq w$, which means $x \rightarrow w = 1$ and hence $y \rightarrow v \leq x \rightarrow w$. On the other hand, if $y \not\leq v$ then $y \rightarrow v = 0$ and then certainly, $y \rightarrow v \leq x \rightarrow w$, regardless.

Finally, to show that $x \rightarrow {\sim}y \leq y \rightarrow {\sim}x$, suppose that $x \leq {\sim}y$, in which case we have $x \rightarrow {\sim}y = 1$. Since negation is order inverting we have ${\sim}{\sim}y \leq {\sim}x$, and since it has period two, we have $y \leq {\sim}x$, and then, $y \rightarrow {\sim}x = 1$, too. On the other hand, if $x \not\leq {\sim}y$, then $x \rightarrow {\sim}y = 0$, and then certainly $x \rightarrow {\sim}y \leq y \rightarrow {\sim}x$, regardless. ■

So, in particular, L_8 is a Brady lattice. It fails to be distributive at the level of the lattice ordering, since $a \,\&\, (b \vee c) = a \,\&\, f = a$, while $(a \,\&\, b) \vee (a \,\&\, c) = t \vee t = t$, and $a \not\leq t$.

However, given that the truth filter T on L_8 is *prime*, distributivity holds in a coarser way: if $x \,\&\, (y \vee z) \in T$ then $(x \,\&\, y) \vee (x \,\&\, z) \in T$, no matter what values x, y and z take. We can see that although $(a \,\&\, b) \vee (a \,\&\, c)$ (which is t) falls below $a \,\&\, (b \vee c)$ (which is a), it does not fall so far as to go outside the truth filter. In this way, we have a concrete representation of the two notions of distribution at work in MC.

So, we have seen that not *every* lattice contains a non-trivial prime filter (O_6 is but one example of many), but some non-distributive lattices do contain one, such as L_8. The behaviour of the filter T in L_8 illustrates a general phenomenon. We can see here that conjunction and disjunction are T-functional, in the following sense: $x \vee y \in T$ iff $x \in T$ or $y \in T$, and $x \,\&\, y \in T$ iff $x \in T$ and $y \in T$. We can take the quotient of our lattice with respect to membership in T, and get a two-element lattice, which is, of necessity, distributive.[13] Not all lattices (such as O_6) allow for such an operation, since any non-trivial filters on O_6 fail to be prime.

It is worth reflecting on the other distinctive features of MC, to see whether this construction on bounded lattices can shed further light on MC. First, we should notice that the evaluation conditions for the conditional validate the contraction rule as well as conjunctive syllogism (since $x \to (x \to y) = 1$ iff $x \leq x \to y$ which means that either $x = 0$ (in which case $x \to y = 1$, or $x \neq 0$, in which case since $x \leq x \to y$ we must have $x \to y = 1$ regardless. So, $x \to (x \to y) \leq x \to y$ in these algebras), so *this* construction gives no insight into the difference between conjunctive syllogism and contraction. Second, this construction works only on bounded lattices (we use 0 and 1 essentially in the semantics of the conditional), so there is no insight into the absence of Church constants.

Now consider fusion, which is a conjunction-like operator that stands to $\to$ as regular conjunction stands to material implication in classical or intuitionistic logic. The defining conditions are that $\Rightarrow A \circ B \to C$ iff $\Rightarrow A \to (B \to C)$. In a Brady algebra, a fusion operator is at least *implicitly* present if we can find some way to *define* $x \circ y$ such that $x \circ y \leq z$ iff $x \leq y \to z$. In any Brady algebra defined in the construction of Theorem 3 the fusion operator is, in fact, definable. Set $x \circ y = y$ when $y \neq 0$, and $x \circ 0 = 0$. It is straightforward to show that the fusion rules are satisfied: if $x \circ y \leq z$ then either $x = 0$—and hence, $x \leq y \to z$ as desired—or $x \neq 0$—and hence, $x \circ y = y \leq z$ and hence, $y \to z = 1$, so $x \leq y \to z$, as desired. Conversely, if $x \leq y \to z$, either $x = 0$—in which case $x \circ y = 0$ and $x \circ y \leq z$—or $x \neq 0$—and then $x \leq y \to z$ means that $y \to z = 1$ (since conditionals are either 0 or 1), which means that $y \leq z$, but if $x \neq 0$ then $x \circ y = y$ and $x \circ y \leq z$ as desired. So, in all of these models fusion is *definable*, and so, the models do not give us any particular insight into how we might have a logic in which fusion is absent.

Furthermore, in the lattice L_8 (and in any other finite examples we can construct) the truth filter is *principal*: T has the form $\{x : a \leq x\}$ for some given item a. (In any finite algebra, we take the conjunction of all of the elements in T: this is in T, and is our target value a.) This element a can be the semantic value of the Ackermann constant t, since if we assign t this value in every valuation, it is straight forward to verify the rules $A \Rightarrow t \to A$ and its converse, $t \to A \Rightarrow A$. Not every prime filter in a lattice is principal, but it would help to see exactly how such a thing could arise in a concrete example, so I will turn to this in the next section, and then leave the final section to discuss models without the Church

[13] This operation of taking the quotient does not preserve $\sim$ or $\to$. Notice that $\sim$ on L_8 is not T-functional: ~ 1 is 0, which is outside T, while $\sim f$ is t, which remains inside. The same goes for implications.

constants and without fusion, and in which we reject contraction while retaining conjunctive syllogism.

4.2 A TRUTH FILTER WITHOUT t

The aim is to find a concrete Brady algebra in which the truth filter is *prime*, but not principal: in such an algebra, there is no way to interpret the Ackermann constant t. As we have seen above, this means that our target is an *infinite* algebra, whose truth filter is closed under finitary conjunctions (as all filters must be), but not arbitrary conjunctions. In this section, I will construct a simple countable Brady algebra in which the truth filter is not principal.[14]

To start, recall that algebras arising out of possible worlds semantics take each proposition to be a set of worlds, where conjunction is represented by intersection and disjunction by union. The *true* propositions, from the point of view of a given world w, then, are the sets that contain w. If this family of propositions contains *every* set of worlds, then indeed, the filter of all true propositions contains a smallest member $\{w\}$. If, on the other hand, not every set of worlds counts as a proposition, then perhaps our truth filter contains no minimal element.

Here is one example of such a structure. Take ω, the set of natural numbers, and consider the set of *cofinite* subsets of ω. (A set α is *cofinite* in ω if its complement $\omega\backslash\alpha$ is finite.) The cofinite sets in ω are closed under union and under intersection. They form a (distributive) lattice under the subset ordering $\subseteq$, and this lattice contains a greatest element ω, but no least element. The set consisting of all of those cofinite subsets of ω containing 0 is a filter in this lattice (if α and β both contain 0 then so does $\alpha \cap \beta$, and if $\alpha \subseteq \alpha'$, then α' contains 0, too), and it is prime (if $\alpha \cup \beta$ contains 0, then so must one of α and β). However, the filter is not principal. For any cofinite α containing 0, we can find the first n (other than 0) in α and $\alpha\backslash\{n\}$ is *also* cofinite and in the filter, and is a proper subset of α.

So, this structure can give us some insight into how we might have a non-principal truth filter. The target value $\{0\}$ cannot be precisely individuated by any one proposition. However, it is not quite the structure we need to apply Theorem 3 to simply make a Brady lattice. The family of cofinite subsets of ω is not a bounded lattice, and we cannot define on it an order inverting negation of period two. (We could do so only if, from the point of view of the order, the structure is up-down symmetric, and the cofinite subsets of ω ordered by subsethood are anything but up-down symmetric.) However, it is easy enough to use what we have learned as

[14] Of course, the Lindenbaum algebra for MC is another example of such an algebra, but recall, our aim is to find some independent grasp on the distinctive properties of MC. Simply appealing to the algebra arising out of the axiomatic system of MC and leaving it at that gives little *independent* insight. After all, had we included the truth constant t in the axiomatisation of MC, then the truth filter in its Lindenbaum algebra would have been principal.

the basis for defining such a structure.

Consider the set of all pairs $\langle \alpha, \epsilon \rangle$ where α is either a *cofinite* subset of ω, or is $\{\,\}$, and ϵ is either a *finite* subset of ω, or is ω itself. This structure forms a lattice, under the pairwise subset order: $\langle \alpha, \epsilon \rangle \leq \langle \alpha', \epsilon' \rangle$ iff $\alpha \subseteq \alpha'$ and $\epsilon \subseteq \epsilon'$, and joint and meet are pairwise union and intersection respectively. The lattice is *bounded* with minimal element $\langle \{\,\}, \{\,\} \rangle$ and maximal element $\langle \omega, \omega \rangle$. In addition, we now have a natural order-inverting negation of period two, by setting $\sim\langle \alpha, \epsilon \rangle = \langle \omega \backslash \epsilon, \omega \backslash \alpha \rangle$. (That was the point of using pairs of cofinite and finite sets.) The set T of all pairs $\langle \alpha, \epsilon \rangle$ where $0 \in \alpha$ form a non-trivial prime filter in this lattice, so the conditions of Theorem 3 apply, and with the conditional defined in the usual way, the result is a Brady lattice.[15] So, we have constructed an independent example of how we might have a Brady lattice in which the Ackermann constant t is not only absent, but in which there is no semantic value that such a constant could take.

4.3 CHURCH CONSTANTS, CONTRACTION AND FUSION

The examples we have seen so far have not provided any insight into the absence of Church constants, how *contraction* might fail while preserving conjunctive syllogism, and the absence of fusion in MC. Theorem 3 has its uses in constructing Brady lattices, but these examples obscure other important features of MC. What can we say about these remaining distinctives?

For the absence of Church constants, it is natural to think of Meyer and Slaney's *Abelian Logic* [9]. Abelian logic is a contra-classical substructural logic, extending the multiplicative and additive fragment of linear logic with the distribution of the additive (lattice) connectives (so, unlike MC, abelian logic validates the distribution axiom), and containing the non-classical principle

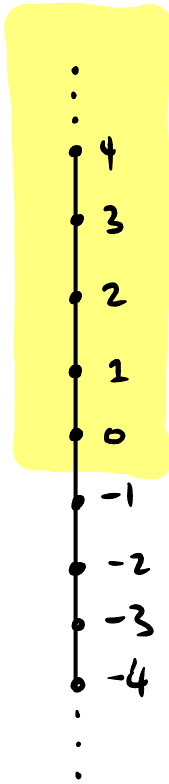
$$A \to B \to B \to A$$

(remember, without parentheses, the arrow associates to the left). This can is a generalisation of $A \to \bot \to \bot \to A$, which is a notational variant of $\sim\sim A \to A$, if we understand $\sim A$ as $A \to \bot$. In abelian logic, *any* proposition B might be understood as a '*falsum*', defining a kind of order inverting involutive negation. It follows immediately that the Church constant $\top$ (a proposition for which $A \to \top$ always holds) trivialises abelian logic, if present, since $A \to \top \to \top$ must hold, and by *modus ponens*, we have A, for any A whatsoever. Abelian logic resists

[15] Notice that the traditional four-valued semantics for first-degree entailment embeds nicely in this Brady algebra, with the values $\langle \omega, \omega \rangle$ for `true`, $\langle \{\,\}, \{\,\} \rangle$ for `false`, $\langle \omega, \{\,\} \rangle$ for `both` and $\langle \{\,\}, \omega \rangle$ for `neither`. Both $\langle \omega, \{\,\} \rangle$ and $\langle \{\,\}, \omega \rangle$ are fixed-points for negation, while the other two values are the top and bottom elements of the algebra.

the Church constant $\top$ and dually, also the Church constant $\bot$, since $\bot \to \bot$ is equivalent to $\top$ since $\bot \to (A \to \bot)$ entails $A \to (\bot \to \bot)$, by permutation.

The integers $\mathbb{Z}$ under the natural ordering forms the standard algebraic model for abelian logic. By design, there is no greatest and no least element, and the implication connective is modelled by subtraction $[\![A \to B]\!] = [\![B]\!] - [\![A]\!]$. Here, the truth filter is the set of non-negative numbers, since $[\![A \to B]\!] \geq 0$ iff $[\![A]\!] \leq [\![B]\!]$, and we can see that $[\![A \to B \to B \to A]\!] = 0$, so the generalised double negation axiom is always inside the truth filter (if only barely).



This model for abelian logic is a natural way to understand logics without Church constants. This model does not merely fail to assign a value to a Church constant $\top$ or $\bot$—there is no value in the algebra to assign any formula with the desired properties. No semantic value is at the limit of truth or falsity, as they extend indefintely in either direction.

Alas, $\mathbb{Z}$, understood in this way, is *not* a model of MC. We can see that the conjunctive syllogism axiom A10 fails in this model. If we assign $[\![p]\!] = 1$, $[\![q]\!] = 2$ and $[\![r]\!] = 3$, then $[\![p \to q]\!] = [\![q \to r]\!] = -1$ and so $[\![(p \to q) \,\&\, (q \to r)]\!] = -1$, but $[\![p \to r]\!] = -2$, so $[\![(p \to q) \,\&\, (q \to r) \to (p \to r)]\!] = -1$, and this value falls outside of the truth filter. This algebra fails to model MC.

Nonetheless, we can revise the evaluation conditions for the conditional to find values that are more amenable to MC. We see in this counterexample to conjunctive syllogism that the problem arises with the values for *false* conditionals. If we cap the value of false conditionals at -1, not allowing falsity to pile up any more than that, then the conditional respects the constraints required for a Brady algebra.

$$x \to y = \begin{cases} y - x & \text{if } x \leq y \\ -1 & \text{otherwise} \end{cases}$$

Let us call the resulting structure $\mathbb{BZ}$, for *Brady's Integers*. It is straightforward to check that $\to$ reflects the ordering in $T = \{x : 0 \leq x\}$ on $\mathbb{BZ}$, since $0 \leq x \to y$ if and only if $x \leq y$. T is indeed a prime filter on $\mathbb{Z}$ (as *every* filter on $\mathbb{Z}$ ordered in the standard way is prime). The negation conditions are satisfied if we define ${\sim}x$ to be $-x$ (this is order inverting and period two, and $x \to {\sim}y = y \to {\sim}x$, since $(-y) - x = (-x) - y$, and $x \leq -y$ iff $y \leq -x$, for every x and y.

The conditions on implication in Brady lattices are easy to verify for $\mathbb{BZ}$. We will check the ordering condition (if $x \leq y$ and $v \leq w$ then $y \to v \leq x \to w$) and conjunctive syllogism, and the others are verified just as simply. First the ordering condition. Let's suppose that $x \leq y$ and $v \leq w$. Then, if $y \leq v$ then

$y \to v = v - y \leq w - x = x \to w$ as desired. On the other hand, if $y \not\leq v$ then $y \to v = -1$ which is the lowest possible value for any conditional, and so, $y \to v \leq x \to w$.

Second, the conjunctive syllogism condition: Suppose $x \leq y$ and $y \leq z$. Then $(x \to y) \,\&\, (y \to z)$ is the smaller of $y - x$ and $z - x$. Since $x \leq y \leq z$, this is less than or equal to $z - x$, which is $x \to z$, as desired. Suppose, on the other hand, that either $x \not\leq y$ or $y \not\leq z$. Then $(x \to y) \,\&\, (y \to z) = -1$, which is the smallest value any conditional can take, and so, it is less than or equal to $x \to z$, as desired.

The other verifications are just as simple, so I will declare that $\mathbb{BZ}$ is, therefore, a Brady algebra, and so, is a model for MC. It is a model that is robustly resistant to the definition of the Church constants, since it has no truest truth and no falsest falsehood.

$\mathbb{BZ}$ illustrates the remaining distinctive features of MC, too. As we have seen, the revised clause for the conditional was enough to validate conjunctive syllogism. The variation, however, does not validate contraction. If we assign $[\![A]\!] = -2$ and $[\![B]\!] = -1$, then $[\![A \to B]\!] = -1 - (-2) = 1$, while $[\![A \to. A \to B]\!] = 1 - (-2) = 3$, so this model is a counterexample to the inference from $A \to. A \to B$ to $A \to B$. Similarly, the depth-irrelevant *modus ponens* formula $B \,\&\, (B \to A) \to A$ is refuted when we assign A and B the same values: $[\![B \,\&\, (B \to A)]\!] = -1$ while $[\![A]\!] = -2$ so the whole formula has value -1, which is outside the truth filter.

We have a model in which contraction (including the *modus ponens* formula) fails but conjunctive syllogism is verified. One diagnosis of this fact, consonant with the general motivation for MC, is that in this model conditional formulas are somewhat special: they have only a limited range of values they can take (never dipping below -1). This means that there are some formulas that are not equivalent to any conditional formulas. In linear logic, or in the relevant logics R and RW, any formula A is equivalent to $A \to A \to A$, and so, if conjunctive syllogism were to hold, then we could infer an instance

$$(A \to A \to A) \,\&\, (A \to B) \to. A \to A \to B$$

which, would then entail

$$A \,\&\, (A \to B) \to. A \to A \to B$$

(since $A \to A \to A$ is equivalent to A), and then, since we can in linear logic, and stronger logics, prove $A \to A \to B \to B$,[16] we have the *modus ponens* axiom $A \,\&\, (A \to B) \to B$. In MC this argument is blocked from the beginning.

[16] Remember, conditionals associate to the left.

Conditional formulas (understood as statements of meaning containment) may have a special semantic status. There is no guarantee that every statement may be equivalent to a conditional. In each of the Brady algebras discussed in this paper, in fact, conditional formulas take special values. In the products of Theorem 3, conditionals have the value 0 or 1. In this algebra on $\mathbb{Z}$ conditionals are limited to never take a value lower than -1. In each case, there are algebra elements that are never fated to be the values of conditionals.

The final distinctive feature of MC to explore is the absence of fusion. Recall, a fusion operator is definable on a structure if we can find for each x and y a value $x \circ y$ such that, for every z, $x \circ y \leq z$ if and only if $x \leq y \to z$. There cannot be such an operator on $\mathbb{BZ}$, given the values of conditional formulas. If there were such an operator, we would have, for example, for every z, $-2 \circ 1 \leq z$ iff $-2 \leq 1 \to z$. But $-2 \leq 1 \to z$ *always* (whatever value z takes), so we must have $-2 \circ 1 \leq z$ for every z. But this is impossible, since there is no least element in $\mathbb{BZ}$. So, fusion is not definable in $\mathbb{BZ}$. The intrinsic structure of the ordering (here, having no least elements) and the distinctive behaviour of conditionals (having values restricted to -1 and above) conspire to ensure that there is no way to define a fusion connective. The structure $\mathbb{BZ}$ prohibits it.

* * *

I started this paper describing the different ways that I did not understand or appreciate the distinctive features of Brady's logic MC of meaning containment. Having spent the time to reflect on those properties and constructing concrete models that exhibit each of those features, I have come to understand some of these features a little more. If this helps others engage with Brady's contributions to logic, this paper will have done its work.

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