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RELEVANCE LOGIC

1 INTRODUCTION

1.1 Delimiting the topic

The title of this piece is not 'A Survey of Relevance Logic'. Such a project was impossible in the mid 1980s when the first version of this article was published, due to the development of the field and even the space limitations of the *Handbook*. The situation is if anything, more difficult now. For example Anderson and Belnap and Dunn's two volume [1975, 1992] work *Entailment: The Logic of Relevance and Necessity*, runs to over 1200 pages, and is their summary of just some of the work done by them and their co-workers up to about the late 1980s. Further, the comprehensive bibliography (prepared by R. G. Wolf) contains over 3000 entries in work on relevance logic and related fields.

So, we need some way of delimiting our topic. To be honest the fact that *we* are writing this is already a kind of delimitation. It is natural that you shall find emphasised here the work that we happen to know best. But still rationality demands a less subjective rationale, and so we will proceed as follows.

Anderson [1963] set forth some open problems for his and Belnap's system **E** that have given shape to much of the subsequent research in relevance logic (even much of the earlier work can be seen as related to these open problems, e.g. by giving rise to them). Anderson picks three of these problems as major: (1) the admissibility of Ackermann's rule γ (the reader should not worry that he is expected to already know what this means), (2) the decision problems, (3) the providing of a semantics. Anderson also lists additional problems which he calls 'minor' because they have no 'philosophical bite'. We will organise our remarks on relevance logic around three major problems of Anderson. The reader should be told in advance that each of these problems are closed (but of course 'closed' does not mean 'finished'—closing one problem invariably opens another related problem). This gives then three of our sections. It is obvious that to these we must add an introduction setting forth at least some of the motivations of relevance logic and some syntactical specifications. To the end we will add a section which situates work in relevance logic in the wider context of study of other logical systems, since in the recent years it has become clear that relevance logics fit well among a wider class of 'resource-conscious' or 'substructural' logics [Schroeder-Heister and Došen, 1993, Restall, 2000] [and cite the S–H article in this volume]. We thus have the following table of contents:

We should add a word about the delimitation of our topic. There are by now a host of formal systems that can be said with some justification to be 'relevance logics'. Some of these antedate the Anderson–Belnap approach, some are more recent. Some have been studied somewhat extensively, whereas others have been discussed for only a few pages in some journal. It would be impossible to describe all of these, let alone to assess in each and every case how they compare with the Anderson–Belnap approach. It is clear that the Anderson–Belnap-style logics have been the most intensively studied. So we will concentrate on the research program of Anderson, Belnap and their co-workers, and shall mention other approaches only insofar as they bear on this program. By way of minor recompense we mention that Anderson and Belnap [1975] have been good about discussing related approaches, especially the older ones.

Finally, we should say that our paradigm of a relevance logic throughout this essay will be the Anderson–Belnap system **R** or relevant implication (first devised by Belnap—see [Belnap, 1967a, Belnap, 1967b] for its history) and not so much the Anderson–Belnap favourite, their system **E** of entailment. There will be more about each of these systems below (they are explicitly formulated in Section 1.3), but let us simply say here that each of these is concerned to formalise a species of implication (or the conditional—see Section 1.2) in which the antecedent suffices *relevantly* for the consequent. The system **E** differs from the system **R** primarily by adding necessity to this relationship, and in this **E** is a modal logic as well as a relevance logic. This by itself gives good reason to consider **R** and not **E** as the paradigm of a relevance logic.[1]

[1]It should be entered in the record that there are some workers in relevance logic who consider both **R** and **E** too strong for at least some purposes (see [Routley, 1977], [Routley *et al.*, 1982], and more recently, [Brady, 1996]).

1.2 Implication and the Conditional

Before turning to matters of logical substance, let us first introduce a framework for grammar and nomenclature that is helpful in understanding the ways that writers on relevance logic often express themselves. We draw heavily on the 'Grammatical Propaedeutic' appendix of [Anderson and Belnap, 1975] and to a lesser extent on [Meyer, 1966], both of which are very much recommended to the reader for their wise heresy from logical tradition.

Thus logical tradition (think of [Quine, 1953]) makes much of the grammatical distinction between 'if, then' (a connective), and 'implies' or its rough synonym 'entails' (transitive verbs). This tradition opposes

1. If today is Tuesday, then this is Belgium

to the pair of sentences

2. 'Today is Tuesday' implies 'This is Belgium',
3. That today is Tuesday implies that this is Belgium.

And the tradition insists that (1) be called a *conditional*, and that (2) and (3) be called *implications*.

Sometimes much philosophical weight is made to rest on this distinction. It is said that since 'implies' is a verb demanding nouns to flank it, that implication must then be a relation between the objects stood for by those nouns, whereas it is said that 'if, then' is instead a connective combining that implication (unlike 'if, then') is really a metalinguistic notion, either overtly as in (2) where the nouns are names of sentences, or else covertly as in (3) where the nouns are naming propositions (the 'ghosts' of linguistic entities). This last is then felt to be especially bad because it involves ontological commitment to propositions or some equally disreputable entities. The first is at least free of such questionable ontological commitments, but does raise real complications about 'nested implications', which would seem to take us into a meta-metalanguage, etc.

The response of relevance logicians to this distinction has been largely one of 'What, me worry?' Sometime sympathetic outsiders have tried to apologise for what might be quickly labelled a 'use–mention confusion' on the part of relevance logicians [Scott, 1971]. But 'hard-core' relevance logicians often seem to luxuriate in this 'confusion'. As Anderson and Belnap [1975, p. 473] say of their 'Grammatical Propaedeutic': "the principle aim of this piece is to convince the reader that it is philosophically respectable to 'confuse' implication or entailment with the conditional, and indeed philosophically suspect to harp on the dangers of such a 'confusion'. (The suspicion is that such harpists are plucking a metaphysical tune on merely grammatical strings.)"

The gist of the Anderson–Belnap position is that there is a generic conditional-implication notion, which can be carried into English by a variety of grammatical constructions. Implication itself can be viewed as a connective requiring prenominalisation: 'that __ implies that __', and as such it nests. It is an incidental feature of English that it favours sentences with main subjects and verbs, and 'implies' conforms to this reference by the trick of disguising sentences as nouns by prenominalisation. But such grammatical prejudices need not be taken as enshrining ontological presuppositions.

Let us use the label 'Correspondence Thesis' for the claim that Anderson and Belnap come close to making (but do not actually make), namely, that *in general* there is nothing other than a purely grammatical distinction between sentences of the forms

4. If A, then B, and
5. That A implies that B.

Now undoubtedly the Correspondence Thesis overstates matters. Thus, to bring in just one consideration, [Castañeda, 1975, pp. 66 ff.] distinguishes 'if A then B' from 'A only if B' by virtue of an essentially pragmatic distinction (frozen into grammar) of 'thematic' emphases, which cuts across the logical distinction of antecedent and consequent. Putting things quickly, 'if' introduces a sufficient condition for something happening, something being done, etc. whereas 'only if' introduces a necessary condition. Thus 'if' (by itself or prefixed with 'only') always introduces the state of affairs thought of as a condition for something else, then something else being thus the focus of attention. Since 'that A implies that B' is devoid of such thematic indicators, it is not equivalent at *every* level of analysis to either 'if A then B' or 'A only if B'.

It is worth remarking that since the formal logician's $A \to B$ is equally devoid of thematic indicators, 'that A implies that B' would seem to make a better reading of it than either 'if A then B' or 'A only if B'. And yet it is almost universally rejected by writers of elementary logic texts as even an acceptable reading.

And, of course, another consideration against the Correspondence Thesis is produced by notorious examples like Austin's

6. There are biscuits on the sideboard if you want some,

which sounds very odd indeed when phrased as an implication. Indeed, (6) poses perplexities of one kind or another for any theory of the conditional, and so should perhaps best be ignored as posing any special threat tot he Anderson and Belnap account of conditionals. Perhaps it was Austin-type examples that led Anderson and Belnap [1975, pp. 491–492] to say "we think every use of 'implies' or 'entails' as a connective can be replaced by a suitable 'if-then'; however, the converse may not be true". They go on to say "But with reference to the uses in which we are primarily interested, we feel free to move back and forth between 'if-then' and 'entails' in a free-wheeling manner".

Associated with the Correspondence Thesis is the idea that just as there can be contingent conditionals (e.g. (1)), so then the corresponding implications (e.g. (3)) must also be contingent. This goes against certain Quinean tendencies to 'regiment' the English word 'implies' so that it stands only for *logical* implication. Although there is no objection to thus giving a technical usage to an ordinary English word (even requiring in this technical usage that 'implication' be a metalinguistic relation between sentences), the point is that relevance logicians by and large believe we are using 'implies' in the ordinary non-technical sense, in which a sentence like (3) might be true without there being any logical (or even necessary) implication from 'Today is Tuesday' to 'This is Belgium'.

Relevance logicians are not themselves free of similar regimenting tendencies. Thus we tend to differentiate 'entails' from 'implies' on precisely the ground that 'entails', unlike 'implies', stands only for *necessary* implication [Meyer, 1966]. Some writings of Anderson and Belnap even suggest a more restricted usage for just *logical* implication, but we do not take this seriously. There does not seem to be any more linguistic evidence for thus restricting 'entails' than there would be for 'implies', though there may be at least more excuse given the apparently more technical history of 'entails' (in its logical sense—cf. The OED).

This has been an explanation of, if not an apology for, the ways in which relevance logicians often express themselves. but it should be stressed that the reader need not accept all, or any, of this background in order to make sense of the basic aims of the relevance logic enterprise. Thus, e.g. the reader may feel that, despite protestations to the contrary, Anderson, Belnap and Co. are hopelessly confused about the relationships among 'entails', 'implies', and 'if-then', but still think that their system **R** provides a good formalisation of the properties of 'if-then' (or at least 'if-then relevantly'), and that they system **E** does the same for some strict variant produced by the modifier 'necessarily'.

One of the reasons the recent logical tradition has been motivated to insist on the fierce distinction between implications and conditionals has to do with the awkwardness of reading the so-called 'material conditional' $A \to B$ as corresponding to any kind of implication (cf. [Quine, 1953]).

The material conditional $A \to B$ can of course be defined as $\neg A \vee B$, and it certainly does seem odd, modifying an example that comes by oral tradition from Anderson, to say that:

7. Picking a guinea pig up by its tail implies that its eyes will fall out.

just on the grounds that its antecedent is false (since guinea pigs have no tails). But then it seems equally false to say that:

8. If one picks up a guinea pig by its tail, then its eyes will fall out.

And also both of the following appear to be equally false:

9. Scaring a pregnant guinea pig implies that all of her babies will be born tailless.

10. If one scares a pregnant guinea pig, then all of her babies will be born tailless.

It should be noted that there are other ways to react to the oddity of sentences like the ones above other than calling them simply false. Thus there is the reaction stemming from the work of Grice [1975] that says that at least the conditional sentences (8) and (10) above are true though nonetheless pragmatically odd in that they violate some rule based on conversational co-operation to the effect that one should normally say the strongest thing relevant, i.e. in the cases above, that guinea pigs have no tails (cf. [Fogelin, 1978, p. 136 ff.] for a textbook presentation of this strategy).

Also it should be noted that the theory of the 'counterfactual' conditional due to Stalnaker–Thomason, D. K. Lewis and others (cf. Chapter [[??]] of this *Handbook*), while it agrees with relevance logic in finding sentences like (8) (not (10) *false*, disagrees with relevance logic in the formal account it gives of the conditional.

It would help matters if there were an extended discussion of these competing theories (Anderson–Belnap, Grice, Stalnaker-Thomason-Lewis), which seem to pass like ships in the night (can three ships do this without strain to the image?) but there is not the space here. Such a discussion might include an attempt to construct a theory of a relevant counterfactual conditional (if A were to be the case, then *as a result* B would be the case). The rough idea would be to use say The Routley–Meyer semantics for relevance logic (cf. Section 3.7) in place of the Kripke semantics for modal logic, which plays a key role in the Stalnaker–Thomason–Lewis semantical account of the conditional (put the 3-placed alternativeness relation in the role of the usual 2-placed one). Work in this area is just starting. See the works of [Mares and Fuhrmann, 1995] and [Akama, 1997] which both attempt to give semantics for relevant counterfactuals.

Also any discussion relating to Grice's work would surely make much of the fact that the theory of Grice makes much use of a basically unanalysed notion of relevance. One of Grice's chief conversational rules is 'be relevant', but he does not say much about just what this means. One could look at relevance logic as trying to say something about this, at least in the case of the conditional.

Incidentally, as Meyer has been at great pains to emphasise, relevance logic gives, on its face anyway, no separate account of relevance. It is not as if there is a unary relevance operator ('relevantly').

One last point, and then we shall turn to more substantive issues. Orthodox relevance logic differs from classical logic not just in having an additional logical connective ($\rightarrow$) for the conditional. If that was the only difference relevance logic would just be an 'extension' of classical logic, using the notion of Haack [1974], in much the same way as say modal logic is an extension of classical logic by the addition of a logical connective $\Box$ for necessity. The fact is (cf. Section 1.6) that although relevance logic contains all the same theorems as classical logic in the classical vocabulary say, $\wedge, \vee, \neg$ (and the quantifiers), it nonetheless does not validate the same inferences. Thus, most notoriously, the disjunctive syllogism (cf. Section 2) is counted as invalid. Thus, as Wolf [1978] discusses, relevance logic does not fit neatly into the classification system of [Haack, 1974], and might best be called 'quasi-extension' of classical logic, and hence 'quasi-deviant'. Incidentally, all of this applies only to 'orthodox' relevance logic, and not to the 'classical relevance logics' of Meyer and Routley (cf. Section 3.11).

1.3 Hilbert-style Formulations

We shall discuss first the pure implicational fragments, since it is primarily in the choice of these axioms that the relevance logics differ one from the other. We shall follow the conventions of Anderson and Belnap [Anderson and Belnap, 1975], denoting by '$\mathbf{R}_{\rightarrow}$' what might be called the 'putative implicational fragment of $\mathbf{R}$'. Thus $\mathbf{R}_{\rightarrow}$ will have as axioms all the axioms of $\mathbf{R}$ that only involve the implication connective. That $\mathbf{R}_{\rightarrow}$ is in fact the implicational fragment of $\mathbf{R}$ is much less than obvious since the possibility exists that the proof of a pure implicational formula could detour in an essential way through formulas involving connectives other than implication. In fact Meyer has shown that this does not happen (cf. his Section 28.3.2 of [Anderson and Belnap, 1975]), and indeed Meyer has settled in almost every interesting case that the putative fragments of the well-known relevance logics (at least $\mathbf{R}$ and $\mathbf{E}$) are the same as the real fragments. (Meyer also showed that this does not happen in one interesting case, **RM**, which we shall discuss below.)

For $\mathbf{R}_{\rightarrow}$ we take the rule *modus ponens* ($A, A \rightarrow B \vdash B$) and the following axiom schemes.

$$A \to A \quad \text{Self-Implication} \quad (1)$$
$$(A \to B) \to [(C \to A) \to (C \to B)] \quad \text{Prefixing} \quad (2)$$
$$[A \to (A \to B)] \to (A \to B) \quad \text{Contraction} \quad (3)$$
$$[A \to (B \to C)] \to [B \to (A \to C)] \quad \text{Permutation.} \quad (4)$$

A few comments are in order. This formulation is due to Church [1951b] who called it 'The weak implication calculus'. He remarks that the axioms are the same as those of Hilbert's for the positive implicational calculus (the implicational fragment of the intuitionistic propositional calculus **H**) except that (1) is replaced with

$$A \to (B \to A) \quad \text{Positive Paradox.} \quad (1')$$

(Recent historical investigation by Došen [1992] has shown that Orlov constructed an axiomatisation of the implication and negation fragment of **R** in the mid 1920s, predating other known work in the area. Church and Moh, however, provided a Deduction Theorem (see Section 1.4) which is absent from Orlov's treatment.)

The choice of the implicational axioms can be varied in a number of informative ways. Thus putting things quickly, (2) Prefixing may be replaced by

$$(A \to B) \to [(B \to C) \to (A \to C)] \quad \text{Suffixing.} \quad (2')$$

(3) Contraction may be replaced by

$$[A \to (B \to C)] \to [(A \to B) \to (A \to C)] \quad \text{Self- Distribution,} \quad (3')$$

and (4) Permutation may be replaced by

$$A \to [(A \to B) \to B] \quad \text{Assertion.} \quad (4')$$

These choices of implicational axioms are 'isolated' in the sense that one choice does not affect another. Thus

THEOREM 1 $\mathbf{R}_\to$ *may be axiomatised with* modus ponens*, (1) Self-Implication and any selection of one from each pair* $\{(2), (2')\}, \{(3), (3')\}$*, and* $\{(4), (4')\}$.

Proof. By consulting [Anderson and Belnap, 1975, pp. 79–80], and fiddling. ■

There is at least one additional variant of $\mathbf{R}_{\rightarrow}$ that merits discussion. It turns out that it suffices to have Suffixing, Contraction, and the pair of axiom schemes

$$[(A \rightarrow A) \rightarrow B] \rightarrow B \quad \text{Specialised Assertion,} \tag{4a}$$

$$A \rightarrow [(A \rightarrow A) \rightarrow A] \quad \text{Demodaliser.} \tag{4b}$$

Thus (4b) is just an instance of Assertion, and (4a) follows from Assertion by substitution $A \rightarrow A$ for A and using Self-Implication to detach. That (4a) and (4b) together with Suffixing and Contraction yield Assertion (and, less interestingly, Self-Implication) can be shown using the fact proven in [Anderson and Belnap, 1975, Section 8.3.3], that these yield (letting $\vec{A}$ abbreviate $A_1 \rightarrow A_2$)

$$\vec{A} \rightarrow [(\vec{A} \rightarrow B) \rightarrow B] \quad \text{Restricted-Assertion.} \tag{4''}$$

The point is that (4a) and (4b) in conjunction say that A is equivalent to $(A \rightarrow A) \rightarrow A$, and so every formula A has an equivalent form $\vec{A}$ and so 'Restricted Assertion' reduces to ordinary Assertion.[2]

Incidentally, no claim is made that this last variant of $\mathbf{R}_{\rightarrow}$ has the same isolation in its axioms as did the previous axiomatisations. Thus, e.g. that Suffixing (and not Prefixing) is an axiom is important (a matrix of J. R. Chidgey's (cf. [Anderson and Belnap, 1975, Section 8.6]) can be used to show this.

The system **E** of entailment differs primarily from the system **R** in that it is a system of relevant strict implication. Thus **E** is both a relevance logic and a modal logic. Indeed, defining $\Box A =_{\text{df}} (A \rightarrow A) \rightarrow A$ one finds **E** has something like the modality structure of **S4** (cf. [Anderson and Belnap, 1975, Sections 4.3 and 10]).

This suggests that $\mathbf{E}_{\rightarrow}$ can be axiomatised by dropping Demodaliser from the axiomatisation of $\mathbf{R}_{\rightarrow}$, and indeed this is right (cf. [Anderson and Belnap, 1975, Section 8.3.3], for this and all other claims about axiomatisations of $\mathbf{E}_{\rightarrow}$).[3]

The axiomatisation above is a 'fixed menu' in that Prefixing cannot be replaced with Suffixing. There are other 'à la carte' axiomatisations in the style of Theorem 1.

THEOREM 2 $\mathbf{E}_{\rightarrow}$ *may be axiomatised with* modus ponens*, Self-Implication and any selection from each of the pairs* {*Prefixing, Suffixing*}, {*Contraction, Self-Distribution*} *and* {*Restricted-Permutation, Restricted-Assertion*} *(one from each pair).*

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[2]There are some subtleties here. Detailed analysis shows that both Suffixing *and Prefixing* are needed to replace $\vec{A}$ with A (cf. Section 1.3). Prefixing can be derived from the above set of axioms (cf. [Anderson and Belnap, 1975, pp. 77–78 and p. 26].

[3]The actual history is backwards to this, in that the system **R** was first axiomatised by [Belnap, 1967a] by adding Demodaliser to **E**.

Another implicational system of less central interest is that of 'ticket entailment' $\mathbf{T}_\to$. It is motivated by Anderson and Belnap [1975, Section 6] as deriving from some ideas of Ryle's about 'inference tickets'. It was motivated in [Anderson, 1960] as 'entailment shorn of modality'. The thought behind this last is that there are two ways to remove the modal sting from the characteristic axiom of alethic modal logic, $\Box A \to A$. One way is to add Demodaliser $A \to \Box A$ so as to destroy all modal distinctions. The other is to drop the axiom $\Box A \to A$. Thus the essential way one gets $\mathbf{T}_\to$ from $\mathbf{E}_\to$ is to drop Specialised Assertion (or alternatively to drop Restricted Assertion or Restricted Permutation, depending on which axiomatisation of $\mathbf{E}_\to$ one has). But before doing so one must also add whichever one of Prefixing and Suffixing was lacking, since it will no longer be a theorem otherwise (this is easiest to visualise if one thinks of dropping Restricted permutation, since this is the key to getting Prefixing from Suffixing and *vice versa*). Also (and this is a strange technicality) one must replace Self-Distribution with its permuted form:

$$(A \to B) \to [[A \to (B \to C)] \to (A \to C)] \quad \text{Permuted Self-Distribution.} \tag{3''}$$

This is summarised in

THEOREM 3 (Anderson and Belnap [Section 8.3.2, 1975]) $\mathbf{T}_\to$ *is axiomatised using Self-Implication, Prefixing, Suffixing, and either of* {*Contraction, Permuted Self-Distribution*}, *with* modus ponens.

There is a subsystem of $\mathbf{E}_\to$ called $\mathbf{TW}_\to$ (and $\mathbf{P}{-}\mathbf{W}$, and $\mathbf{T}{-}\mathbf{W}$ in earlier nomenclature) axiomatised by dropping Contraction (which corresponds to the combinator $\mathbf{W}$) from $\mathbf{T}_\to$. This has obtained some interest because of an early conjecture of Belnap's (cf. [Anderson and Belnap, 1975, Section 8.11]) that $A \to B$ and $B \to A$ are both theorems of $\mathbf{TW}_\to$ only when A is the same formula as B. That Belnap's Conjecture is now Belnap's Theorem is due to the highly ingenious (and complicated) work of E. P. Martin and R. K. Meyer [1982] (based on the earlier work of L. Powers and R. Dwyer). Martin and Meyer's work also highlights a system $\mathbf{S}_\to$ (for Syllogism) in which Self-Implication is dropped from $\mathbf{TW}_\to$.

Moving on now to adding the positive extensional connectives $\wedge$ and $\vee$, in order to obtain $\mathbf{R}_{\to,\wedge,\vee}$ (denoted more simply as $\mathbf{R}^+$) one adds to $\mathbf{R}_\to$

the axiom schemes

$A \wedge B \to A,\ \ A \wedge B \to B$	Conjunction Elimination	(5)
$[(A \to B) \wedge (A \to C)] \to (A \to B \wedge C)$	Conjunction Introduction	(6)
$A \to A \vee B, \quad B \to A \vee B$	Disjunction Introduction	(7)
$[(A \to C) \wedge (B \to C)] \to (A \vee B \to C)$	Disjunction Elimination	(8)
$A \wedge (B \vee C) \to (A \wedge B) \vee C$	Distribution	(9)

plus the rule of adjunction ($A, B \vdash A \wedge B$). One can similarly get the positive intuitionistic logic by adding these all to $\mathbf{H}_{\to}$.

Axioms (5)–(8) can readily be seen to be encoding the usual elimination and introduction rules for conjunction and disjunction into axioms, giving $\wedge$ and $\vee$ what might be called 'the lattice properties' (cf. Section 3.3). It might be thought that $A \to (B \to A \wedge B)$ might be a better encoding of conjunction introduction than (6), having the virtue that it allows for the dropping of adjunction. This is a familiar axiom for intuitionistic (and classical) logic, but as was seen by Church [1951b], it is only a hair's breadth away from Positive Paradox ($A \to (B \to A)$), and indeed yields it given (5) and Prefixing. For some mysterious reason, this observation seemed to prevent Church from adding extensional conjunction/disjunction to what we now call $\mathbf{R}_{\to}$ (and yet the need for adjunction in the Lewis formulations of modal logic where the axioms are al strict implications was well-known).

Perhaps more surprising than the need for adjunction is the need for axiom (9). It would follow from the other axioms if only we had Positive Paradox among them. The place of Distribution in $\mathbf{R}$ is continually problematic. It causes inelegancies in the natural deduction systems (cf. Section 1.5) and is an obstacle to finding decision procedures (cf. Section 4.8). Incidentally, all of the usual distributive laws follow from the somewhat 'clipped' version (9).

The rough idea of axiomatising $\mathbf{E}^+$ and $\mathbf{T}^+$ is to add axiom schemes (5)–(9) to $\mathbf{E}_{\to}$ and $\mathbf{T}_{\to}$. This is in fact precisely right for $\mathbf{T}^+$, but for $\mathbf{E}^+$ one needs also the axiom scheme (remember $\Box A =_{\mathrm{df}} (A \to A) \to A$):

$$\Box A \wedge \Box B \to \Box(A \wedge B) \tag{10}$$

This is frankly an inelegance (and one that strangely enough disappears in the natural deduction context of Section 1.5). It is needed for the inductive proof that necessitation ($\vdash C \Rightarrow\ \vdash \Box C$) holds, handling he case where C just came by adjunction (cf. [Anderson and Belnap, 1975, Sections 21.2.2 and 23.4]). There are several ways of trying to conceal this inelegance, but they are all a little *ad hoc*. Thus, e.g. one could just postulate the rule of necessitation as primitive, or one could strengthen the axiom of Restricted

Permutation (or Restricted Assertion) to allow that $\vec{A}$ be a conjunction $(A_1 \to A_1) \wedge (A_2 \to A_2)$.

As Anderson and Belnap [1975, Section 21.2.2] remark, if propositional quantification is available, $\Box A$ could be given the equivalent definition $\forall p(p \to p) \to A$, and then the offending (10) becomes just a special case of Conjunction Introduction and becomes redundant.

It is a good time to advertise that the usual zero-order and first-order relevance logics can be outfitted with a couple of optional convenience features that come with the higher-priced versions with propositional quantifiers. Thus, e.g. the propositional constant t can be added to $\mathbf{E}^+$ to play the role of $\forall p(p \to p)$, governed by the axioms.

$$(t \to A) \to A \tag{11}$$

$$t \to (A \to A), \tag{12}$$

and again (10) becomes redundant (since one can easily show $(t \to A) \leftrightarrow [(A \to A) \to A]$).

Further, this addition of t is conservative in the sense that it leads to no new t-free theorems (since in any given proof t can always be replaced by $(p_1 \to p_1) \wedge \cdots \wedge (p_n \to p_n)$ where $p_1, \ldots, p_n$ are all the propositional variables appearing in the proof — cf. [Anderson and Belnap, 1975]).

Axiom scheme (11) is too strong for $\mathbf{T}^+$ and must be weakened to

$$t. \tag{11$\mathbf{T}$}$$

In the context of $\mathbf{R}^+$, (11) and (11**T**) are interchangeable. and in $\mathbf{R}^+$, (12) may of course be permuted, letting us characterise t in a single axiom as 'the conjunction of all truths':

$$A \leftrightarrow (t \to A) \tag{13}$$

(in **E**, t may be thought of as 'the conjunction of all necessary truths').

'Little t' is distinguished from 'big T', which can be conservatively added with the axiom scheme

$$A \to T \tag{14}$$

(in intuitionistic or classical logic t and T are equivalent).

Additionally useful is a binary connective $\circ$, labelled variously 'intensional conjunction', 'fusion', 'consistency' and 'cotenability'. these last two labels are appropriate only in the context of **R**, where one can define $A \circ B =_{\text{df}} \neg(A \to \neg B)$. One can add $\circ$ to $\mathbf{R}^+$ with the axiom scheme:

$$[(A \circ B) \to C] \leftrightarrow [A \to (B \to C)] \quad \text{Residuation (axiom).} \tag{15}$$

This axiom scheme is too strong for other standard relevance logics, but Meyer and Routley [1972] discovered that one can always add conservatively the two way rule

$$(A \circ B) \to C \dashv\, \vdash A \to (B \to C) \quad \text{Residutation (rule)} \tag{16}$$

(in $\mathbf{R}^+$ (16) yields (15)). Before adding negation, we mention the positive fragment $\mathbf{B}^+$ of a kind of minimal (Basic) relevance logic due to Routley and Meyer (cf. Section 3.9). $\mathbf{B}^+$ is just like $\mathbf{TW}^+$ except for finding the axioms of Prefixing and Suffixing too strong and replacing them by rules:

$$A \to B \vdash (C \to A) \to (C \to B) \quad \text{Prefixing (rule)} \tag{17}$$

$$A \to B \vdash (B \to C) \to (A \to C) \quad \text{Suffixing (rule)} \tag{18}$$

As for negation, the full systems **R**, **E**, etc. may be formed adding to the axiom schemes for $\mathbf{R}^+$, $\mathbf{E}^+$, etc. the following [4]

$$(A \to \neg A) \to \neg A \quad \text{Reductio} \tag{19}$$

$$(A \to \neg B) \to (B \to \neg A) \quad \text{Contraposition} \tag{20}$$

$$\neg\neg A \to A \quad \text{Double Negation.} \tag{21}$$

Axiom schemes (19) and (20) are intuitionistically acceptable negation principles, but using (21) one can derive forms of reductio and contraposition that are intuitionistically rejectable. Note that (19)–(21) if added to $\mathbf{H}^+$ would give the full intuitionistic propositional calculus **H**.

In **R**, negation can alternatively be defined in the style of Johansson, with $\neg A =_{\text{df}} (A \to f)$, where f is a false propositional constant, cf. [Meyer, 1966]. Informally, f is the disjunction of all false propositions (the 'negation' of t). Defining negation thus, axiom schemes (19) and (20) become theorems (being instances of Contraction and Permutation, respectively). But scheme (21) must still be taken as an axiom.

Before going on to discuss quantification, we briefly mention a couple of other systems of interest in the literature.

Given that **E** has a theory of necessity riding piggyback on it in the definition $\Box A =_{\text{df}} (A \to A) \to A$, the idea occurred to Meyer of adding to **R** a primitive symbol for necessity $\Box$ governed by the **S4** axioms.

$$\Box A \to A \tag{$\Box$1}$$

$$\Box(A \to B) \to (\Box A \to \Box B) \tag{$\Box$2}$$

$$\Box A \wedge \Box B \to \Box(A \wedge B) \tag{$\Box$3}$$

$$\Box A \to \Box\Box A, \tag{$\Box$4}$$

[4] Reversing what is customary in the literature, we use $\neg$ for the standard negation of relevance logic, reserving $\sim$ for the 'Boolean negation' discussed in Section 3.11. We do this so as to follow the notational policies of the *Handbook*.

and the rule of Necessitation ($\vdash A \Rightarrow\ \vdash \Box A$).

His thought was that **E** could be exactly translated into this system $\mathbf{R}^{\Box}$ with entailment defined as strict implication. That this is subtly not the case was shown by Maksimova [1973] and Meyer [1979b] has shown how to modify $\mathbf{R}^{\Box}$ so as to allow for an exact translation.

Yet one more system of interest is **RM** (cf. Section 3.10) obtained by adding to **R** the axiom scheme

$$A \to (A \to A) \quad \text{Mingle.} \tag{22}$$

Meyer has shown somewhat surprisingly that the pure implicational system obtained by adding Mingle to **R** is not the implicational fragment of **RM**, and he and Parks have shown how to axiomatise this fragment using a quite unintelligible formula (cf. [Anderson and Belnap, 1975, Section 8.18]). Mingle may be replaced equivalently with the converse of Contraction:

$$(A \to B) \to (A \to (A \to B)) \quad \text{Expansion.} \tag{23}$$

Of course one can consider 'mingled' versions of **E**, and indeed it was in this context that McCall first introduced mingle, albeit in the strict form (remember $\vec{A} = A_1 \to A_2$),

$$\vec{A} \to (\vec{A} \to \vec{A}) \quad \vec{\text{M}}\text{ingle} \tag{24}$$

(cf. [Dunn, 1976c]).

We finish our discussion of axiomatics with a brief discussion of first-order relevance logics, which we shall denote by **RQ**, **EQ**, etc. We shall presuppose a standard definition of first-order formula (with connectives $\neg, \wedge, \vee, \to$ and quantifiers $\forall, \exists$). For convenience we shall suppose that we have two denumerable stocks of variables: the bound variables x, y, etc. and the free variables (sometimes called parameters) a, b, etc. The bound variables are never allowed to have unbound occurrences.

The quantifier laws were set down by Anderson and Belnap in accord with the analogy of the universal quantifier with a conjunction (or its instances), and the existential quantifier as a disjunction. In view of the validity of quantifier interchange principles, we shall for brevity take only the universal quantifier $\forall$ as primitive, defining $\exists xA =_{\text{df}} \neg\forall x\neg A$. We thus need

$$\forall xA \to A(a/x) \qquad \forall\text{-elimination} \tag{25}$$

$$\forall x(A \to B) \to (A \to \forall xB) \qquad \forall\text{- introduction} \tag{26}$$

$$\forall x(A \vee B) \to A \vee \forall xB \qquad \text{Confinement.} \tag{27}$$

If there are function letters or other term forming operators, then (25) should be generalised to $\forall xA \to A(t/x)$, where t is any term (subject to our conventions that the 'bound variables' x, y, etc. do not occur ('free') in it). Note well that because of our convention that 'bound variables' do not occur free, the usual proviso that x does not occur free in A in (26) and (27) is automatically satisfied. (27) is the obvious 'infinite' analogy of Distribution, and as such it causes as many technical problems for **RQ** as does Distribution for **R** (cf. Section 4.8). Finally, as an additional rule corresponding to adjunction, we need:

$$\frac{A(a/x)}{\forall x A} \quad \text{Generalisation.} \tag{28}$$

There are various more or less standard ways of varying this formulation. Thus, e.g. (cf. Meyer, Dunn and Leblanc [1974]) one can take all universal generalisations of axioms, thus avoiding the need for the rule of Generalisation. Also (26) can be 'split' into two parts:

$$\forall x(A \to B) \to (\forall x A \to \forall x B) \tag{26a}$$

$$A \to \forall x A \qquad \text{Vacuous Quantification} \tag{26b}$$

(again note that if we allowed x to occur free we would have to require that x not be free in A).

The most economical formulation is due to Meyer [1970]. It uses only the axiom scheme of $\forall$-elimination and the rule.

$$\frac{A \to B \vee C(a/x)}{A \to B \vee \forall x C} \ (a \text{ cannot occur in } A \text{ or } B) \tag{29}$$

which combines (26)–(28).

1.4 Deduction Theorems in Relevance Logic

Let **X** be a formal system, with certain formulas of **X** picked out as *axioms* and certain (finitary) relations among the formulas of **X** picked out as *rules*. (For the sake of concreteness, **X** can be thought of as any of the Hilbert-style systems of the previous section.) Where Γ is a list of formulas of **X** (thought of as *hypotheses*) it is customary to define a *deduction from* Γ to be a sequence $B_1, \ldots, B_n$, where for each $B_i (1 \leq i \leq n)$, either (1) B_i is in Γ, or (2) B is an axiom of **X**, or (3) B_i 'follows from' earlier members of the sequence, i.e. $R(B_{j_1}, \ldots, B_{j_k}, B_i)$ holds for some $(k+1)$—any rule R of **X** and $B_{j_1}, \ldots, B_{j_k}$ all precede B_i in the sequence $B_1, \ldots, B_n$. A formula A is then said to be *deducible* from Γ just in case there is some deduction from Γ terminating in A. We symbolise this as $\Gamma \vdash_{\mathbf{X}} A$ (often suppressing the subscript).

A *proof* is of course a deduction from the empty set, and a theorem is just the last item in a proof. There is the well-known

DEDUCTION THEOREM (Herbrand). *If* $A_1, \ldots, A_n, A \vdash_{\mathbf{H}_\rightarrow} B$*, then we have also* $A_1, \ldots, A_n \vdash_{\mathbf{H}_\rightarrow} A \rightarrow B$*.*

This theorem is proven in standard textbooks for classical logic, but the standard inductive proof shows that in fact the Deduction Theorem holds for any formal system $\mathbf{X}$ having *modus ponens* as its sole rule and $\mathbf{H}_\rightarrow \subseteq \mathbf{X}$ (i.e. each instance of an axiom scheme of $\mathbf{H}_\rightarrow$ is a theorem of $\mathbf{X}$). Indeed $\mathbf{H}_\rightarrow$ can be motivated as the minimal pure implicational calculus having modus ponens as its sole rule and satisfying the Deduction Theorem. This is because the axioms of $\mathbf{H}_\rightarrow$ can all be derived as theorems in any formal system $\mathbf{X}$ using merely *modus ponens* and the supposition that $\mathbf{X}$ satisfies the Deduction Theorem. Thus consider as an example:

(1)	$A, B \vdash A$	Definition of $\vdash$
(2)	$A \vdash B \rightarrow A$	(1), Deduction Theorem
(3)	$\vdash A \rightarrow (B \rightarrow A)$	(2), Deduction Theorem.

Thus the most problematic axiom of $\mathbf{H}_\rightarrow$ has a simple '*a priori* deduction', indeed one using only the Deduction Theorem, not even *modus ponens* (which is though needed for more sane axioms like Self-Distribution).

It might be thought that the above considerations provide a very powerful argument for motivating intuitionistic logic (or at least some logic having he same implicational fragment) as The One True Logic. For what else should an implication do but satisfy *modus ponens* and the Deduction Theorem?

But it turns out that there is another sensible notion of deduction. This is what is sometimes called a *relevant deduction.*(Anderson and Belnap [1975, Section 22.2.1] claim that this is the *only* sensible notion of deduction, but we need not follow them in that). If there is anything that sticks out in the *a priori* deduction of Positive Paradox above it is that in (1), B was not *used* in the deduction of A.

A number of researchers have been independently bothered by this point and have been motivated to study a relevant implication that goes hand in hand with a notion of relevant deduction. This, in this manner Moh [1950] and Church [1951b] came up with what is in effect $\mathbf{R}_\rightarrow$. And Anderson and Belnap [1975, p. 261] say "In fact, the search for a suitable deduction theorem for Ackermann's systems . . . provided the impetus leading us to the research reported in this book." This research program begun in the late 1950s took its starting point in the system(s) of Ackermann [1956], and the bold stroke separating the Anderson–Belnap system $\mathbf{E}$ from Ackermann's system Π' was basically the dropping of Ackermann's rule γ so as to have an appropriate deduction theorem (cf. Section 2.1).

Let us accordingly define a deduction of B from $A_1, \ldots, A_n$ to be *relevant with respect to a given hypothesis* A_i just in case A_i is actually *used* in the given deduction of B in the sense (paraphrasing [Church, 1951b]) that there is a chain of inferences connecting A_i with the final formula B. This last can be made formally precise in any number of ways, but perhaps the most convenient is to flag A_i with say a $\sharp$ and to pass the flag along in the deduction each time *modus ponens* is applied to two items at least one of which is flagged. It is then simply required that the last step of the deduction (B) be flagged. Such devices are familiar from various textbook presentations of classical predicate calculus when one wants to keep track whether some hypothesis $A_i(x)$ was used in the deduction of some formula $B(x)$ to which one wants to apply Universal Generalisation.

We shall define a deduction of B from $A_1, \ldots, A_n$ to be *relevant* simpliciter just in case it is relevant with respect to each hypothesis A_i. A practical way to test for this is to flag each A_i with a different flag (say the subscript i) and then demand that all of the flags show up on the last step B.

We can now state a version of the

RELEVANT DEDUCTION THEOREM (Moh, Church). *If there is a deduction in $\mathbf{R}_{\rightarrow}$ of B from $A_1, \ldots, A_n, A$ that is relevant with respect to A, then there is a deduction in $\mathbf{R}_{\rightarrow}$ of $A \rightarrow B$ from $A_1, \ldots, A_n$. Furthermore the new deduction will be 'as relevant' as the old one, i.e. any A_i that was used in the given deduction will be used in the new deduction.*

Proof. Let the given deduction be $B_1, \ldots, B_k$, and let it be given with a particular analysis as to how each step is justified. By induction we show for each B_i that if A was used in obtaining B_i (B_i is flagged), then there is a deduction of $A \rightarrow B_i$ from $A_1, \ldots, A_n$, and otherwise there is a deduction of B_i from those same hypotheses. The tedious business of checking that the new deduction is as relevant as the old one is left to the reader. We divide up cases depending on how the step B_i is justified.

Case 1. B_i was justified as a hypothesis. Then neither B_i is A or it is some A_j. But $A \rightarrow A$ is an axiom of $\mathbf{R}_{\rightarrow}$ (and hence deducible from $A_1, \ldots, A_n$), which takes care of the first alternative. And clearly on the second alternative B_i is deducible from $A_1, \ldots, A_n$ (being one of them).

Case 2. B_i was justified as an axiom. Then A was not used in obtaining B_i, and of course B_i is deducible (being an axiom).

Case 3. B_i was justified as coming from preceding steps $B_j \rightarrow B_i$ and B_j by *modus ponens*. There are four subcases depending on whether A was used in obtaining the premises.

Subcase 3.1. A was used in obtaining both $B_j \rightarrow B_i$ and B_j. Then by inductive hypothesis $A_1, \ldots, A_n \vdash_{\mathbf{R}_{\rightarrow}} A \rightarrow (B_j \rightarrow B_i)$ and $A_1, \ldots, A_n \vdash_{\mathbf{R}_{\rightarrow}} A \rightarrow B_j$. So $A \rightarrow B$ may be obtained using the axiom of Self-Distribution.

Subcase 3.2. A was used in obtaining $B_j \rightarrow B_i$ but not B_j. Use the axiom of Permutation to obtain $A \rightarrow B_i$ from $A \rightarrow (B_j \rightarrow B_i)$ and B_j.

Subcase 3.3. A was not used in obtaining $B_j \to B_i$ but was used for B_j. Use the axiom of Prefixing to obtain $A \to B_i$ from $B_j \to B_i$ and $A \to B_j$.

Subcase 3.4. A was not used in obtaining either $B_j \to B_i$ nor B_j. Then B_i follows form these using just *modus ponens*.

Incidentally, $\mathbf{R}_{\to}$ can easily be verified to be the minimal pure implicational calculus having *modus ponens* as sole rule and satisfying the Relevant Deduction Theorem, since each of the axioms invoked in the proof of this theorem can be easily seen to be theorems in any such system (cf. the next section for an illustration of sorts).

There thus seem to be at least two natural competing pure implicational logics $\mathbf{R}_{\to}$ and $\mathbf{H}_{\to}$, differing only in whether one wants one's deductions to be relevant or not.[5] ■

Where does the Anderson–Belnap's [1975] preferred system $\mathbf{E}_{\to}$ fit into all of this? The key is that the implication of $\mathbf{E}_{\to}$ is both a strict and a relevant implication (cf. Section 1.3 for some subtleties related to this claim). As such, and since Anderson and Belnap have seen fit to give it the modal structure of the Lewis system **S4**, it is appropriate to recall the appropriate deduction theorem for **S4**.

MODAL DEDUCTION THEOREM [Barcan Marcus, 1946] *If* $A_1 \to B_1, \ldots, A_n \to B_n, A \vdash_{\mathbf{S4}} B$ *(→ here denotes strict implication), then* $A_1 \to B_1, \ldots, A_n \to B_n \vdash_{\mathbf{S4}} A \to B$.

The idea here is that in general in order to derive the strict (*necessary*) implication $A \to B$ one must not only be able to deduce B from A and some other hypotheses but furthermore those other hypotheses must be supposed to be necessary. And in **S4** since $A_i \to B_j$ is equivalent to $\Box(A_i \to B_j)$, requiring those additional hypotheses to be strict implications at least suffices for this.

Thus we could only hope that $\mathbf{E}_{\to}$ would satisfy the

MODAL RELEVANT DEDUCTION THEOREM [Anderson and Belnap, 1975] *If there is a deduction in* $\mathbf{E}_{\to}$ *of* B *from* $A_1 \to B_1, \ldots, A_n \to B_n, A$ *that is relevant with respect to* A*, then there is a deduction in* $\mathbf{E}_{\to}$ *of* $A \to B$ *from* $A_1 \to B_1, \ldots, A_n \to B_n$ *that is as relevant as the original.*

The proof of this theorem is somewhat more complicated than its unmodalised counterpart which we just proved (cf. [Anderson and Belnap, 1975, Section 4.21] for a proof).

We now examine a subtle distinction (stressed by Meyer—see, for example, [Anderson and Belnap, 1975, pp. 394–395]), postponed until now for

[5]This seems to differ from the good-humoured polemical stand of Anderson and Belnap [1975, Section 22.2.1], which says that the first kind of 'deduction', which they call (pejoratively) 'Official deduction', is no kind of deduction at all.

pedagogical reasons. We must ask, how many hypotheses can dance on the head of a formula? The question is: given the list of hypotheses A, A, do we have one hypothesis or two? When the notion of a *deduction* was first introduced in this section and a 'list' of hypotheses Γ was mentioned, the reader would naturally think that this was just informal language for a set. And of course the set $\{A, A\}$ is identical to the set $\{A\}$. Clearly A is relevantly deducible from A. The question is whether it is so deducible from A, A. We have then two different criteria of use, depending on whether we interpret hypotheses as grouped together into lists that distinguish multiplicity of occurrences (sequences)[6] or sets. This issue has been taken up elsewhere of late, with other accounts of deduction appealing to 'resource consciousness' [Girard, 1987, Troelstra, 1992, Schroeder-Heister and Došen, 1993] as motivating some non-classical logics. Substructural logics in general appeal to the notion that the number of times a premise is used, or even more radically, the *order* in which premises are used, matter.

At issue in **R** and its neighbours is whether $A \to (A \to A)$ is a correct relevant implication (coming by two applications of 'The Deduction Theorem' from $A, A \vdash A$). This is in fact not a theorem of **R**, but it is the characteristic axiom of **RM** (cf. Section 1.3). So it is important that in the Relevant Deduction Theorem proved for $\mathbf{R}_\to$ that the hypotheses $A_1, \ldots, A_n$ be understood as a sequence in which the same formula may occur more than once. One can prove a version of the Relevant Deduction Theorem with hypotheses understood as collected into a set for the system $\mathbf{RMO}_\to$, obtained by adding $A \to (A \to A)$ to $\mathbf{R}_\to$ (but the reader should be told that Meyer has shown that $\mathbf{RMO}_\to$, is *not* the implicational fragment of **RM**, cf. [Anderson and Belnap, 1975, Section 8.15]).[7]

Another consideration pointing to the *naturalness* of $\mathbf{R}_\to$ is its connection to the λI-calculus. A formula is a theorem of $\mathbf{R}_\to$ if and only if it is the type of a closed term of the λI-calculus as defined by Church. A λI term is a λ term in which every lambda abstraction binds at least one free variable. So, $\lambda x.\lambda y.xy$ has type $A \to \big((A \to B) \to B\big)$, and so, is a theorem of $\mathbf{R}_\to$, while $\lambda x.\lambda y.x$, has type $A \to (B \to A)$, which is an intuitionistic theorem, but not an $\mathbf{R}_\to$ theorem. This is reflected in the λ term, in which the λy does not bind a free variable.

We now briefly discuss what happens to deduction theorems when the

[6]Sequences are not quite the best mathematical structures to represent this grouping since it is clear that the order of hypotheses makes no difference (at least in the case of **R**). Meyer and McRobbie [1979] have investigated 'firesets' (finitely repeatable sets) as the most appropriate abstraction.

[7]Arnon Avron has defended this system, $\mathbf{RMO}_\to$, as a natural way to characterise relevant implication [Avron, 1986, Avron, 1990a, Avron, 1990b, Avron, 1990c, Avron, 1992]. In Avron's system, conjunction and disjunction are *intensional* connectives, defined in terms of the implication and negation of $\mathbf{RMO}_\to$. As a result, they do not have all of the distributive lattice properties of traditional relevance logics.

pure implication systems $\mathbf{R}_{\rightarrow}$ and $\mathbf{E}_{\rightarrow}$ are extended to include other connectives, especially $\wedge$. $\mathbf{R}$ will be the paradigm, its situation extending straight-forwardly to $\mathbf{E}$. The problem is that the full system $\mathbf{R}$ seems not to be formulable with *modus ponens* as the sole rule; there is also need for adjunction ($A, B \vdash A \wedge B$) (cf. Section 1.3).

Thus when we think of proving a version of the Relevant Deduction Theorem for the full system $\mathbf{R}$, it would seem that we are forced to think through once more the issue of when a hypothesis is used, this time with relation to adjunction. It might be thought that the thing to do would be to pass the flag $\sharp$ along over an application of adjunction so that $A \wedge B$ ends up flagged if either of the premises A or B was flagged, in obvious analogy with the decision concerning *modus ponens*.

Unfortunately, that decision leads to disaster. For then the deduction $A, B \vdash A \wedge B$ would be a relevant one (both A and B would be 'used'), and two applications of 'The Deduction Theorem' would lead to the thesis $A \rightarrow (B \rightarrow A \wedge B)$, the undesirability of which has already been remarked.

A more appropriate decision is to count hypotheses as used in obtaining $A \wedge B$ just when they were used to obtain both premises. This corresponds to the axiom of Conjunction Introduction $(C \rightarrow A) \wedge (C \rightarrow B) \rightarrow (C \rightarrow A \wedge B)$, which thus handles the case in the inductive proof of the deduction theorem when the adjunction is applied. This decision may seem *ad hoc* (perhaps 'use' simpliciter is not quite the right concept), but it is the only decision to be made unless one wants to say that the hypothesis A can (in the presence of the hypothesis B) be 'used' to obtain $A \wedge B$ and hence B (passing on the flag from A this way is something like laundering dirty money).

This is the decision that was made by Anderson and Belnap in the context of natural deduction systems (see next section), and it was applied by Kron [1973, 1976] in proving appropriate deduction theorems for $\mathbf{R}$, $\mathbf{E}$ (and $\mathbf{T}$). It should be said that the appropriate Deduction Theorem requires simultaneous flagging of the hypothesis (distinct flags being applied to each formula occurrence, say using subscripts in the manner of the 'practical suggestion' after our definition of relevant deduction for $\mathbf{R}_{\rightarrow}$), with the requirement that all of the subscripts are passed on to the conclusion. So the Deduction Theorem applies only to *fully* relevant deductions, where every premise is used (note that no such restriction was placed on the Relevant Deduction Theorem for $\mathbf{R}_{\rightarrow}$).

An alternative stated in Meyer and McRobbie [1979] would be to adjust the definition of deduction, modifying clause (2) so as to allow as a step in a deduction any theorem (not just axiom) of $\mathbf{R}$, and to restrict clause (3) so that the only rule allowed in moving to later steps is *modus ponens*.[8] This

[8]Of course this requires we give an independent characterisation of *proof* (and *theorem*), since we can no longer define a *proof* as a deduction from zero premisses. We thus

is in effect to restrict adjunction to theorems, and reminds one of similar restrictions in the context of deduction theorems of similarly restricting the rules of necessitation and universal generalisation. It has the virtue that the Relevant Deduction Theorem and its proof are the same as for $\mathbf{R}_{\rightarrow}$. (Incidentally, Meyer's and Kron's sense of deduction coincide when all of $A_1, \ldots, A_n$ are used in deducing B; this is obvious in one direction, and less than obvious in the other.)

There are yet two other versions of the deduction theorem that merit discussion in the context of relevance logic (relevance logic, as Meyer often points out, allows for many distinctions).

First in Belnap [1960b] and Anderson and Belnap [1975], there is a theorem (stated for **E**, but we will state it for our paradigm **R**) called The Entailment Theorem, which says that $A_1, \ldots, A_n$ 'entails' B iff $\vdash_{\mathbf{R}} (A_1 \wedge \ldots \wedge A_n) \rightarrow B$. A formula B is defined in effect to be *entailed* by hypothesis $A_1, \ldots, A_n$ just in case there is a deduction of B using their conjunction $A_1 \wedge \ldots \wedge A_n$. Adjunction is allowed, but subject to the restriction that the conjunctive hypothesis was used in obtaining both premises. The Entailment Theorem is clearly implied by Kron's version of the Deduction Theorem.

The last deduction theorem for **R** we wish to discuss is the

Enthymematic Deduction Theorem (Meyer, Dunn and Leblanc [1974]). *If* $A_1, \ldots, A_n, A \vdash_{\mathbf{R}} B$, *then* $A_1, \ldots, A_n \vdash_{\mathbf{R}} A \wedge t \rightarrow B$.

Here ordinary deducibility is all that is at issue (no insistence on the hypotheses being used). It can either be proved by induction, or cranked out of one of the more relevant versions of the deduction theorem. Thus it falls out of the Entailment Theorem that

$$\vdash_{\mathbf{R}} X \wedge A \wedge T \rightarrow B,$$

where X is the conjunction of $A_1, \ldots, A_n$, and T is the conjunction of all the axioms of **R** used in the deduction of B. But since $\vdash_{\mathbf{R}} t \rightarrow T$, we have $\vdash_{\mathbf{R}} X \wedge A \wedge t \rightarrow B$.

However, the following **R** theorem holds:

$$\vdash_{\mathbf{R}} (X \wedge A \wedge t \rightarrow B) \rightarrow (X \wedge t \rightarrow (A \wedge t \rightarrow B)).$$

So $\vdash_{\mathbf{R}} X \wedge t \rightarrow (A \wedge t \rightarrow B)$, which leads (using $\vdash_{\mathbf{R}} t$) to $X \vdash_{\mathbf{R}} A \wedge t \rightarrow B$, which dissolving the conjunction gives the desired

$$A_1, \ldots, A_n \vdash_{\mathbf{R}} A \wedge t \rightarrow B.$$

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define a proof as a sequence of formulas, each of which is either an axiom or follows from preceding items by either *modus ponens* or adjunction (!).

In view of the importance of the notion, let us symbolise $A \wedge t \to B$ as $A \to_t B$. This functions as a kind of 'enthymematic implication' (A and some truth *really implies* B) and there will be more about Anderson, Belnap and Meyer's investigations of this concept in Section 1.7. Let us simply note now that in the context of deduction theorems, it functions like intuitionistic implication, and allows us in $\mathbf{R}_{\to}$ to have two different kinds of implication, each well motivated in its relation to the two different kinds of deducibility (ordinary and relevant).[9] For a more extensive discussion of deduction theorems in relevance logics and related systems, more recent papers by Avron [1991] and Brady [1994] should be consulted.

1.5 Natural Deduction Formulations

We shall be very brief about these since natural deduction methods are amply discussed by Anderson and Belnap [1975], where such methods in fact are used s a major motivation for relevance logic. Here we shall concentrate on a natural deduction system $N\mathbf{R}$ for $\mathbf{R}$.

The main idea of natural deduction (cf. Chapters [[were I.1 and I.2]] of the *Handbook*) of course is to allow the making of temporary hypotheses, with some device usually being provided to facilitate the book-keeping concerning the use of hypotheses (and when their use is 'discharged'). Several textbooks (for example, [Suppes, 1957] and [Lemmon, 1965])[10] have used the device of in effect subscripting each hypothesis made with a distinct numeral, and then passing this numeral along with each application of a rule, thus keeping track of which hypothesis are used. When a hypothesis is discharged, the subscript is dropped. A line obtained with no subscripts is a 'theorem' since it depends on no hypotheses.

Let us then let α, β, etc. range over classes of numerals. The rules for $\to$ are then naturally:

$$\frac{\begin{array}{c} A \to B_\alpha \\ A_\beta \end{array}}{B_{\alpha\cup\beta}} \quad [\to E] \qquad\qquad \frac{\begin{array}{c} A_{\{k\}} \\ \vdots \\ B_\alpha \end{array}}{A \to B_{\alpha - \{k\}}} \quad (\text{provided } k \in \alpha) \qquad [\to I]$$

Two fussy, really incidental remarks must be made. First, in the rule $\to E$ it is to be understood that the premises need not occur in the order listed, nor need they be adjacent to each other or to the conclusion. Otherwise we would need a rule of 'Repetition', which allows the repeating of a formula with its subscripts as a later line. (Repetition is trivially derivable given

[9] In $\mathbf{E}$ enthymematic implication is like $\mathbf{S4}$ strict implication. See [Meyer, 1970a].

[10] The idea actually originates with [Feys and Ladrière, 1955].

our 'non-adjacent' understanding of $\to E$—in order to repeat A_α, just prove $A \to A$ and apply $\to E$.) Second, it is understood that we have what one might call a rule of 'Hypothesis Introduction': anytime one likes one can write a formula as a line with a new subscript (perhaps most conveniently, the line number).

Now a non-fussy remark must be made, which is really the heart of the whole matter. In the rule for $\to I$, a *proviso* has been attached which has the effect of requiring that the hypothesis A was actually used in obtaining B. This is precisely what makes the implication relevant (one gets the intuitionistic implication system $\mathbf{H}_\to$ if one drops this requirement). The reader should find it instructive to attempt a proof of Positive Paradox ($A \to (B \to A)$) and see how it breaks down for $N\mathbf{R}_\to$ (but succeeds in $N\mathbf{H}_\to$. The reader should also construct proofs in $N\mathbf{R}_\to$ of all the axioms in one of the Hilbert-style formulations of $\mathbf{R}_\to$ from Section 1.3.

Then the equivalence of $\mathbf{R}_\to$ in its Hilbert-style and natural deduction formulations is more or less self-evident given the Relevant Deduction Theorem (which shows that the rule $\to I$ can be 'simulated' in the Hilbert-style system, the only point at issue).

Indeed it is interesting to note that Lemmon [1965], who *seems* to have the same *proviso* on $\to I$ that we have for $N\mathbf{R}_\to$ (his actual language is a bit informal), does not prove Positive Paradox until his *second* chapter adding conjunction (and disjunction) to the implication-negation system he developed in his first chapter. His proof of Positive Paradox depends finally upon an 'irrelevant' $\wedge I$ rule. The following is perhaps the most straightforward proof in his system (differing from the proof he actually gives):

(1)	A_1	Hyp
(2)	B_2	Hyp
(3)	$A \wedge B_{1,2}$	$1, 2, \wedge I?$
(4)	$A_{1,2}$	$3, \wedge E$
(5)	$B \to A_1$	$2, 4, \to I$
(6)	$A \to (B \to A)$	$1, 5, \to I.$

We think that the manoeuvre used in getting B's 2 to show up attached to A in line (4) should be compared to laundering dirty money by running it through an apparently legitimate business. The correct 'relevant' versions of the conjunction rules are instead

$$\frac{\begin{array}{c}A_\alpha\\ B_\alpha\end{array}}{A \wedge B_\alpha}\ [\wedge I] \qquad \frac{A \wedge B_\alpha}{A_\alpha} \quad \frac{A \wedge B_\alpha}{B_\alpha}\ [\wedge E]$$

What about disjunction? In $\mathbf{R}$ (also $\mathbf{E}$, etc.) one has de Morgan's Laws and Double Negation, so one can simply define $A \vee B = \neg(\neg A \wedge \neg B)$. One might think that settling down in separate int-elim rules for $\vee$ would then only be a matter of convenience. Indeed, one can find in [Anderson and Belnap, 1975] in effect the following rules:

$$\frac{A_\alpha}{A \vee B_\alpha} \qquad \frac{B_\alpha}{A \vee B_\alpha} \quad [\vee I] \qquad\qquad \begin{array}{c} A \vee B_\alpha \\ \vdots \\ A_k \\ \vdots \\ C_{\beta\cup\{k\}} \\ B_h \\ \vdots \\ C_{\beta\cup\{h\}} \\ \hline C_{\alpha\cup\beta} \end{array} \quad [\vee E]$$

But (as Anderson and Belnap point out) these rules are insufficient. ¿From them one cannot derive the following

$$\frac{A \wedge (B \vee C)_\alpha}{(A \wedge B) \vee C_\alpha} \text{ Distribution.}$$

And so it must be taken as an additional rule (even if disjunction is defined from conjunction and negation).

This is clearly an unsatisfying, if not unsatisfactory, state of affairs. The customary motivation behind int-elim rules is that they show how a connective may be introduced into and eliminated from argumentative discourse (in which it has no essential occurrence), and thereby give the connective's role or meaning. In this context the Distribution rule looks very much to be regretted.

One remedy is to modify the natural deduction system by allowing hypotheses to be introduced in two different ways, 'relevantly' and 'irrelevantly'. The first way is already familiar to us and is what requires a subscript to keep track of the relevance of the hypothesis. It requires that the hypotheses introduced this way will *all* be used to get the conclusion. The second way involves only the weaker promise that at least *some* of the hypotheses so introduced will be used. This suggestion can be formalised by allowing several hypotheses to be listed on a line, but with a single relevance numeral attached to them as a bunch. Thus, schematically, an argument of the form

$$\begin{array}{ll} (1) & A, B_1 \\ (2) & C, D_2 \\ & \quad\vdots \\ (k) & E_{1,2} \end{array}$$

should be interpreted as establishing

$$A \wedge B \to (C \wedge D \to E).$$

Now the natural deduction rules must be stated in a more general form allowing for the fact that more than one formula can occur on a line. Key among these would be the new rule:

$$\begin{array}{c} \Gamma, A \vee B_\alpha \\ \vdots \\ \Gamma, A_k \\ \vdots \\ \Delta_{\beta\cup\{k\}} \\ \Gamma, B_l \\ \vdots \\ \Delta_{\beta\cup\{l\}} \\ \hline \Delta_{\alpha\cup\beta} \end{array} \qquad [\vee E']$$

It is fairly obvious that this rule has Distribution built into it. Of course, other rules must be suitably modified. It is easiest to interpret the formulas on a line as grouped into a set so as not to have to worry about 'structural rules' corresponding to the commutation and idempotence of conjunction.

The rules $\to I, \to E, \vee I, \vee E, \wedge I$, and $\wedge E$ can all be left as they were (or except for $\to I$ and $\to E$, trivially generalised so as to allow for the fact that the premises might be occurring on a line with several other 'irrelevant' premises), but we do need one new structural rule:

$$\begin{array}{c} \Gamma_\alpha \\ \Delta_\alpha \\ \hline \Gamma, \Delta_\alpha \end{array} \quad [\text{Comma } I]$$

Once we have this it is natural to take the conjunction rules in 'Ketonen form':

$$\frac{\Gamma, A, B_\alpha}{\Gamma, A \wedge B_\alpha}\,[\wedge I']$$

$$\frac{\Gamma, A \wedge B_\alpha}{\Gamma, A, B_\alpha}\,[\wedge E']$$

with the rule

$$\frac{\Gamma, \Delta_\alpha}{\Gamma_\alpha}\,[\text{Comma } E]$$

It is merely a tedious exercise for the reader to show that this new system $N'\mathbf{R}$ is equivalent to $N\mathbf{R}$. Incidentally, $N'\mathbf{R}$ was suggested by reflection upon the Gentzen System $L\mathbf{R}^+$ of Section 4.9.

Before leaving the question of natural deduction for $\mathbf{R}$, we would like to mention one or two technical aspects. First, the system of Prawitz [1965] differs from $\mathbf{R}$ in that it lacks the rule of Distribution. This is perhaps compensated for by the fact that Prawitz can prove a normal form theorem for proofs in his system. A different system yet is that of [Pottinger, 1979], based on the idea that the correct $\wedge I$ rule is

$$\frac{\begin{array}{c}A_\alpha\\ B_\beta\end{array}}{A \wedge B_{\alpha\cup\beta}}$$

He too gets a normal form theorem. We conjecture that some appropriate normal form theorem is provable for the system $N'\mathbf{R}^+$ on the well-known analogy between cut-elimination and normalisation and the fact that cut-elimination has been proven for $L\mathbf{R}^+$ (cf. Section 4.9). Negation though would seem to bring extra problems, as it does when one is trying to add it to $L\mathbf{R}^+$.

One last set of remarks, and we close the discussion of natural deduction. The system $N\mathbf{R}$ above differs from the natural deduction system for $\mathbf{R}$ of Anderson and Belnap [1975]. Their system is a so-called 'Fitch-style' formalism, and so named $F\mathbf{R}$. The reader is presumed to know that in this formalism when a hypothesis is introduced it is thought of as starting a subproof, and a line is drawn along the left of the subproof (or a box is drawn around the subproof, or some such thing) to demarcate the scope of the hypothesis. If one is doing a natural deduction system for classical or intuitionistic logic, subproofs or dependency numerals can either one be used to do essentially the same job of keeping track the use of hypotheses (though dependency numerals keep more careful track, and that is why they are so useful for relevant implication).

Mathematically, a Fitch-style proof is a nested structure, representing the fact that subproofs can contain further subproofs, etc. But once one has dependency numerals, this extra structure, at least for $\mathbf{R}$, seems otiose, and so we have dispensed with it. The story for $\mathbf{E}$ is more complex, since on the Anderson and Belnap approach $\mathbf{E}$ differs from $\mathbf{R}$ only in what is allowed to be 'reiterable' into subproof. Since implication in $\mathbf{E}$ is necessary as well as relevant, the story is that in deducing B from A in order to show $A \to B$, one should only be allowed to use items that have been assumed to be necessarily true, and that these can be taken to be formulas of the form $C \to D$. So only formulas of this form can be reiterated for use in the subproof from A to B. Working out how best to articulate this idea using only dependency numerals (no lines, boxes, etc.) is a little messy. This concern to keep track of how premises are used in a proof by way of labels has been taken up in a general way by recent work on *Labelled Deductive Systems* [D'Agostino and Gabbay, 1994, Gabbay, 1997].

We would be remiss not to mention other formulations of natural deduction systems for relevance logics and their cousins. A different generalisation of Hunter's natural deduction systems (which follows more closely the Gentzen systems for positive logics — see Section 4.9) is in [Read, 1988, Slaney, 1990].[11]

1.6 Basic Formal Properties of Relevance Logic

This section contains a few relatively simple properties of relevance logics, proofs for which can be found in [Anderson and Belnap, 1975]. With one exception (the 'Ackermann Properties'—see below), these properties all hold for both the system **R** and **E**, and indeed for most of the relevance logics defined in Section 1.3. For simplicity, we shall state these properties for sentential logics, but appropriate versions hold as well for their first-order counterparts.

First we examine the REPLACEMENT THEOREM *For both* **R** *and* **E**,

$$\vdash (A \leftrightarrow B) \wedge t \to (\chi(A) \leftrightarrow \chi(B)).$$

Here $\chi(A)$ is any formula with perhaps some occurrences of A and $\chi(B)$ is the result of perhaps replacing one or more of those occurrences by B. The proof is by a straightforward induction on the complexity of $\chi(A)$, and one clear role of the conjoined t is to imply $\chi \to \chi$ when $\chi(= \chi(A))$ contains no occurrences of A, or does but none of them is replaced by B. It might be thought that if these degenerate cases are ruled out by requiring that some actual occurrence of A be replaced by B, then the need for t would vanish. This is indeed true for the implication-negation (and of course the pure implication) fragments of **R** and **E**, but not for the whole systems in virtue of the non-theoremhood of what V. Routley has dubbed 'Factor':

1. $(A \to B) \to (A \wedge \chi \to B \wedge \chi)$.

Here the closest one can come is to

2. $(A \to B) \wedge t \to (A \wedge \chi \to B \wedge \chi)$,

[11] The reader should be informed that still other natural deduction formalisms for **R** of various virtues can be found in [Meyer, 1979b] and [Meyer and McRobbie, 1979].

the conjoined g giving the force of having $\chi \to \chi$ in the antecedent, and the theorem $(A \to B) \wedge (\chi \to \chi) \to (A \wedge \chi \to B \wedge \chi)$ getting us home. (2) of course is just a special case of the Replacement Theorem. Of more 'relevant' interest is the

VARIABLE SHARING PROPERTY. If $A \to B$ is a theorem of **R** (or **E**), then there exists some sentential variable p that occurs in both A and B. This is understood by Anderson and Belnap as requiring some commonality of meaning between antecedent and consequent of logically true relevant implications. The proof uses an ingenious logical matrix, having eight values, for which see [Anderson and Belnap, 1975, Section 22.1.3]. There are discussed both the original proof of Belnap and an independent proof of Dončenko, and strengthening by Maksimova. Of modal interest is the

ACKERMANN PROPERTY. No formula of the form $A \to (B \to C)$ (A containing no $\to$) is a theorem of **E**. The proof again uses an ingenious matrix (due to Ackermann) and has been strengthened by Maksimova (see [Anderson and Belnap, 1975, Section 22.1.1 and Section 22.1.2]) (contributed by J. A. Coffa) on 'fallacies of modality'.

1.7 First-degree Entailments

A *zero degree formula* contains only the connectives $\wedge, \vee$, and $\neg$, and can be regarded as either a formula of relevance logic or of classical logic, as one pleases. A *first degree implication* is a formula of the form $A \to B$, where both A and B are zero-degree formulas: Thus first degree implications can be regarded as either a restricted fragment of some relevance logic (say **R** or **E**) or else as expressing some metalinguistic logical relation between two classical formulas A and B. This last is worth mention, since then even a classical logician of Quinean tendencies (who remains unconverted by the considerations of Section 1.2 in favour of nested implications) can still take first degree logical relevant implications to be legitimate.

A natural question is what is the relationship between the provable first-degree implications of **R** and those of **E**. It is well-known that the corresponding relationship between classical logic and some normal modal logic, say **S4** (with the $\to$ being the material conditional and strict implication, respectively), is that they are identical in their first degree fragments. The same holds of **R** and **E** (cf. [Anderson and Belnap, 1975, Section 2.42]).

This fragment, which we shall call $\mathbf{R_{fde}}$ (Anderson and Belnap [1975] call it $\mathbf{E_{fde}}$) is stable (cf. [Anderson and Belnap, 1975, Section 7.1]) in the sense that it can be described from a variety of perspectives. For some semantical perspectives see Sections 3.3 and 3.4. We now consider some syntactical perspectives of more than mere 'orthographic' significance.

The perhaps least interesting of these perspectives is a 'Hilbert-style' presentation of $\mathbf{R_{fde}}$ (cf. [Anderson and Belnap, 1975, Section 15.2]). It has the following axioms:

3. $A \wedge B \to A, A \wedge B \to B$ Conjunction Elimination
4. $A \to A \vee B, B \to A \vee B$ Disjunction Introduction
5. $A \wedge (B \vee C) \to (A \wedge B) \vee C$ Distribution
6. $A \to \neg\neg A, \neg\neg A \to A$ Double Negation

It also has gobs of rules:

7. $A \to B, B \to C \vdash A \to C$ Transitivity
8. $A \to B, A \to C \vdash A \to B \wedge C$ Conjunction Introduction
9. $A \to C, B \to C \vdash A \vee B \to C$ Disjunction Introduction
10. $A \to B \vdash \neg B \to \neg A$ Contraposition.

More interesting is the characterisation of Anderson and Belnap [1962b, 1975] of $\mathbf{R_{fde}}$ as 'tautological entailments'. The root idea is to consider first the 'primitive entailments'.

11. $A_1 \wedge \ldots \wedge A_m \to B_1 \vee \ldots \vee B_n$,

where each A_i and B_j is either a sentential variable or its negate (an 'atom') and make it a necessary and sufficient criterion for such a primitive entailment to hold that same A_i actually be identically the same formula as some B_j (that the entailment be 'tautological' in the sense that A_i is repeated). This rules out both

12. $p \wedge \neg p \to q$,
13. $p \to q \vee \neg q$,

where there is no variable sharing, but also such things as

14. $p \wedge \neg p \wedge q \to \neg q$,

where there is (of course all of (12)–(14) are valid classically, where a primitive entailment may hold because of atom sharing or because either the antecedent is contradictory or else the consequent is a logical truth).

Now the question remains as to which non-primitive entailments to count as valid. Both relevance logic and classical logic agree on the standard count as valid. Both relevance logic and classical logic agree on the standard 'normal form equivalences': commutation, association, idempotence, distribution, double negation, and de Morgan's laws. So the idea is, given a candidate entailment $A \to B$, by way of these equivalences, A can be put into disjunctive normal form and B may be put into conjunctive normal form, reducing the problem to the question of whether the following is a valid entailment:

15. $A_1 \vee \cdots \vee A_k \to B_1 \wedge \cdots \wedge B_h$.

But simple considerations (on which both classical and relevance logic agree) having to do with conjunction and disjunction introduction and elimination show that (15) holds if for each disjunct A_i and conjunct B_j, the primitive entailment $A_i \to B_j$ is valid. For relevance logic this means that there must be atom sharing between the conjunction A_i and the disjunction B_j.

This criterion obviously counts the Disjunctive Syllogism

16. $\neg p \wedge (p \vee q) \to q$,

as an invalid entailment, for using distribution to put its antecedent into disjunctive normal form, (16) is reduced to

16′ $(\neg p \wedge p) \vee (\neg p \wedge q) \to q$.

But by the criterion of tautological entailments,

17. $\neg p \wedge p \to q$,

which is required for the validity of (16′), is rejected.

Another pleasant characterisation of $\mathbf{R_{fde}}$ is contained in [Dunn, 1976a] using a simplification of Jeffrey's 'coupled trees' method for testing classically valid entailments. The idea is that to test $A \to B$ one works out a truth-tree for A and a truth tree for B. One then requires that every branch in the tree for A 'covers' some branch in the tree for B in the sense that every atom in the covered branch occurs in the covering branch. This has the intuitive sense that every way in which A might be true is also a way in which B would be true, whether these ways are logically possible or not, since 'closed' branches (those containing contradictions) are not exempt as they are in Jeffrey's method for classical logic. This coupled-trees approach is ultimately related to the Anderson–Belnap tautological entailment method, as is also the method of [Dunn, 1980b] which explicates an earlier attempt of Levy to characterise entailment (cf. also [Clark, 1980]).

1.8 Relations to Familiar Logics

There is a sense in which relevance logic contains classical logic.

ZDF THEOREM (Anderson and Belnap [1959a]). *The zero-degree formulas (those containing only the connectives $\wedge, \vee, \neg$) provable in* **R** *(or* **E***) are precisely the theorems of classical logic.*

The proof went by considering a 'cut-free' formulation of classical logic whose axioms are essentially just excluded middles (which are theorems of **R** / **E**) and whose rules are all provable first-degree relevant entailments (cf. Section 2.7). This result extends to a first-order version [Anderson and Belnap Jr., 1959b]. (The admissibility of γ (cf. Section 2) provides another route to the proof to the ZDF Theorem.)

There is however another sense in which relevance logic does not contain classical logic:

FACT (Anderson and Belnap [1975, Section 25.1]). **R** (and **E**) lack as a derivable rule Disjunctive Syllogism:

$$\neg A, A \vee B \vdash B.$$

This is to say there is no deduction (in the standard sense of Section 1.4) of B from $\neg A$ and $A \vee B$ as premises. This is of course the most notorious feature of relevance logic, and the whole of Section 2 is devoted to its discussion.

Looking now in another direction, Anderson and Belnap [1961] began the investigation of how to translate intuitionistic and strict implication into **R** and **E**, respectively, as 'enthymematic' implication. Anderson and Belnap's work presupposed the addition of propositional quantifies to, let us say **R**, with the subsequent definition of 'A intuitionistically implies B' (in symbols $A \supset B$) as $\exists p(p \wedge (A \wedge p \to B))$. This has the sense that A together with some truth relevantly implies B, and does seem to be at least in the neighbourhood of capturing Heyting's idea that $A \supset B$ should hold if there exists some 'construction' (the p) which adjoined to A 'yields' (relevant implication) B. Meyer in a series of papers [1970a, 1973] has extended and simplified these ideas, using the propositional constant t in place of propositional quantification, defining $A \supset B$ as $A \wedge t \to B$. If a propositional constant F for the intuitionistic absurdity is introduced, then intuitionistic negation can be defined in the style of Johansson as $\neg A =_{\text{df}} A \supset F$. As Meyer has discovered one must be careful what axiom one chooses to govern F. $F \to A$ or even $F \supset A$ is too strong. In intuitionistic logic, the absurd proposition *intuitionistically* implies only the *intuitionistic* formulas, so the correct axiom is $F \supset A^*$, where A^* is a translation into **R** of an intuitionistic formula. Similar translations carry **S4** into **E** and classical logic into **R**.

2 THE ADMISSIBILITY OF γ

2.1 Ackermann's Rule γ

The first mentioned problem for relevance logics in Anderson's [1963] seminal 'open problems' paper is the question of 'the admissibility of γ'. To demystify things a bit it should be said that γ is simply *modus ponens* for the material conditions $(\neg A \vee B)$:

$$1.\quad \frac{\begin{array}{c} A \\ \neg A \vee B \end{array}}{B}\,.$$

It was the third listed rule of Ackermann's [1956] system of *strenge Implikation* (α, β, γ; 1st, 2nd, 3rd). This was the system Anderson and Belnap 'tinkered with' to produce **E** (Ackermann also had a rule δ which they replaced with an axiom).

The major part of Anderson and Belnap's 'tinkering' was the extremely bold step of simply deleting γ as a primitive rule, on the well- motivated ground that the corresponding object language formula

2. $A \wedge (\neg A \vee B) \to B$

is not a theorem of **E**.

It is easy to see that (2) could not be a theorem of either **E** or **R**, since it is easy to prove in those systems

3. $A \wedge \neg A \to A \wedge (\neg A \vee B)$

(largely because $\neg A \to \neg A \vee B$ is an instance of an axiom), and of course (3) and (2) yield by transitivity the 'irrelevancy'

4. $A \wedge \neg A \to B$.

The inference (1) is obviously related to the Stoic principle of the *disjunctive syllogism*:

$$5.\quad \frac{\begin{array}{c} \neg A \\ A \vee B \end{array}}{B}\,.$$

Indeed, given the law of double negation (and replacement) they are equivalent, and double negation is never at issue in the orthodox logics. Thus **E** and **R** reject

6. $\neg A \wedge (A \vee B) \to B$

as well as (2).

This rejection is typically the hardest thing to swallow concerning relevance logics. One starts off with some pleasant motivations about relevant implication and using subscripts to keep track of whether a hypothesis has actually been used (as in Section 1.5), and then one comes to the point where one says 'and of course we have to give up the disjunctive syllogism' and one loses one's audience. Please do not stop reading! We shall try to make this rejection of *disjunctive syllogism* as palatable as we can.

(See [Belnap and Dunn, 1981, Restall, 1999] for related discussions, and also discussion of [Anderson and Belnap, 1975, Section 16.1]); see Burgess [1981] for an opposing point of view.

2.2 The Lewis 'Proof'

One reason that *disjunctive syllogism* has figured so prominently int he controversy surrounding relevance logic is because of the use it was put to by C. I. Lewis [Lewis and Langford, 1932] in his so-called 'independent proof': that a contradiction entails any sentence whatsoever (taken by Anderson and Belnap as a clear breakdown of relevance). Lewis's proof (with our notations of justification) goes as follows:

(1)	$p \wedge \neg p$	
(2)	p	2, $\wedge$-Elimination
(3)	$\neg p$	1, $\wedge$-Elimination
(4)	$p \vee q$	2, $\vee$-Introduction
(5)	q	3, 4 *disjunctive syllogism*

Indeed one can usefully classify alternative approaches to relevant implication according to how they reject the Lewis proof. Thus, e.g. Nelson rejects $\wedge$-Elimination and $\vee$-Introduction, as does McCall's connexive logic. Parry, on the other hand, rejects only $\vee$-Introduction. Geach, and more recently, Tennant [1994],accept each step, but says that 'entailment' (relevant implication) is not transitive. It is the genius of the Anderson–Belnap approach to see disjunctive syllogism as the culprit and the sole culprit.[12]

Lewis concludes his proof by saying, "If by (3), p is false; and, by (4), at least one of the two, p and q is true, then q must be true". As is told in [Dunn, 1976a], Dunn was saying such a thing to an elementary logic class

[12] Although this point is complicated, especially in some of their earlier writings (see, e.g. [Anderson and Belnap Jr., 1962a]) by the claim that there is a kind of fallacy of ambiguity in the Lewis proof. the idea is that if $\vee$ is read in the 'intensional' way (as $\neg A \to B$), then the move from (3) and (4) to (5) is OK (it's just *modus ponens* for the relevant conditional), but the move from (2) to (4) is not (now being a paradox of implication rather than ordinary disjunction introduction).

one time (with no propaganda about relevance logic) when a student yelled out, "But p was the true one—look again at your assumption".

That student had a point. Disjunctive syllogism is not obviously appropriate to a situation of inconsistent information—where p is assumed (given, believed, etc.) to be both true and false. This point has been argued strenuously in, e.g. [Routley and Routley, 1972, Dunn, 1976a] and Belnap [1977b, 1977a]. The first two of these develop a semantical analysis that lets both p and $\neg p$ receive the value 'true' (as is appropriate to model the situation where $p \wedge \neg p$ has been assumed true), and there will be more about these ideas in Section 3.4. The last is particularly interesting since it extends the ideas of Dunn [1976a] so as to provide a model of how a computer might be programmed as to make inferences from its (possibly inconsistent) database. One would not want trivially inconsistent information about the colour of your car that somehow got fed into the FBI's computer (perhaps by pooled databases) to lead to the conclusion that you are Public Enemy Number One.

We would like to add yet one more criticism of *disjunctive syllogism*, which is sympathetic to many of the earlier criticisms.

We need as background to this criticism the natural deduction framework of [Gentzen, 1934] as interpreted by [Prawitz, 1965] and others. the idea (as in Section 1.5) is that each connective should come with rules that introduce it into discourse(as principal connective of a conclusion) and rules that eliminate it from discourse (as principal connective of a premise). further the 'normalisation ideas of Prawitz, though of great technical interest and complication, boil down philosophically to the observation that an elimination rule should not be able to get out of a connective more than an introduction rule can put into the connective. This is just the old conservation Principle, 'You can't get something for nothing', applied to logic.

The paradigm here is the introduction and elimination rules for conjunction. The introduction rule, from A, B to infer $A \wedge B$ packs into $A \wedge B$ precisely what the elimination rule, from $A \wedge B$ to infer either A or B (separately), then unpacks.

Now the standard introduction rule for disjunction is this: from either A or B separately, infer $A \vee B$. We have no quarrel with an introduction rule. an introduction rule gives meaning to a connective and the only thing to watch out for is that the elimination rule does not take more meaning from a connective than the introduction rule gives to it (of course, one can also worry about the usefulness and/or naturalness of the introduction rules for a given connective, but that (*pace* [Parry, 1933]) seems not an issue in the case of disjunction.

In the Lewis 'proof' above, it is then clear that the *disjunctive syllogism* is the only conceivably problematic rule of inference. Some logicians (as indicated above) have queried the inferences from (1) to (2) and (4), and from (2) to (3), but from the point of view that we are now urging, this is simply wrongheaded. Like Humpty Dumpty, we use words to mean what *we* say. So there is nothing wrong with introducing connectives $\wedge$ and $\vee$ *via* the standard introduction rules. Other people may want connectives for which *they* provide different introduction (and matching elimination) rules, but that is *their* business. We want the standard ('extensional') senses of $\wedge$ and $\vee$.

Now the d.s. is a very odd rule when viewed as an elimination rule for $\vee$ parasitical upon the standard introduction rules (whereas the constructive dilemma, the usual $\vee$-Elimination rule is not at all odd). Remember that the introduction rules provide the actual inferences that are to be stored in the connective's battery as potential inferences, perhaps later to be released again as actual inferences by elimination rules. The problem with the *disjunctive syllogism* is that it can release inferences from $\vee$ that it just does not contain. (In another context, [Belnap, 1962] observed that Gentzen-style rules for a given connective should be 'conservative', i.e. they should not create new inferences not involving the given connective.)

Thus the problem with the *disjunctive syllogism* is just that $p \vee q$ might have been introduced into discourse (as it is in the Lewis 'proof') by $\vee$-Introduction from p. So then to go on to infer q from $p \vee q$ and $\neg p$ by the *disjunctive syllogism* would be legitimate only if the inference from $p, \neg p$ to q were legitimate. But this is precisely the point at issue. At the very least the Lewis argument is circular (and not independent).[13]

2.3 The Admissibility of γ

Certain rules of inference are sometimes 'admissible' in formal logics in the sense that whenever the premises are *theorems*, so is the conclusion a *theorem*, although these rules are nonetheless invalid in the sense that the premises may be true while the conclusion is not. Familiar examples are the rule of substitution in propositional logic, generalisation in predicate logic, and necessitation in modal logic. Using this last as paradigm, although the inference from A to $\Box A$ (necessarily A) is clearly invalid and would indeed vitiate the entire point of modal logic, still for the ('normal') modal logics, whenever A is a theorem so is $\Box A$ (and indeed their motivation would be somehow askew if this did not hold).

Anderson [1963] speculated that something similar was afoot with respect to the rule γ and relevance logic. Anderson hoped for a 'sort of lucky accident', but the admissibility of γ seems more crucial to the motivation of **E** and **R** than that. Kripke [1965] gives a list of four conditions that a

[13]This is a new argument on the side of Anderson and Belnap [1962b, pp. 19, 21].

propositional calculus must meet in order to have a normal characteristic matrix, one of which is the admissibility of γ.[14] 'Normal' is meant in the sense of Church, and boils down to being able to divide up its elements into the 'true' and the 'false' with the operations of conjunction, disjunction, and negation treating truth and falsity in the style of the truth tables (a conjunction is true if both components are true, etc.). If one thinks of **E** (as Anderson surely did) as the logic of propositions with the logical operations, and surely this should divide itself up into the true and the false propositions.[15]

2.4 Proof(s) of the Admissibility of γ

There are by now at least four variant proofs of the admissibility of γ for **E** and **R**. The first three proofs (in chronological order: [Meyer and Dunn, 1969], [Routley and Meyer, 1973] and [Meyer, 1976a]) are all basically due to Meyer (with some help from Dunn on the first, and some help from Routley on the second), and all depend on the same first lemma. The last proof was obtained by Kripke in 1978 and is unpublished (see [Dunn and Meyer, 1989]).

All of the Meyer proofs are what Smullyan [1968] would call 'synthetic' in style, and are inspired by Henkin-style methods. The Kripke proof is 'analytic' in style, and is inspired by Kanger–Beth–Hintikka tableau-style methods. In actual detail, Kripke's argument is modelled on completeness proofs for tableau systems, wherein a partial valuation for some open branch is extended to a total valuation. As Kripke has stressed, this avoids the apparatus of inconsistent theories that has hitherto been distinctive of the various proofs of γ's admissibility.

We shall sketch the third of Meyer's proofs, leaving a brief description of the first and second for Section 3.11. Since they depend on semantical notions introduced there.

The strategy of all the Meyer proofs can be divided into two segments: The Way Up and The Way Down. Of course we start with the hypotheses that $\vdash A$ and $\vdash \neg A \vee B$, yet assume not $\vdash B$ for the sake of *reduction*. We shall be more precise in a moment, but The Way Up involves constructing in a Henkin-like manner a maximal theory T (containing all the logical theorems) with $B \notin T$. The problem though is that T may be inconsistent in the sense of having both $C, \neg C \in T$ for some formula C. (Of course this could not happen in classical logic, for by virtue of the paradox of implica-

[14]The other conditions are that it be consistent, that it contain all classical tautologies, and that it be 'complete in the sense of Halldén'. **R** and **E** can be rather easily seen to have the first two properties (see Section 1.8 for the bit about classical tautologies), but the last is rather more difficult (see Section 3.11).

[15]This would be less obvious to Routley and Meyer [1976], and Priest [1987, 1995] who raise the 'consistency of the world' as a real problem.

tion $C \wedge \neg C \to B, B$ would be a member of T contrary to construction.) The Way Down fixes this by finding in effect some subtheory $T' \subseteq T$ that is both complete and consistent, and indeed is a 'truth set' in the sense of [Smullyan, 1968] (Meyer has labelled it the Converse Lindenbaum Lemma). Thus for all formulas X and Y, $\neg X \in T'$ iff $X \notin T'$, and $X \vee Y \in T'$ iff at least one of X and Y is in T'. So since $\neg A \vee B \in T'$, at least one of $\neg A$ and B is in T'. But since $A \in T'$, then $\neg A$ is not in T'. So B must be in T'.[16] But T' is a subset of T, which was constructed to keep B out. So B cannot be in T', and so by *reductio* we obtain B as desired.

Enough of strategy! We now collect together a few notions needed for a more precise statement of The Way Up Lemma. Incidentally, we shall from this point on in our discussion of γ consider only the case of **R**. Results for **E** (and a variety of neighbours) hold analogously.

By an '**R**-theory' we mean a set of formulas T of **R** closed under adjunction And logical relevant implication, i.e. such that

1. if $A, B \in T$, then $A \wedge B \in T$;
2. if $\vdash_{\mathbf{R}} A \to B$ and $A \in T$, then $B \in T$.

Note that an arbitrary **R**-theory may lack some or all of the theorems of **R** (in classical logic and most familiar logics this would be impossible because of the paradox of strict implication which says that a logical theorem is implied by everything). We thus need a special name for those **R**–theories that contain all of the **R**-theorems—those are called *regular*.[17] In this section, since we have no use of irregular theories and shall be talking only of **R**, by a *theory* we shall always mean a regular **R** theory (irregular **R**-theories however play a great role in the completeness theorems of Section 3 below and there we shall have to be more careful about our distinctions).

A theory T is called *prime* if whenever $A \vee B \in T$, then $A \in T$ or $B \in T$. The converse of this holds for *any* theory T in virtue of the **R**-axioms $A \to A \vee B$ and $B \to A \vee B$ and property (2). A theory T is called *complete* if for every formula $A, A \in T$ or $\neg A \in T$, and called *consistent* if

[16] The proof as given here would appear to use *disjunctive syllogism* in the metalanguage at just this point, but it can be restructured (indeed we so restructured the original proofs [Meyer and Dunn, 1969]) so as to avoid at least such an explicit use of *disjunctive syllogism*. The idea is to obtain by distribution ($A \in T'$ and $A \notin T'$) or ($B \in T'$ and $B \notin T'$) from the hypothesis $B \notin T'$. The whole question of a 'relevant' version of the admissibility of γ is a complicated one, and admits of various interpretations. See [Belnap and Dunn, 1981, Meyer, 1978].

[17] It is interesting to note for regular theories, condition (2) may be replaced with the condition

(2′) if $A \in T$ and $(A \to B) \in T$, then $B \in T$, in virtue of the **R**-theorem $A \wedge (A \to B) \to B$.

for no formula A do we have both $A, \neg A \in T$. In virtue of the **R**-theorem $A \vee \neg A$, we have that all prime theories are complete. A consistent prime theory is called *normal*, and it should by now be apparent that a normal theory is a truth set in the sense of Smullyan given above.

Where Γ is a set of formulas, we write $\Gamma \vdash_{\mathbf{R}} A$ to mean that A is deducible from Γ in the 'official sense' of there being a finite sequence $B_1, \ldots, B_n$, with $B_n = A$ and each B_i being either a member of Γ, or an axiom of **R**, or a consequence of earlier terms by *modus ponens* or adjunction (in context we shall often omit the subscript **R**). We write $\Gamma \vdash_{\Delta} A$ to mean that $\Gamma \cup \Delta \vdash_{\mathbf{R}} A$, and quite standardly we write things like $\Gamma, A \vdash_{\mathbf{R}} B$ in place of the more formal $\Gamma \cup \{A\} \vdash_{\mathbf{R}} B$. Note that for any theory T, writing $\vdash_T A$ in place of $\phi \vdash_T A$ boils down to saying that A is a theorem of $T (A \in T)$. Where Δ is a set of formulas not necessarily a theory, $\vdash_{\Delta} A$ can be thought of as saying that A is deducible from the 'axioms' Δ. The set $\{A : \vdash_{\Delta} A\}$ is pretty intuitively the smallest theory containing the axioms Δ, and we shall label it as $Th(\Delta)$.

We can now state and sketch a proof of the

WAY UP LEMMA. *Suppose not* $\vdash_{\mathbf{R}} A$. *Then there exists a prime theory* T *such that not* $\vdash_T A$.

Proof. Enumerate the formulas of $\mathbf{R} : X_1, X_2, \ldots$. Define a sequence of sets of formulas by induction as follows.

$T_0 =$ set of theorems of **R**.

$T_{i+1} = Th(T_i \cup \{X_{i+1}\})$ if it is not the case that $T_i, X_{i+1} \vdash A$;
T_i, otherwise.

Let T be the union of all these T_n's. It is easy to see as is standard that T is a theory not containing A. Also we can show that T is prime.

Thus suppose $\vdash_T X \vee Y$ and yet $X, Y \notin T$. Then it is easy to se that since neither X nor Y could be added to the construction when their turn came up without yielding A, we have both

1. $X \vdash_T A$,
2. $Y \vdash_T A$.

But by reasonably standard moves (**R** has distribution), we get

3. $X \vee Y \vdash_T A$,

and so $\vdash_T A$ contrary to the construction. ∎

THE WAY DOWN LEMMA. *Let T' be a prime theory. Then there exists a normal theory $T \subseteq T'$.*

The concept we need is that of a 'metavaluation' (more precisely as we use it here a '*quasi*-metavaluation', but we shall not bother the reader with such detail). The concept and its use *re* γ may be found in [Meyer, 1976a]. (See also Meyer [1971, 1976b] for other applications.) For simplicity we assume for a while that the only primitive connectives are $\neg, \vee$ and $\rightarrow$ ($\wedge$ can be defined *via* de Morgan). A metavaluation v is a function from the set of formulas into the truth values $\{0, 1\}$, such that

1. for a propositional variable $p, v(p) = 1$ iff $p \in T$;
2. $v(\neg A) = 1$ iff both (a) $v(A) = 0$ and (b) $\neg A \in T$;
3. $v(A \vee B) = 1$ iff either $v(A) = 1$ or $v(B) = 1$.
4. $v(A \rightarrow B) = 1$ iff both (a) $v(A) = 0$ or $v(B) = 1$, and (b) $A \rightarrow B \in T$.

One surprising aspect of these conditions is the double condition in (2) that must be met for $\neg A$ to be assigned the value 1. Not only must (a) A be assigned 0 (the usual 'extensional condition'), but also (b) $\neg A$ must be a theorem of T (the 'intensional condition'). and of course there are similar remarks about (4). The condition in (1) also relies upon G (actually to a lesser extent than it might seem—when both $p, \neg p \in T$, it would not hurt to let $v(p) = 0$).

We now set $T' = \{A : v(A) = 1\}$. The following lemma is useful, and has an easy proof by induction on complexity of formulas (the case when A is a negation evaluated as 0 uses the completeness of T).

COMPLETENESS LEMMA. *If $v(A) = 1$, then $A \in T$. If $v(A) = 0$, then $\neg A \in T$.*

It is reasonably easy to see that T' is in fact a truth set. That it behaves OK with respect to disjunction can be read right off of clause (3) in the definition of v, so we need only look at negation where the issue is whether T' is both consistent and complete. It is clear from clause (2) that T' is consistent, but T' is also complete. Thus, suppose $A \notin T'$, then by the Completeness Lemma $\neg A \in T$. This is the intensional condition for $v(\neg A) = 1$, but our supposition that $A \notin T'$ is just the extensional condition that $v(A) = 0$. Hence $v(\neg A) = 1$, i.e. $\neg A \in T'$ as desired.

It is also reasonably easy to check that T' is an **R**-theory. It is left to the reader to do the easy calculation that T' is closed under adjunction and **R**-implication, i.e. that these preserve assignments by v of the value 1. Here we will illustrate the more interesting verification that the **R**-axioms all get assigned the value 1. We shall not actually check all of them, but rather consider several typical ones.

First we check suffixing: $(A \to B) \to [(B \to C) \to (A \to C)]$. Suppose v assigns it 0. Since it is a theorem of **R** and *a fortiori* of T, then it satisfies the intensional condition and so must fail to satisfy the extensional condition. So $v(A \to B) = 1$ and $v((B \to C) \to (A \to C)) = 0$. By the Completeness Lemma, then $(A \to B) \in T$, and so by *modus ponens* from the very axiom in question (Suffixing) we have that $(B \to C) \to (A \to C) \in T$. So $v((B \to C) \to (A \to C))$ satisfies the intensional condition, and so must fail to satisfy the extensional condition since it is 0. So $v(B \to C) = 1$ and $v(A \to C) = 0$. By reasoning analogous to that above (one more *modus ponens*) we derive that $v(A \to C)$ must finally fail to satisfy the extensional condition, i.e. $v(A) = 1$ and $v(C) = 0$. But clearly since all of $v(A \to B) = 1$, $v(B \to C) = 1$, $v(A) = 1$, then by the extensional condition, $v(C) = 1$, and we have a contradiction.

The reader might find it instructive in seeing how negation is handled to verify first the intuitionistically acceptable form of the Reductio axioms $(A \to \neg A) \to \neg A$, and then to verify its classical variant (used in some axiomatisations of **R**), $(\neg A \to A) \to A$. The first is easier. Also Classical Double Negation, $\neg\neg A \to A$ is fun.

This completes the sketch of Meyer's latest proof of the admissibility of γ for **R**.

2.5 γ for First-order Relevance Logics

The first proof of the admissibility of γ for first-order **R**, **E**, etc. (which we shall denote as **RQ**, etc.) was in Meyer, Dunn and Leblanc [1974], and uses algebraic methods analogous to those used for the propositional relevance logic in [Meyer and Dunn, 1969]. The proof we shall describe here though will again be Meyer's metavaluation-style proof.

The basic trick needed to handle first-order quantifiers is to produce this time a *first-order truth set*. Assuming that only the universal quantifier $\forall$ is primitive (the existential can be defined: $\exists x =_{\text{df}} \neg\forall x\neg$), this means we need

($\forall$) $\quad \forall x A \in T$ iff $A(a/x) \in T$ for all parameters (free variables) a.

This is easily accommodated by adding a clause to the definition of the metavaluation v so that

5. $v(\forall x A) = 1$ iff $v(A(a/x)) = 1$ for all parameters a.

This does not entirely fix things, for in proving the Completeness Lemma we have now in the induction to consider the case when A is of the form

$\forall xB$. If $v(\forall xB) = 1$, then (by (5)), $va(B(a/x)) = 1$ for all parameters a. By inductive hypothesis, for all $a, B(a/x) \in T$. But, and here's the rub, this does not guarantee that $\forall xB(a/x) \in T$. We need to have constructed on The Way Up a theory T that is 'ω-complete' in just the sense that this guarantee is provided. ([Meyer *et al.*, 1974] call such a theory 'rich'.) Of course it is understood by 'theory' we now mean a 'regular **RQ**-theory', i.e. one containing all of the axioms of **RQ** and closed under its rules (see Section 1.3). Actually things can be arranged as in [Meyer *et al.*, 1974] so that generalisation is in effect built into the axioms so that the only rules can continue to be adjunction and *modus ponens.*

Thus we need the following

WAY UP LEMMA FOR **RQ**. *Suppose A is not a theorem of first-order* **RQ**. *Then there exists a prime rich theory T so that $A \notin T$.*

This lemma is Theorem 3 of [Meyer *et al.*, 1974], and its proof is of basically a Henkin style with one novelty. In usual Henkin proofs one can assure ω-completeness by building into the construction of T that whenever $\neg\forall xB$ is put in, then so is $\neg B(a/x)$ for some *new* parameter a. This guarantees ω-completeness since if $B(a) \in T$ for all a, but $\forall xB \notin T$, then by completeness $\neg\forall xB \in T$ and so by the usual construction $\neg B(a) \in T$ for some a, and so *by consistency* (??) $B(a) \notin T$ for some a, contradicting the hypothesis for ω-completeness. But we of course have for relevance logics no guarantee that T is consistent, as has been remarked above.

The novelty then was to modify the construction so as to keep things out as well as put things in, though this last still was emphasised. Full symmetry with respect to 'good guys' and 'bad guys' was finally obtained by Belnap, [18] in what is called the Belnap Extension Lemma, which shall be stated after a bit of necessary terminology.

We shall call an ordered pair (Δ, Θ) of sets of formulas of **RQ** and '**RQ**-pair'. We shall say that one **RQ** pair (Δ_1, Θ_1) *extends* another (Δ_0, Θ_0) if $\Delta_0 \subseteq \Delta_1$ and $\Theta_0 \subseteq \Theta_1$. An **RQ** pair is defined to be *exclusive* if for no $A_1, \ldots, A_m \in \Delta, B_1, \ldots, B_n \in \Theta$ do we have $\vdash A_1 \wedge \cdots \wedge A_m \to B_1 \vee \cdots \vee B_n$. It is called *exhaustive* if for every formula A, either $A \in \Delta$ or $A \in \Theta$.[19] It is now easiest to assume that $\wedge$ and $\exists$ are back as primitive. We call a set of formulas Γ $\vee$-*prime* ($\wedge$-*prime*) if whenever $A \vee B \in \Gamma(A \wedge B \in \Gamma)$,

[18] Belnap's result is unpublished, although he communicated it to Dunn in 1973. Dunn circulated a write-up of it about 1975. It is cited in some detail in [Dunn, 1976d]. Gabbay [1976] contains an independent but precise analogue for the first-order intuitionistic logic with constant domain.

[19] We choose our terminology carefully, not calling (Δ, Θ) a 'theory', not using 'consistency' for exclusiveness, and not using 'completeness' for exhaustiveness. We do this so as to avoid conflict with our earlier (and more customary) usage of these terms and in this we differ on at least one term from usages on other occasions by Gabbay, Belnap, or Dunn.

at least one of A or $B \in \Gamma$ (clearly $\vee$-primeness is the same as primeness). Analogously, we call Γ $\exists$-*prime* ($\forall$- *prime*) if whenever $\exists xA \in \Gamma(\forall xA \in \Gamma)$, then $A(a/x) \in \Gamma$ for some a. Given an **RQ** pair (Δ, Θ) we shall call $\Delta(\Theta)$ *completely prime* if Δ is both $\vee$- and $\exists$-prime (Θ is both $\wedge$- and $\forall$-prime). the pair (Δ, Θ) is called completely prime if both Δ and Θ are completely prime. We can now state the

Belnap Extension Lemma. *Let* (Δ, Θ) *be an exclusive* **RQ** *pair. Then* (Δ, Θ) *can be extended to an exclusive, exhaustive, completely prime* **RQ** *pair* (T, F) *in a language just like the language of* **RQ** *except for having denumerably many new parameters.*

We shall not prove this lemma here, but simply remark that it is a surprisingly straightforward application of Henkin methods to construct a maximal **RQ**-pair and show it has the desired properties (indeed it simply symmetrises the usual Henkin construction of first-order classical logic).

In order to derive the **RQ** Way Up Lemma we simply set $\Delta = \mathbf{RQ}$ and $\Theta = \{A\}$ and extend it to the pair (T, F) using the Belnap Extension Lemma. It is easy to see that T is a (*regular*) **RQ**-theory, and clearly G is prime. but also T is ω-complete. Thus suppose $B(a/x) \in T$ for all a, but $\forall xB \notin T$. Then by exhaustiveness $\forall xB \in FR$. Then by $\forall$-primeness, $B(a/x) \in FR$ for some a. But since $\vdash_{\mathbf{RQ}} B(a/x) \to B(a/x)$, this contradicts the exclusiveness of the pair (T, F).

2.6 γ for Higher-order Relevance Logics and Relevant Arithmetic

The whole point about γ being merely an *admissible* rule is that it might not hold for various extensions of **F** (cf. [Dunn, 1970] for actual counter examples). Thus, as we just saw, it was an achievement to show that γ continues to e admissible in **R** when it is extended to include first-order quantification. The question of the admissibility of γ naturally has great interest when **R** is further extended to include theories in the foundations of mathematics such as type theory (set theory) and arithmetic.

Meyer [1976a] contains investigations of the admissibility of γ for relevant type theory ($\mathbf{R}^{\omega}$). We shall report nothing in the way of detail here except to observe that Meyer's result is invariant among various restrictions of the formulas A in the Comprehension Axiom scheme:

$$\exists X^{x+1} \forall y^n (X^{n+1}(y^n) \leftrightarrow A).$$

As for relevantly formulated arithmetic, most work has gone on in studying Meyer's systems $\mathbf{R}^{\sharp}$, $\mathbf{R}^{\sharp\sharp}$ and their relatives, based on Peano arithmetic, though Dunn has also considered a relevantly formulated version of Robinson Arithmetic [Anderson *et al.*, 1992]. Here we will recount the results for $\mathbf{R}^\sharp$ and $\mathbf{R}^{\sharp\sharp}$ for they are rather surprising. In a nutshell, γ is admissible in relevant arithmetics with the infinitary ω-rule (from $A(0)$, $A(1)$, $A(2)$, ... to infer $\forall x A(x)$), but not without it [Friedman and Meyer, 1992, Meyer, 1998].

The system $\mathbf{R}^\sharp$ is given by rewriting the traditional axioms of Peano arithmetic with relevant implication instead of material implication in the natural places. You get the following list of axioms

Identity	$y = z \to (x = y \to x = z)$
Successor	$x' = y' \to x = y$
	$x = y \to x' = y'$
	$0 \neq x'$
Addition	$x + 0 = x$
	$x + y' = (x + y)'$
Multiplication	$x0 = 0$
	$xy' = xy + x$
Induction	$A(0) \wedge \forall x\big(A(x) \to A(x')\big) \to \forall x A(x)$

which you add to those of **RQ** in order to obtain an arithmetic theory. The question about the admissibility of γ was open for many years, until Friedman teamed up with Meyer to show that it is not [Friedman and Meyer, 1992]. The proof does not provide a direct counterexample to γ. Instead, it takes a more circuitous route. First, we need Meyer's classical containment result for $\mathbf{R}^\sharp$. When we map formulae in the extensional vocabulary of arithmetic to the language of $\mathbf{R}^\sharp$ by setting $\tau(x = y)$ to $(x = y) \vee (0 \neq 0)$ and leaving the rest of the map to respect truth functions (so $\tau(A \wedge B) = \tau(A) \wedge \tau(B)$, $\tau(\neg A) = \neg\tau(A)$ and $\tau(\forall x A) = \forall x \tau(A)$) then we have the following theorem:

> $\tau(A)$ is a theorem of $\mathbf{R}^\sharp$ iff A is a theorem of classical Peano arithmetic.

This is a subtle result. The proof goes through by showing, by induction, that $\tau(A)$ is equivalent either to $(A \wedge (0 = 0)) \vee (0 \neq 0)$ or to $(A \vee (0 \neq 0)) \wedge (0 = 0)$, and then that γ and the classical form of induction (with material implication in place of relevant implication) is valid for formulae of this form in $\mathbf{R}^\sharp$. Then, if we had the admissibility of γ for $\mathbf{R}^\sharp$, we could infer A from $\tau(A)$. (If $\tau(A)$ is equivalent to $(A \wedge (0 = 0)) \vee (0 \neq 0)$, then we can use $0 = 0$ and γ to derive $A \wedge (0 = 0)$, and hence A. Similarly for the other case).

The next significant result is that not all theorems of classical Peano arithmetic are theorems of $\mathbf{R}^\sharp$. Friedman provided a counterexample, which is simple enough to explain here. First, we need some simple preparatory results.

- $\mathbf{R}^\sharp$ is a conservative extension of the theory $\mathbf{R}^{\sharp+}$ axiomatised by the negation free axioms of $\mathbf{R}^\sharp$ [Meyer and Urbas, 1986].
- If classical Peano theorem is to be provable in $\mathbf{R}^\sharp$ and if it contains no negations, then it must be provable in $\mathbf{R}^{\sharp+}$.
- Any theorem provable in $\mathbf{R}^{\sharp+}$ must be provable in the classical positive system $\mathbf{PA}^+$ which is based on classical logic, instead of $\mathbf{R}$.

The proofs of these results are relatively straightforward. The next result is due to Friedman, and it is much more surprising.

- The ring of complex numbers is a model of $\mathbf{PA}^+$.

The only difficult thing to show is that it satisfies the induction axiom. For any formula $A(x)$ in the vocabulary of arithmetic, the set of complex numbers α such that $A(\alpha)$ is true is either finite or cofinite. If $A(x)$ is atomic, then it is equivalent to a polynomial of the form $f(x) = 0$, and f must either have finitely many roots or be 0 everywhere. But the set of either finite or cofinite sets is closed under boolean operations, so no $A(x)$ we can construct will have an extension which is neither finite or cofinite.) As a result, the induction axiom must be satisfied. For if $A(0)$ holds and if $A(x) \supset A(x')$ holds then there are infinitely many complex numbers α such that $A(\alpha)$. So the extension of A is at least cofinite. But if there is a point α such that $A(\alpha)$ fails, then so would $A(\alpha - 1)$, $A(\alpha - 2)$ and so on by the induction step $A(x) \supset A(x')$, and this contradicts the confinitude of the extension of A. As a result, $A(\alpha)$ holds for *every* α.

We can then use this surprising model of positive Peano arithmetic to construct a Peano theorem which is not a theorem of $\mathbf{R}^\sharp$. It is known that for any odd prime p, there is a positive integer y which is not a quadratic residue mod p. That is, $\exists y \forall z \neg(y \equiv z^2 \bmod p)$ is provable in Peano arithmetic. This formula can be rewritten in the language or arithmetic with a little work. However, the corresponding formula is false in the complex numbers, so it is not a theorem of $\mathbf{PA}^+$. Therefore it isn't a theorem of $\mathbf{R}^\sharp$ $^+$, and by the conservative extension result, it is not a theorem of $\mathbf{R}^\sharp$. As a consequence, $\mathbf{R}^\sharp$ is not closed under γ.

Where is the counterexample to γ? Meyer's containment result provides a proof of $\tau(B)$, where B is the quadratic residue formula. The γ rule would allow us to derive B from $\tau(B)$, and it is here that γ must fail.

If we replace the induction axiom by the infinitary rule ω, we can prove the admissibility of γ using a modification of the Belnap Extension Lemma for the Way Up and using the standard metavaluation technique for the Way Down. The modification of the Belnap Extension Lemma is due to Meyer [1998].

> Belnap Extension Lemma, with Witness Protection: Let (Δ, Θ) be an exclusive $\mathbf{R}^{\sharp\sharp}$ pair in the language of arithmetic (that is, with 0 as the only constant). Then (Δ, Θ) can be extended to an exclusive, exhaustive, completely prime $\mathbf{R}^{\sharp\sharp}$ pair (T, F) in the same language.

This lemma requires the ω-rule for its proof. Consider the induction stage in which you wish to place $\forall x A(x)$ in Θ_i. The witness condition dictates that there be some term t such that $A(t)$ also appear in Θ_i. The ω-rule ensures that we can do this without the need for a new term, for if no term $0''^{\cdots\prime}$ could be consistently added to Θ_i, then each $A(0''^{\cdots\prime})$ is a consequence of Δ_i, and by the ω-rule, so is $\forall x A(x)$, contradicting the fact that we can add $\forall x A(x)$ to Θ_i. So, we know that some $0''^{\cdots\prime}$ will do, and as a result, we need add no new constants to form the complete theory T. The rest of the way up lemma and the whole of the way down lemma can then be proved with little modification. (for details, see [Meyer, 1998]). Consequently, γ is admissible in $\mathbf{R}^{\sharp\sharp}$.

These have been surprising results, and important ones, for relevant arithmetic is an important 'test case' for accounts of relevance. It is a theory in which we can have some fairly clear idea of what it is for one formulae to properly *follow from* another. In $\mathbf{R}^{\sharp}$ and $\mathbf{R}^{\sharp\sharp}$, we have $0 = 2 \to 0 = 4$ because there is an 'arithmetically appropriate' way to derive $0 = 4$ from $0 = 2$ — by multiplying both sides by 2. However, we cannot derive $0 = 2 \to 0 = 3$, and, correspondingly, there is no way to derive $0 = 3$ from $0 = 2$ using the resources of arithmetic. The only way to do it within the vocabulary is to appeal to the falsity of $0 = 3$, and this is not a relevantly acceptable move. $0 \neq 3 \to (0 = 3 \to 0 = 2)$ does not have much to recommend as pattern of reasoning which respects the canons of relevance.

We are left with important questions. Are there axiomatisable extensions of $\mathbf{R}^{\sharp}$ which are closed under γ? Can theories like $\mathbf{R}^{\sharp}$ and $\mathbf{R}^{\sharp\sharp}$ be extended to deal with more interesting mathematical structures, while keeping account of some useful notion of relevance? Early work on this area, from a slightly different motivation (paraconsistency, not relevance) indicates that there are some interesting results at hand, but the area is not without its difficulties [Mortensen, 1995].

The admissibility of γ would also seem to be of interest for relevant type theory (even relevant second-order logic) with an axiom of infinity (see [Dunn, 1979b]).

One of the chief points of philosophical interest in showing the admissibility of γ for some relevantly formulated version of a classical theory relates to the question of the consistency of the classical theory (this was first pointed out in Meyer, Dunn and Leblanc [1974]). As we know from Gödel's work, interesting classical theories cannot be relied upon to prove their own consistency. To exaggerate perhaps only a little, the consistency of systems like Peano (even Robinson) arithmetic must be taken in faith.

But using relevance logic in place of classical logic in formulating such theories gives us a new strategy of faith. It is conceivable that since relevance logic is weaker than classical logic, the consistency of the resultant theory might be easier to demonstrate. This has proved true at least in the sense of absolute consistency (some sentence is unprovable) as shown by [Meyer, 1976c] for Peano arithmetic using elementary methods. Classically of course there is no difference between absolute consistency and ordinary (negation) consistency (for no sentence are both A and $\neg A$ provable), and if γ is admissible for the theory, then this holds for relevance logic, too. The interesting thing then would be to produce a proof of the admissibility of γ, which we know from Gödel would itself have to be non-elementary.

One could then imagine arguing with a classical mathematician in the following Pascal's Wager sort of way [Dunn, 1980a].

> Look. You have equally good reason to believe in the negation consistency of the classical system and the (relative) completeness of the relevant system. In both cases you have a non-elementary proof which secures your belief, but which might be mistaken. Consider the consequences in each case if it is mistaken. If you are using the classical system, disaster! Since even one contradiction classically implies everything, for each theorem you have proven, you might just as well have proven its negation. But if you are using the relevant system, things are not so bad. For at least large classes of sentences, it can be shown by elementary methods (Meyer's work) that not both the sentences and their negations are theorems.

2.7 Ackermann's γ and Gentzen's Cut: Gentzen Systems as Relevance Logic

In [Meyer *et al.*, 1974] an analogy was noted between the role that the admissibility of γ plays in relevance logic and the role that cut elimination plays in Gentzen calculi (even those for classical systems). For the reader unfamiliar with Gentzen calculi, this subsection will make more sense after she has read Sections 4.6 and 4.7. The Gentzen system for the classical propositional calculus $L\mathbf{K}$ with the material conditional and negation as primitive (as is well-known, all of the other truth-functional connectives can be defined from these) may be obtained by adding to the rules of $L\mathbf{R}^{\neg}_{\rightarrow}$ of Section 4.7. the rule of Thinning (see Section 4.6) on both the left and right. Gentzen also had as a primitive rule:

$$\frac{\alpha \vdash A, B \quad \gamma, A \vdash \delta}{\alpha, \gamma \vdash B, \delta}, \tag{Cut}$$

which has as a special case

$$\frac{\vdash \quad A \vdash B}{\vdash B}. \tag{1}$$

Since $A \vdash B$ is derivable just when $\vdash A \to B$ is derivable, and since in classical logic $A \to B$ is equivalent to $\neg A \vee B$, (1) above is in effect

$$\frac{\vdash A \quad \vdash \neg A \vee B}{\vdash B}, \tag{1'}$$

which is just γ.

All of the Gentzen rules except Cut have the Subformula Property: Every formula that occurs in the premises also occurs in the conclusion, though perhaps there as a subformula. Gentzen showed *via* his *Hauptsatz* that Cut was redundant—it could be eliminated without loss (hence this is often called the Elimination Theorem). Later writers have tended to think of Gentzen systems as lacking the Cut Rule, and so the Elimination Theorem is stated as showing that Cut is admissible in the sense that whenever the premises are derivable so is the conclusion. There is thus even a parallel historical development with Ackermann's rule γ in relevance logic, since writers on relevance logic have tended to follow Anderson and Belnap's decision to drop γ as a primitive rule.

Note that the Subformula Property can be thought of as a kind of relation of relevance between premises and conclusion. Thus Cut as primitive destroys a certain kind of relevance property of Gentzen systems, just as γ as primitive destroys the relevance of premises to conclusion in relevance logic. The analogies become even clearer if we reformulate Gentzen's system according to the following ideas of [Schütte, 1956].

The basic objects of Gentzen's calculus $L\mathbf{K}$ were the *sequents* $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$, where the A_i's and B_j's are formulas (any or all of which might be missing). Such a sequent may be interpreted as a statement to the effect that either one of the A_i's is false or one of the B_j's is true. To every such sequent there corresponds what we might as well call its 'right-handed counterpart':

$$\vdash \neg A_1, \ldots, \neg A_m, B_1, \ldots, B_n$$

It is possible to develop a calculus parallel to Gentzen's using only 'right-handed' sequents, i.e. those with empty left side. This is in effect what

Schütte did, but with one further trick. Instead of working with a right-handed sequent $\vdash A_1, \ldots, A_m$, which can be thought of as a sequence of formulas, he in effect replaced it with the single formula $A_1 \vee \cdots \vee A_m$.[20]

With these explanations in mind, the reader should have no trouble in perceiving Schütte's calculus $\mathbf{K}_1$ as 'merely' a notational variant of Gentzen's original calculus $L\mathbf{K}$ (albeit, a highly ingenious one). Also Schütte's system had the existential quantifier which we have omitted here purely for simplicity. Dunn and Meyer [1989] treats it as well.

The axioms of $\mathbf{K}_1$ are all formulas of the form $A \vee \neg A$. The inference rules divide themselves into two types:

Structural rules:

$$\frac{\mathcal{M} \vee A \vee B \vee \mathcal{N}}{\mathcal{M} \vee B \vee A \vee \mathcal{N}}\text{[Interchange]} \qquad \frac{\mathcal{N} \vee A \vee A}{\mathcal{N} \vee A}\text{[Contraction]}$$

Operational rules:

$$\frac{\mathcal{N}}{\mathcal{N} \vee B}\text{[Thinning]} \quad \frac{\mathcal{N} \vee \neg A \quad \mathcal{N} \vee \neg B}{\mathcal{N} \vee \neg(A \vee B)}\text{[de Morgan]} \quad \frac{\mathcal{N} \vee A}{\mathcal{N} \vee \neg\neg A}\text{[Double Negation]}$$

It is understood in every case but that of Thinning that either both of $\mathcal{M}$ and $\mathcal{N}$ may be missing. Also there is an understanding in multiple disjunctions that parentheses are to be associated to the right.

In [Meyer *et al.*, 1974] it was said that the rule Cut is just γ 'in peculiar notation'. In the context of Schütte's formalism the notation is not even so different. Thus:

$$\frac{\mathcal{M} \vee A \quad \neg A \vee \mathcal{N}}{\mathcal{M} \vee \mathcal{N}}\text{[Cut]} \qquad \frac{A \quad \neg A \vee B}{B}[\gamma].$$

Since either $\mathcal{M}$ or $\mathcal{N}$ may be missing, obviously γ is just a special case of Cut.

It is pretty easy to check that each of the rules above corresponds to a provable first-degree relevant implication. Indeed [Anderson and Belnap Jr., 1959a] with their 'Simple Treatment' formulation of classical logic (extended to quantifiers in [Anderson and Belnap Jr., 1959b]) independently arrived at a Cut-free system for classical logic much like Schütte's (but with some improvements, i.e. they have more general axioms and avoid the need for structural rules). They used this to show that $\mathbf{E}$ contains all the classical tautologies as theorems, the point being that the Simple Treatment rules are all provable entailments in $\mathbf{E}$ (unlike the usual rule for axiomatic formulations of classical logic, *modus ponens* for the material conditional, i.e.

[20]It ought be noted that similar "single sided" Gentzen systems find extensive use in the proof theory for Linear Logic [Girard, 1987, Troelstra, 1992].

γ). Thus the later proven admissibility of γ was not needed for this purpose, although it surely can be so used. Schütte's system can also clearly be adapted to the purpose of showing that classical logic is contained in relevance logic, and indeed [Belnap, 1960a] used $\mathbf{K}_1$ (with its quantificational rules) to show that **EQ** contains all the theorems of classical first-order logic.

It turns out that one can give a proof of the admissibility of Cut for a classical Gentzen-style system, say Schütte's $\mathbf{K}_1$, along the lines of a Meyer-style proof of the admissibility of γ (see [Dunn and Meyer, 1989], first reported in 1974).[21] We will not give many details here, but the key idea is to treat the rules of $\mathbf{K}_1$ as rules of deducibility and not merely as theorem generating devices. Thus we define a *deduction* of A from a set of formulas Γ as a finite tree of formulas with A as its origin, members of Γ or axioms of $\mathbf{K}_1$ at its tips, and such that each point that is not a tip follows from the points just above it by one of the rules of $\mathbf{K}_1$ (this definition has to be slightly more complicated if quantifiers are present due to usual problems caused by generalisation). We can then inductively build a prime complete theory (closed under deducibility) on The Way up, which will clearly be inconsistent since because of the 'Subformula Property' clearly, e.g. q is not deducible from $p, \neg p$. but this can be fixed on The Way Down by using metavaluation techniques so as to find a complete consistent subtheory.

In 1976 E. P. Martin, Meyer and Dunn extended and analogised the result of Meyer concerning the admissibility of γ for relevant type theory described in the last subsection, in much the same way as the γ argument for the first-order logic has been analogised here, so as to obtain a new proof of Takeuti's Theorem (Cut-elimination for simple type theory). This unpublished proof dualises the proof of Takahashi and Prawitz (cf. [Prawitz, 1965]) in the same way that the proof here dualises the usual semantical proofs of Cut-elimination for classical first-order logic. This dualisation is vividly described by saying that in place of 'Schütte's Lemma' that every semi-(partial-) valuation may be extended to a (total) valuation, there is instead the 'Converse Schütte Lemma' that every 'ambi-valuation' (sometimes assigns a sentence both the values 0, 1) may be restricted to a (consistent) valuation.

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[21] We hasten to acknowledge the nonconstructive character of this prof. In this our proof compares with that of Schütte [1956] (also proofs for related formalisms due to Anderson and Belnap, Beth, Hintikka, Kanger) in its uses of semantical (model-theoretic) notions, and differs from Gentzen's. Like the proofs of Schütte *et al.* this proof really provides a completeness theorem. We may briefly label the difference between this proof and those of Schütte and the others by using (loosely) the jargon of Smullyan [1968]. Calling both Hilbert-style formalisms and their typical Henkin-style completeness proofs 'synthetic', and calling both Gentzen-style formalisms and their typical Schütte-style completeness proof 'analytic', it looks as if we can be said to have given an synthetic completeness proof for an analytic formalism.

3 SEMANTICS

3.1 Introduction

In Anderson's [1963] 'open problems' paper, the last major question listed, almost as if an afterthought, was the question of the semantics of **E** and **EQ**. Despite this appearance Anderson said (p. 16) 'the writer does *not* regard this question as "minor"; it is rather the principle large question remaining open'. Anderson cited approvingly some earlier work of Belnap's (and his) on providing an algebraic semantics for first-degree entailments, and said (p. 16), 'But the general problem of finding a semantics for the whole of **E**, with an appropriate completeness theorem, remains unsolved'.

It is interesting to note that Anderson's paper appeared in the same *Acta Filosphica Fennica* volume as the now classic paper of Kripke [1963] which provided what is now simply called 'Kripke-style' semantics for a variety of modal logics (Kripke [1959a] of course provided a semantics for **S5**, but it lacked the accessibility relation R which is so versatile in providing variations).

When Anderson was writing his 'open problems' paper, the paradigm of a semantical analysis of a non-classical logic was probably still something like the work of McKinsey and Tarski [1948], which provided interpretations for modal logic and intuitionistic logic by way of certain algebraic structures analogous to the Boolean algebras that are the appropriate structures for classical logic. But since then the Kripke-style semantics (sometimes referred to as 'possible-worlds semantics', or 'set-theoretical semantics') seems to have become the paradigm. Fortunately, **E** and **R** now have both an algebraic semantics and a Kripke-style semantics. We shall first distinguish in a kind of general way the differences between these two main approaches to semantics, before going on to explain the particular details of the semantics for relevant logics (again **R** will be our paradigm).

3.2 Algebraic vs. Set-theoretical Semantics

It is convenient to think of a logical system as having two distinct aspects syntax (well-formed strings of symbols, e.g. sentences) and semantics (what, e.g. these sentences mean, i.e. propositions). These double aspects compete with one another as can be seen in the competing usages 'sentential calculus' and 'propositional calculus', but we should keep firmly in mind both aspects.

Since sentences can be combined by way of connectives, say he conjunction sign $\wedge$, to form further sentences, typically there is for each logical system at least one natural algebra arising at the level of syntax, the algebra of sentences (if one has a natural logical equivalence relation there is yet another that one obtains by identifying logically equivalent sentences together into equivalence classes—the so-called 'Lindenbaum algebra'). And since propositions can be combined by the corresponding logical operations, say conjunction, to form propositions, here is an analogous algebra of propositions.

Now undoubtedly some readers, who were taught to 'Quine' propositions from an early age, will have troubles with the above story. The same reader would most likely not find compelling any particular metaphysical account we might give of numbers. We ask that reader then to at least suspend *dis*belief in propositions so that we can get on with the mathematics.

There is an alternative approach to semantics which can be described by saying that rather than taking propositions as primitive, it 'constructs' them out of certain other semantical primitives. Thus there is as a paradigm of this approach the so-called 'UCLA proposition' as a set of 'possible worlds'.[22] We here want to stress the general structural idea, not placing much emphasis upon the particular choice of 'possible *worlds*' as the semantical primitive. Various authors have chosen 'reference points', 'cases', 'situations', 'set-ups', etc.—as the name for the semantical primitive varying for sundry subtle reasons from author to author. We have both in relevance logic contexts have preferred 'situations', but in a show of solidarity we shall here join forces with the Routley's [1972] in their use of 'set-ups'.

Such 'set-theoretical' semantical accounts do not always explicitly verify such a construction of propositions. Indeed perhaps the more common approach is to provide an interpretation that says whether a formula A is true and false at a given set-up S writing $\phi(a, S) = T$ or $S \vDash A$ or some such thing. Think of Kripke's [1963] presentation of his semantics for modal logic. But (unless one has severe ontological scruples about sets) one might just as well interpret A by assigning it a class of set-ups, writing $\Phi(A)$ or $|A|$ or some such thing. One can go from one framework to the other by way of equivalence

$$S \in |A| \text{ iff } S \vDash A.$$

3.3 Algebra of First-degree Relevant Implications

Given two propositions a and b, it is natural to consider the implication relation among them, which we write as $a \leq b$ ('a implies b'). It might be thought to be natural to write this the other way around as $a \geq b$ on some intuition that a is the stronger or 'bigger' one if it implies b. Also it suggests $a \supseteq b$ ('b is contained in a'), which is a natural enough way to think of implication. There are good reasons though behind our by now almost universal choice (of course at one level it is just notation, and it doesn't matter what

[22] Actually the germ of this idea was already in Boole (cf. [Dipert, 1978]), although apparently he thought of it as an analogy rather than as a reduction.

your convention is). Following the idea that a proposition might be identified with the set of cases in which it is true, a implies b corresponds to $a \subseteq b$, which has the same direction as $a \leq b$. Then conjunction $\wedge$ corresponds to intersection $\cap$, and they have roughly the same symbol (and similarly for $\vee$ and $\cup$).

It is also natural to assume, as the notation suggests, that implication is a partial order, i.e.

(p.o.1)	$a \leq a$	(Reflexivity),
(p.o.2)	$a \leq b$ and $b \leq a \Rightarrow a = b$	(Antisymmetry),
(p.o.3)	$a \leq b$ and $b \leq c \Rightarrow a \leq c$	(Transitivity).

It is natural also to assume that there are operations of conjunction $\wedge$ and disjunction $\vee$ that satisfy

($\wedge$lb)	$a \wedge b \leq a$, $a \wedge b \leq b$,
($\wedge$glb)	$x \leq a$ and $x \leq b \Rightarrow x \leq a \wedge b$,
($\vee$ub)	$a \leq a \vee b$, $b \leq a \vee b$,
($\vee$lub)	$a \leq x$ and $b \leq x \Rightarrow a \vee b \leq x$.

Note that ($\wedge$lb) says that $a \wedge b$ is a lower bound both of a and of b, and ($\wedge$glb) says it is the greatest such lower bound. Similarly $a \vee b$ is the least upper bound of a and b.

A structure $(L, \leq, \wedge, \vee)$ satisfying all the properties above is a well-known structure called a lattice. Almost any logic would be compatible with the assumption that propositions form a lattice (but there are exceptions, witness Parry's [1933] Analytic Implication which would reject ($\vee$ub)).

Lattices can be defined entirely operationally as structures $(L, \wedge, \vee)$ with the relation $a \leq b$ defined as $a \wedge b = a$. Postulates characterising the operations are:

Idempotence:	$a \wedge a = a$, $a \vee a = a$
Commutativity:	$a \wedge b = b \wedge a$, $a \vee b = b \vee a$
Associativity:	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$
Absorption:	$a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$.

An (upper) *semi-lattice* is a structure $(S, \vee)$, with $\vee$ satisfying Idempotence, Commutativity, and Associativity.

Given two lattices $(L, \wedge, \vee)$ and $(L', \wedge', \vee')$, a function h from L into $'$ is called a (*lattice*) *homomorphism* if both $h(a \wedge b) = h(a) \wedge' h(b)$ and $h(a \vee b) = h(a) \vee' h(b)$. If h is one–one, h is called an *isomorphism.*

Many logics (certainly orthodox relevance logic) would insist as well that propositions form a *distributive* lattice, i.e. that

$$a \wedge (b \vee c) \leq (a \wedge b) \vee c.$$

This implies the usual distributive laws $a \wedge (b \vee c) = (a \wedge b) \wedge (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. (Again there are exceptions, important ones being quantum logic with its weaker orthomodular law, and linear logic with its rejection of even the orthomodular law.)

The paradigm example of a distributive lattice is a collection of sets closed under intersection and union (a so-called 'ring' of sets). Stone [1936] indeed showed that abstractly all distributive lattices can be represented in this way. Although we will not argue this here, it is natural to think that if propositions correspond to classes of cases, then conjunction should carry over to intersection and disjunction to union, and so productions should form a distributive lattice.

Certain subsets of lattices are especially important. A *filter* is a non-empty subset F such that

1. $a, b \in F \Rightarrow a \wedge b \in F$,

2. $a \in F$ and $a \leq b \Rightarrow b \in F$.

Filters are like theories. Note by easy moves that a filter satisfies

1′. $a, b \in F \Leftrightarrow a \wedge b \in F$,

2′. $a \in F$ or $b \in F \Rightarrow a \vee b \in F$.

When a filter also satisfies the converse of (2′) it is called *prime*, and is like a prime theory. A filter that is not the whole lattice is called *proper*. Stone [1936] showed (using an equivalent of the Axiom of Choice) the

PRIME FILTER SEPARATION PROPERTY. In a distributive lattice, if $a \not\leq b$, then there exists a prime filter P with $a \in P$ and $b \notin P$.

This is related to the Belnap Extension Lemma of Section 2.5.

So far we have omitted discussion of negation. This is because there is less agreement among logics as to what properties it should have.[23] There is, however, widespread agreement that it should at least have these:

1. (Contraposition) $a \leq b \Rightarrow \neg b \leq \neg a$,

2. (Weak Double Negation) $a \leq \neg\neg a$.

These can both be neatly packaged in one law:

3. (Intuitionistic Contraposition) $a \leq \neg b \Rightarrow b \leq \neg a$.

We shall call any unary function $\neg$ satisfying (3) (or equivalently (1) and (2)) a *minimal complement*. The intuitionists of course do not accept

[23]Cf. Dunn [1994, 1996] wherein the various properties below are related to various ways of treating incompatibility between states of information.

4. (Classical Contraposition) $\neg a \leq \neg b \Rightarrow b \leq a$, or

5. (Classical Double Negation) $\neg\neg a \leq a$.

If one adds either of (4) or (5) to the requirements for a minimal complement one gets what is called a *de Morgan complement* (or quasi-complement), because, as can be easily verified, it satisfies all of de Morgan's laws

(deM1) $\neg(a \wedge b) = \neg a \vee \neg b$,
(deM2) $\neg(a \vee b) = \neg a \wedge \neg b$.

Speaking in an algebraic tone of voice, de Morgan complement is just a (one–one) order-inverting mapping (a *dual automorphism*) of period two.

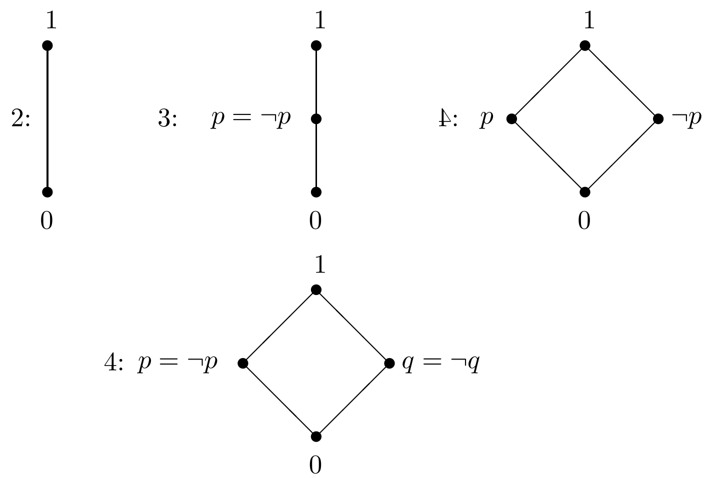
De Morgan complement captures many of the features of classical negation, but it misses

(Irrelevance 1) $a \wedge \neg a \leq b$,
(Irrelevance 2) $a \leq b \vee \neg b$.

If (either of) these are added to a de Morgan complement it becomes a *Boolean complement*. If Irrelevance 1 is added to a minimal complement, it becomes a *Heyting complement* (or *pseudo-complement*).

A structure $(L, \wedge, \vee, \neg)$, where $(L, \wedge, \vee)$ is a *distributive* lattice and $\neg$ is a de Morgan (Boolean) complement is called a *de Morgan* (*Boolean*) lattice. Note that we did not try to extend this terminological framework to 'Heyting lattices', because in the literature a Heyting lattice requires an operation called 'relative pseudo-complementation' in addition to Heyting complementation (plain pseudo-complementation).

As an example of de Morgan lattices consider the following (here we use ordinary Hasse diagrams to display the order; $a \leq b$ is displayed by putting a in a connected path below b):



The backwards numeral labelling the third lattice over is not a misprint. It signifies that not only has the de Morgan complement been obtained by inverting he order of the diagram, as in the order three (of course $\neg I = \Theta$ and *vice versa*), but also by rotating it from right to left at the same time. 2 and 4 are Boolean lattices.

A *homomorphism* (*isomorphism*) h between de Morgan Lattice with de Morgan complements $\neg$ and $\neg'$ respectively is a lattice homomorphism (isomorphism) such that $h(\neg a) = \neg' h(a)$.

A valuation in a lattice outfitted with one or the other of these 'complementations' is a map v from the zero-degree formulas into its elements satisfying

$$\begin{aligned} v(\neg A) &= \neg v(A), \\ v(A \wedge B) &= v(A) \wedge v(B), \\ v(A \vee B) &= v(A) \vee v(B). \end{aligned}$$

Note that the occurrence of '$\neg$' on the left-hand side of the equation denotes the negation connective, whereas the occurrence on the right-hand side denotes the complementation operation in the lattice (similarly for $\wedge$ and $\vee$). Such ambiguities resolve themselves contextually.

A valuation v can be regarded as in interpretation of the formulas as propositions.

De Morgan lattices have become central to the study of relevance of logic, but they were antecedently studied, especially in the late 1950s by Moisil and Monteiro, by Białynicki-Birula and Rasiowa (as 'quasi-Boolean algebras'), and by Kalman (as 'distributive i-lattices') (see Anderson and Belnap [1975] or Rasiowa [1974] for references and information).

Belnap seems to have first recognised their significance for relevance logic, though his research favoured a special variety which he called an *intensionally complemented distributive lattice with truth filter* ('icdlw/tf'), shortened in Section 18 of [Anderson and Belnap, 1975] to just *intensional lattice*. An intensional lattice is a structure $(L, \wedge, \vee, \neg, T)$, where $(L, \wedge, \vee, \neg)$ is a de Morgan lattice and T is a *truth-filter*, i.e. T is a filter which is complete in the sense T contains at least one of a and $\neg a$ for each $a \in L$, and *consistent* in the sense that T contains no more than one of a and $\neg a$.

Belnap and Spencer [1966] showed that a necessary and sufficient condition for a de Morgan lattice to have a truth filter is that negation have no fixed point, i.e. for no element $a, a = \neg a$ (such a lattice was called an *icdl*). For Boolean algebras this is a non-degeneracy condition, assuring that the algebra has more than one element, the one element Boolean algebra being best ignored for many purposes. But the experience in relevance logic has been that de Morgan lattices where some elements are fixed points are extremely important (not all elements can be fixed points or else we do have the one element lattice).

The viewpoint of [Dunn, 1966] was to take general de Morgan lattices as basic to the study of relevance logics (though still results were analogised wherever possible to the more special icdl's). Dunn [1966] showed that upon defining a first-degree implication $A \to B$ to be (*de Morgan*) *valid* iff for every valuation v in a de Morgan lattice, $v(A) \leq v(B), A \to B$ is valid iff $A \to B$ is a theorem of $\mathbf{R_{fde}}$ (or $\mathbf{E_{fde}}$). The analogous result for icdl's (in effect due to Belnap) holds as well.

Soundness ($\vdash_{\mathbf{R_{fde}}} A \to B \Rightarrow A \to B$ is valid) is a more or less trivial induction on the length of proofs in $\mathbf{R}_{\mathrm{fde}}$ fragment—cf. [Anderson and Belnap, 1975, Section 18].

Completeness ($A \to B$ valid $\Rightarrow \vdash_{\mathbf{R}_{\mathrm{fde}}} A \to B$) is established by proving the contrapositive. We suppose not $\vdash_{\mathbf{R}_{\mathrm{fde}}} A \to B$. We then form the 'Lindenbaum algebra', which has as an element for each zero degree formula (zdf) $X, [X] =_{\mathrm{df}} \{Y : Y$ is a zdf and $\vdash_{\mathbf{R}_{\mathrm{fde}}} X \leftrightarrow Y\}$. Operations are defined so that $\neg[X] = [\neg X]$, $[X] \wedge [Y] = [X \wedge Y]$, and $[X] \vee [Y] = [X \vee Y]$, and we set $[X] \leq [Y]$ whenever $\vdash_{\mathbf{R}_{\mathrm{fde}}} X \to Y$. It is more or less transparent, given $\mathbf{R}_{\mathrm{fde}}$ formulated as it is, that the result is a de Morgan lattice. It is then easy to see that $A \to B$ is invalidated by the *canonical* valuation $v_c(X) = [X]$, since clearly $[A] \not\leq [B]$.

The above kind of soundness and completeness result is really quite trivial (though not unimportant), once at least the logic has had its axioms chopped so that they look like the algebraic postulates merely written in a different notation. The next result is not so trivial.

CHARACTERISATION THEOREM OF $\mathbf{R}_{\mathrm{FDE}}$ WITH RESPECT TO 4. $\vdash_{\mathbf{R}_{\mathrm{fde}}} A \to B$ *iff* $A \to B$ *is valid in 4, i.e. for every valuation* v *in 4,* $v(A) \leq v(B)$.

Proof. Soundness follows from the trivial fact recorded above that $\mathbf{R}_{\rm fde}$ is sound with respect to de Morgan lattices in general. For completeness we need the following:

4-*Valued Homomorphism Separation Property.* Let $\mathcal{D}$ be a de Morgan lattice with $a \not\leq b$. Then there exists a homomorphism h of $\mathcal{D}$ into 4 so that $h(a) \not\leq h(b)$.

Completeness will follow almost immediately from this result, for upon supposing that not $\vdash_{\mathbf{R}_{\rm fde}} A \to B$, we have $v(A) = h[A] \not\leq h[B] = v(B)$ (the composition of a homomorphism with a valuation is transparently a valuation). So we go on to establish the Homomorphism Separation Property.

Assume that $a \not\leq b$. By the Prime Filter Separation Property, we know there is a prime filter P with $a \in P$ and $b \notin P$. for a given element x, we define $h(x)$ according to the following four possible 'complementation patterns' with respect to P.

1. $x \in P, \neg x \notin P$: set $h(x) = 1$;
2. $\neg x \in P, x \notin P$: set $h(x) = 0$;
3. $x \in P, \neg x \in P$: set $h(x) = p$;
4. $x \notin P, \neg x \notin P$: set $h(x) = q$.

It is worth remarking that if $\mathcal{D}$ is a Boolean lattice, (3) (inconsistency) and (4) (incompleteness) can never arise, which explains the well-known significance of 2 for Boolean homomorphism theory. Clearly these specifications assure that $h(a) \in \{p, I\}$ and $h(b) \in \{q, 0\}$, and so by inspection $h(a) \not\leq h(b)$. It is left to the reader to verify that h in fact is a homomorphism. (Hint to avoid more calculation: set $[p) = \{p, I\}$ and $[q) = \{q, I\}$ (the principal filters determined by p and q). Observe that the definition of h above is equivalent to requiring of h that $h(x) \in [p)$ iff $x \in P$, and $h(x) \in [q)$ iff $\neg a \notin P$. Observe that if whenever $i = p, q, y \in [i)$ iff $z \in [i)$, then $y = z$. Show for $i = p, q, h(a \wedge b) \in [i)$ iff $h(a) \wedge H(b) \in [i)$, $h(a \vee b) \in [i)$ iff $h(a) \vee h(b) \in [i)$, and $h(\neg a) \in [i)$ iff $\neg h(a) \in [i)$. ■

3.4 Set-theoretical Semantics for First-degree Relevant Implication

Dunn [1966] (cf. also [Dunn, 1967]) considered a variety of (effectively equivalent) representations of de Morgan lattices as structures of sets. We shall here discuss the two of these that have been the most influential in the development of set-theoretical semantics for relevance logic.

The earliest one of these is due to Białynicki-Birula and Rasiowa [1957] and goes as follows. Let U be a non-empty set and let $g : U \to U$ be such that it is of period two, i.e.

1. $g(g(x)) = x$, for all $x \in U$.

(We shall call the pair (U, g) and *involuted set*—g is the *involution*, and is clearly 1–1). Let $Q(U)$ be a 'ring' of subsets of U (closed under $\cap$ and $\cup$) closed as well under the operation of 'quasi-complement'

2. $\neg X = U - g[X](X \subseteq U)$.

$(Q(U), \cup, \cap, \neg)$ is called a *quasi-filed of sets* and is a de Morgan lattice.

QUASI-FIELDS OF SETS THEOREM [Białynicki-Birula and Rasiowa, 1957]. Every de Morgan lattice $\mathcal{D}$ is isomorphic to a quasi-field of sets.

Proof. Let U be the set of all prime filters of $\mathcal{D}$, and let P range over U. Let $\neg P \rightarrow \{\neg a : a \in P\}$, and define $g(P) = \mathcal{D} - \neg P$. We leave to the reader to verify that U is closed under g. For each element $a \in \mathcal{D}$, set

$$f(a) = \{P : a \in P\}.$$

Clearly f is one–one because of the Prime Filter Separation Property, so we need only check that f preserves the operations.

ad$\wedge$: $P \in f(a \wedge b) \Leftrightarrow a \wedge b \in P \Leftrightarrow$ ((1′) of Section 3.3) $a \in P$ and $b \in P \Leftrightarrow P \in f(a)$ and $P \in f(b) \Leftrightarrow P \in f(a) \cap f(b)$. So $f(a \wedge b) = f(a) \cap f(b)$ as desired.

ad$\vee$: The argument that $f(a \vee b) = f(a) \cup f(b)$ is exactly parallel using (2′) (or alternately this can be skipped using the fact that $a \vee b = \neg(\neg a \wedge \neg b)$.

ad$\neg$: $P \in f(\neg a) \Leftrightarrow \neg a \in P \Leftrightarrow a \in \neg P \Leftrightarrow a \notin g(P) \Leftrightarrow g(P) \notin f(a) \Leftrightarrow P \notin g[f(a)] \Leftrightarrow P \in U - g[f(a)]$.

We shall now discuss a second representation. Let U be a non-empty set and let R be a ring of subsets of U (closed under intersection and union, but not necessarily under complement, quasi-complement, etc.). by a *polarity* in R we mean an ordered pair $X = (X_1, X_2)$ such that $X_1, X_2 \in R$. We define a relation and operations as follows, given polarities $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$:

$$X \leq Y \Leftrightarrow X_1 \subseteq Y_1 \text{ and } Y_2 \subseteq X_2$$
$$X \wedge Y = (X_1 \cap Y_1, X_2 \cup Y_2)$$
$$X \vee Y = (X_1 \cup Y_1, X_2 \cap Y_2)$$
$$\neg X = (X_2, X_1).$$

By a field of polarities we mean a structure $(P(R), \leq, \wedge, \vee, \neg)$ where $P(R)$ is the set of all polarities in some ring of sets R, and the other components are defined as above. We leave to the reader the easy verification that every field of polarities is a de Morgan lattice. ■

We shall prove the following

Polarities Theorem [Dunn, 1966]. *Every de Morgan lattice is isomorphic to a field of polarities.*

Proof. Given he previous representation, it clearly suffices to show that every quasi-field of sets is isomorphic to a field of polarities.

The idea is to set $f(X) = (X, U - g[X])$. Clearly f is one–one. We check that it preserves operations.

ad$\wedge$: $f(X \cap Y) = (X \cap Y, U - g[X \cap Y]) = (X \cap Y, (U - g[X]) \cap (U - g[Y])) = (X, U - g[X]) \wedge (Y, U - g[Y]) = f(X) \wedge f(Y)$.

ad$\vee$: Similar.

ad$\neg$: $f(\neg X) = (\neg X, U - g(\neg X)) = (U - g[X], U - g(U - g[X])) = (U - g[X], X) = \neg f(X)$.

■

We now discuss informal interpretations of the representation theorems that relate to semantical treatments of relevant first-degree implications familiar in the literature.

Routley and Routley [1972] presented a semantics for $\mathbf{R}_{\text{fde}}$, the main ingredients of which were a set K of 'atomic set-ups' (to be explained) on which was defined an involution $*$. An 'atomic set-up' is just a set of propositional variables, and it is used to determine inductively when complex formulas are also 'in' a given set-up. A set-up is explained informally as being like a possible world except that it is not required to be either consistent or complete. The Routley's [1972] paper seems to conceive of set-ups very syntactically as literally being sets of formulas, but the Routley and Meyer [1973] paper conceives of them more abstractly. We shall think of them this latter way here so as to simplify exposition. The Routleys' models can then be considered a structure $(K, *, \vDash)$, where K is a non-empty set, $*$ is an involution on K, and $\vDash$ is a relation from K to zero-degree formulas. We read '$a \vDash A$' as the formula A holds at the set-up a:

1. $(\wedge \vDash)$ $\quad a \vDash A \wedge B \Leftrightarrow a \vDash A$ and $a \vDash B$
2. $(\vee \vDash)$ $\quad a \vDash A \vee B \Leftrightarrow a \vDash A$ or $a \vDash B$
3. $(\neg \vDash)$ $\quad a \vDash \neg A \Leftrightarrow$ not $a^* \vDash A$.

The connection of the Routleys' semantics with quasi-fields of sets will become clear if we let $(K, *)$ induce a quasi-field of sets Q with quasi- complement $\neg$, and let $|\ |$ interpret sentences in Q subject to the following conditions:

1′. $|\wedge|$ $|A \wedge B| = |A| \cap |B|$

2′. $|\vee|$ $|A \vee B| = |A| \cup |B|$

3′. $|\neg|$ $|\neg A| = \neg |A|$.

Clause $(\wedge \vDash)$ results from clause $|\wedge|$ by translating $a \in |X|$ as $a \vDash X$ (cf. Section 3.2). Thus clause $|\wedge|$ says

$$a \in |A \wedge B| \;\Leftrightarrow\; a \in |A| \text{ and } a \in |B|,$$

i.e. it translates as clause $(\wedge \vDash)$. The case of disjunction is obviously the same. The case of negation is clearly of special interest, so we write it out.

Thus clause $|\neg|$ says

$$\begin{aligned} a \in |\neg A| &\Leftrightarrow a \in \neg |A|, \\ &\Leftrightarrow a \in K - |A|^*, \\ &\Leftrightarrow a \notin |A|^*, \\ &\Leftrightarrow a^* \notin |A|. \end{aligned}$$

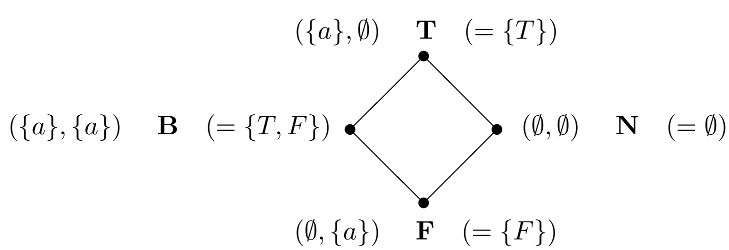
But the translation of this last is just clause $(\neg \vDash)$.

Of course the translation works both ways, so that the Routleys' semantics is just an interpretation in the quasi-fields of sets of Białynicki-Birula and Rasiowa written in different notation. Incidentally soundness and completeness of $\mathbf{R}_{\mathrm{fde}}$ relative to the Routleys' semantics follows immediately *via* the translation above from the corresponding theorem of the previous section *vis à vis* de Morgan lattices together with their representation as quasi-fields of sets. Of course the Routleys' conceived their results and derived them independently from the representation of Białynicki-Birula and Rasiowa.

We will not say very much here about what intuitive sense (if any) can be attached to the Routleys' use of the $*$-operator in their valuational clause for negation. Indeed this question has had little extended discussion in the literature (though see [Meyer, 1979a, Copeland, 1979]). The Routleys' [1972] paper more or less just springs it on the reader, which led Dunn in [Dunn, 1976a] to describe the switching of a with a^* as 'a feat of prestidigitation'. Routley and Meyer [1973] contains a memorable story about how a^* 'weakly asserts', i.e. fails to deny, precisely what a asserts, but one somehow feels that this makes the negation clause vaguely circular. Still, semantics often gives one this feeling and maybe it is just a question of degree. One way of thinking of a and a^* is to regard them as 'mirror images' of one another reversing 'in' and 'out'. Where one is inconsistent (containing both A and $\neg A$), the other is incomplete (lacking both A and $\neg A$), and *vice versa* (when $a = a^*$, a is both consistent and complete and we have a situation appropriate to classical logic). Viewed this way the Routleys' negation clause makes sense, but it does require some anterior intuitions about inconsistent and incomplete set-ups. More about the interpretation of this clause will be discussed in Section 5.1.

Let us now discuss the philosophical interpretation(s) to be placed on the representation of de Morgan lattices as fields of polarities. In Dunn [1966, 1971] the favoured interpretation of a polarity (X_1, X_2) was as a 'proposition surrogate', X_1 consisting of the 'topics' the proposition gives definite positive information about and, X_2 of the topics the proposition gives definite negative information about. A valuation of a zero degree formula in a de Morgan lattice can be viewed after a representation of the elements of the lattice as polarities as an assignment of positive and negative content to the formula. The 'mistake' in the 'classical' Carnap/Bar-Hillel approach to content is to take the content of $\neg A$ to be the set-theoretical complement of the content of A (relative to a given universe of discourse). In general there is no easy relation between the content of A and that of $\neg A$. They may overlap, they may not be exhaustive. Hence the need for the double-entry bookkeeping done by proposition surrogates (polarities). If A is interpreted as (X_1, X_2), $\neg A$ gets interpreted as the interchanged (X_2, X_1).

Another semantical interpretation of the same mathematics is to be found in Dunn [1969, 1976a]. There given a polarity $X = (X_1, X_2), X_1$ is thought of as the set of situations in which X is true and X_2 as the set of situations in which X is false. These situations are conceived of as maybe inconsistent and/or incomplete, and so again X_1 and X_2 need not be set-theoretic complements. This leads in the case when the set of situations being assessed is a singleton $\{a\}$ to a rather simple idea. The field of polarities looks like this



We have taken the liberty of labelling the points so as to make clear the informal meaning. (Thus the top is a polarity that is simply true in a and the bottom is one that is simply false, but the left-hand one is both true and false, and the right-hand one is neither.) Note that the de Morgan complement takes fixed points on both B and $\mathbf{N}$. This is of course our old friend 4, which we know to be characteristic for $\mathbf{R_{fde}}$.

This leads to the idea of an 'ambi-valuation' as an assignment to sentences of one of the four values **T**, **F**, **B**, **N**, conceived either as primitive or realised as sets of the usual two truth values as suggested by the labelling. On this latter plan we have the valuation clauses (with double entry bookkeeping):

$$
\begin{array}{ll}
(\wedge) & T \in v(A \wedge B) \Leftrightarrow T \in v(A) \text{ and } T \in v(B), \\
 & F \in v(A \wedge B) \Leftrightarrow F \in v(A) \text{ or } F \in v(B), \\
(\vee) & T \in v(A \vee B) \Leftrightarrow T \in v(A) \text{ or } T \in v(B), \\
 & F \in v(A \vee B) \Leftrightarrow F \in v(A) \text{ and } F \in v(B), \\
(\neg) & T \in v(\neg A) \Leftrightarrow F \in v(A), \\
 & F \in v(\neg A) \Leftrightarrow T \in v(A).
\end{array}
$$

We stress here (as in [Dunn, 1976a]) that all this talk of something's being both true and false or neither is to be understood epistemically and not ontologically. One can have inconsistent and or incomplete assumptions, information, beliefs, etc. and this is what we are trying to model to see what follows from them in an interesting (relevant!) way. Belnap [1977b, 1977a] calls the elements of the lattice 'told values' to make just this point, and goes on to develop (making connections with Scott's continuous lattices) a theory of 'a useful four-valued logic' for 'how a computer should think' without letting minor inconsistencies in its data lead to terrible consequences.

Before we leave the semantics of first-degree relevant implications, we should mention the interesting semantics of van Fraassen [1969] (see also Anderson and Belnap [1975, Section 20.3.1] and van Fraassen [1973]), which also has a double-entry bookkeeping device. We will not mention details here, but we do think it is an interesting problem to try to give a representation of de Morgan lattices using van Fraassen's facts so as to try to bring it under the same umbrella as the other semantics we have discussed here.

3.5 The Algebra of **R**

This section is going to be brief. Dunn has already exposited on this topic in Section 28.2 of [Anderson and Belnap, 1975] and the interested reader should consult that and then Meyer and Routley [1972] for information about how to algebraise related weaker systems and how to give set-theoretical representations.

De Morgan monoids are a class of algebras that are appropriate to **R** in the sense that (i) the Lindenbaum algebra of **R** is one of them and (ii) all **R** theorems are valid in them ((ii) gives soundness, and of course (i) delivers completeness by way of the canonical valuation). In thinking about de Morgan monoids it is essential that **R** be equipped with the sentential constant $\mathbf{t}$. Also it is nice to think of fusion ($\circ$) as a primitive connective, with even perhaps $\rightarrow$ defined ($A \rightarrow B =_{\mathrm{df}} \neg(A \circ \neg B)$) but this is not essential since in **R** (but not the weaker relevance logics) fusion can be defined as $A \circ B =_{\mathrm{df}} \neg(A \rightarrow \neg B)$.

A de Morgan monoid is a structure $\mathcal{D} = (D, \wedge, \vee, \neg, \circ, e)$ where

(I) $(D, \wedge, \vee, \neg)$ is a de Morgan lattice,

(II) $(D, \circ, e)$ is an Abelian monoid, i.e. $\circ$ is a commutative, associative binary operation on D with e its identity, i.e. $e \in D$ and $e \circ a = a$ for all $a \in D$,

(III) the monoid is ordered by the lattice, i.e. $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$,

(IV) $\circ$ is upper semi-idempotent ('square increasing'), i.e. $a \leq a \circ a$,

(V) $a \circ b \leq c$ iff $a \circ \neg c \leq \neg b$ (Antilogism).

De Morgan monoids were first studied in [Dunn, 1966] (although [Meyer, 1966] already had isolated some of the key structural features of fusion that they abstract). They also were described in [Meyer *et al.*, 1974] and used in showing γ admissible. Similar structures were investigated quite independently by Maksimova [1967, 1971].

The key trick in relating de Morgan monoids to **R** is that they are residuated, i.e. there is a 'residual' operation $\rightarrow$ so that

(VI) $a \circ b \leq c$ iff $a \leq b \rightarrow c$.

Indeed this operation turns out to be $\neg(b \circ \neg c)$ (with the weaker systems or with positive **R** it is important to postulate this law of the residual). Thus

(1)	$a \circ b \leq c \Leftrightarrow b \circ a \leq c$	Commutativity
(2)	$a \circ b \leq c \Leftrightarrow b \circ \neg a$	1, (V)
(3)	$a \circ b \leq c \Leftrightarrow a \leq \neg(b \circ \neg c)$	2, de Morgan lattice.

As an illustration of the power of (VI) we show how the algebraic analogue of the Prefixing axiom follows from Associativity. First note that one can get from (III) the law of

(Monotony) $\quad a \leq b \Rightarrow c \circ a \leq c \circ b.$

Now getting down to Prefixing:

1. $a \rightarrow b \leq a \rightarrow b$
2. $(a \rightarrow b) \circ a \leq b$ 1, (VI)

3. $(c \to a) \circ c \leq a$ 2, Substitution
4. $(a \to b) \circ ((c \to a) \circ c) \leq b$ 2,3, Monotony
5. $((a \to b) \circ (c \to a)) \circ c \leq b$ 4, Associatively
6. $(a \to b) \circ (c \to a) \leq c \to b$ 5, (VI)
7. $a \to b \leq ((c \to a) \to (c \to b))$ 6, (VI).

Incidentally, something better be said at this point about how validity in de Morgan monoids is defined. Unlike the case with $\mathbf{R}_{\text{fde}}$, there are theorems which are of the form $A \to B$, e.g. $A \vee \neg A$. We need some way of defining validity which is broader than insisting that always $v(A) \leq v(B)$. The identity e interprets the sentential constant t. By virtue of the $\mathbf{R}$ axiom $A \leftrightarrow (t \to A)$ characterising t, it makes sense to count all de Morgan monoid elements a such that $e \leq a$ as 'designated', and to define A as valid iff $v(A) \geq e$ for all valuations in all de Morgan monoids. We have the following law

$$a \leq b \Leftrightarrow e \leq a \to b,$$

which follows immediately from (VI) and the fact that e is the identity element. This means that (7) just above can be transformed into

$$e \leq (a \to b) \to ((c \to a) \to (c \to b))$$

validating prefixing as promised.

Other axioms of $\mathbf{R}$ can be validated by similar moves. Commutativity validates Assertion, that e is the identity validates self-implication, square-increasingness validates Contraction, antilogism validates Contraposition, and the other axioms fall out of de Morgan lattice properties with lattice ordering and the residual law pitching in.

We shall not here investigate the 'converse' questions about how the fusion connective in $\mathbf{R}$ is associative, etc. (that the Lindenbaum algebra of $\mathbf{R}$ is indeed a de Morgan monoid (cf. Dunn's Section 28.2.2 of [Anderson and Belnap, 1975])), but the proof is by 'fiddling' with contraposition being the key move.

Not as much is known about the algebraic properties of de Morgan monoids as one would like. Getting technical for a moment and using unexplained but standard terminology from universal algebra, it is known that de Morgan monoids are equationally definable (replace (V) with $a \circ \neg(a \circ \neg b) \leq b$, which can be replaced by the equation $(a \circ \neg(a \circ \neg b)) \vee b = b$). So by a theorem of Birkhoff the class of de Morgan monoids is closed under sub-algebras, homomorphic images, and subdirect products. Further, given a de Morgan monoid $\mathcal{D}$ with a prime filter P with $e \in P$, the relation

$a \approx b \Leftrightarrow (a \to b) \wedge (b \to a) \in P$ is a congruence, and the quotient algebra $\mathcal{D}/\approx$ is subdirectly irreducible, and every de Morgan monoid is a subdirect product of such. It would be nice to have some independent interesting characterisation of the subdirectly irreducibles.

One significant recent result about the algebra of **R** has been provided by John Slaney. He has shown that there are exactly 3088 elements in the free De Morgan monoid generated by the identity e. Or equivalently, in the language of **R** including the constant t, there are exactly 3088 non-equivalent formulae free of propositional variables. The proof technique is quite subtle, as generating a large algebra of 3088 elements is not feasible, even with computer assistance. Instead, Slaney attacked the problem using a "divide and conquer" technique [Slaney, 1985]. Since **R** contains all formulae of the form $A \vee \neg A$, for any A, whenever L is a logic extending **R**, $L = (L + A) \cap (L + \neg A)$, where $L + A$ is the result of adding A as an axiom to L and closing under modus ponens and adjunction. Given this simple result, we can proceed as follows. **R** is $(\mathbf{R} + f \to t) \cap (\mathbf{R} + \neg(f \to t))$. Now it is not difficult to show that the algebra of $\mathbf{R} + (f \to t)$ generated by t is the two element boolean algebra. Then you can restrict your attention to the algebra generated by t in the logic $\mathbf{R} + \neg(f \to t)$. If this has some characteristic algebra, then you can be sure that the elements freely generated by t in **R** are bounded above by the number of elements in the direct product of the two algebras. To get the characteristic algebra of $\mathbf{R} + \neg(f \to t)$, Slaney goes on to divide and conquer again. He ends up considering six matrices, characterising six different extensions of **R**. This would give him an upper bound on the number of constants (the matrices were size 2, 4, 6, 10, 10 and 14, so the bound was their product, 67200, well above 3088). Then you have to consider how many of these elements are generated by the identity in the direct product algebra. A reasonably direct argument shows that there are exactly 3088 elements generated in this way, so the result is proved.

3.6 The Operational Semantics (Urquhart)

This set-theoretical semantics is based upon an idea that occurred independently to Urquhart and Routley in the very late 1960s and early 70s. We shall discuss Routley's contribution (as perfected by Meyer) in the next section and also just mention some related independent work of [Fine, 1974]. Here we concentrate upon the version of Urquhart [1972c] (cf. also Urquhart [1972b, 1972a, 1972d]).

Common to all the versions is the idea that one has some set K whose elements are 'pieces of information', and that there is a binary operation $\circ$ on K that combines pieces of information. Also there is an 'empty piece of information' $0 \in K$. We shall write $x \vDash A$ to mean intuitively 'A holds according to the piece of information x'. The whole point of the semantics is disclosed in the valuational clause

$$(\rightarrow) \quad x \vDash A \rightarrow B \text{ iff } \forall y \in K \text{ (if } y \vDash A \text{, then } x \circ y \vDash B).$$

The idea of the clause from left-to-right is immediately clear: if $A \rightarrow B$ is given by the information x, then if A is given by y, then the combined piece of information $x \circ y$ ought to give B (by *modus ponens*). The idea of the clause from right-to-left is to say that if this happens for all pieces of information y, this can only be because x gives us the information that $A \rightarrow B$.

Perhaps saying the whole point of the semantics is given in the clause $(\rightarrow)$ along is an exaggeration. There are at least two quick surprises. The first is that we do not require (or want) a certain condition analogised from a condition required by Kripke's (relational) semantics for intuitionistic logic:

$$\text{(The Hereditary Condition)} \quad \text{If } x \vDash A \text{, then } x \circ y \vDash A.$$

This would yield that if $x \vDash A$, then $x \vDash B \rightarrow A$, i.e. if $y \vDash B$, ten $x \circ y \vDash A$. This would quickly involve us in irrelevance.

The other surprise is related to the failure of the Hereditary Condition: Validity cannot be defined as a formula's holding at *all* pieces of information in all models, since even $A \rightarrow A$ would not then turn out to be valid. Thus $x \vDash A \rightarrow A$ requires that if $y \vDash A$ then $x \circ y \vDash A$. But this last is just a commuted form of the rejected Hereditary Condition, and there is no more reason to think it holds. We shall see in a moment that the appropriate definition of validity is to require that $0 \vDash A$ for the empty piece of information in all models.

Enough talk of what properties $\circ$ does not have! What property does it have? We have just been flirting with one of them. Clearly $0 \vDash A \rightarrow A$ requires that if $x \vDash A$ then $0 \circ x \vDash A$, and how more naturally would that be obtained than requiring that 0 be a (left) identity?

$$0 \circ x = x. \qquad \text{(Identity)}$$

This then seems the minimal algebraic condition on a model. Urquhart in fact requires others, all naturally motivated by the idea that $\circ$ is the 'union' of pieces of information.

$$x \circ y = y \circ x \qquad \text{(Commutativity)}$$
$$x \circ (y \circ z) = (x \circ y) \circ z \qquad \text{(Associativity)}$$
$$x \circ x = x. \qquad \text{(Idempotence)}$$

These conditions combined may be expressed by saying that $(K, \circ, 0)$ is a '(join) semi-lattice with least element 0', and accordingly Urquhart's semantics is often referred to as the 'semi-lattice semantics'. It is well-known that every semi-lattice is isomorphic to a collection of sets with union as $\circ$ and the empty set as 0 (map x to $\{y : x \circ y = y\}$ so that henceforward $\circ$ will be denoted by $\cup$.

Each of the conditions above of course corresponds to an axiom of $\mathbf{R}_{\rightarrow}$ when it is nicely axiomatised. Thus commutativity plays a natural role in verifying the validity of assertion. The following use of natural deduction in the metalanguage makes this point nicely (we write 'A, x' rather than $x \vDash A$ for a notational analogy):

1.	A, x	Hypothesis
2.	$A \rightarrow B, z$	Hypothesis
3.	$B, x \cup z$	1, 2, $(\rightarrow)$
4.	$B, z \cup x$	3, Comm.
5.	$(A \rightarrow B) \rightarrow B, x$	2, 4$(\rightarrow)$
6.	$(A \rightarrow B) \rightarrow B, 0 \cup x$	5, Identity
7.	$A \rightarrow ((A \rightarrow B) \rightarrow B), 0.$	

The reader may find it amusing to write out an analogous pair of proofs for Prefixing, seeing how Associativity of $\cup$ enters in, and for Contraction watching the Idempotence.[24]

The game has now been given away. There is some fiddling to be sure in proving a completeness theorem for $\mathbf{R}_{\rightarrow}$ *re* the semi-lattice semantics, but basically the idea is that the semi-lattice semantics is just the system $F\mathbf{R}_{\rightarrow}$ 'written in the metalanguage'.

There is not a problem in extending the semi-lattice semantics so as to accommodate conjunction. The clause

$$x \vDash A \wedge B \text{ iff } x \vDash A \text{ and } x \vDash B \qquad (\wedge)$$

does nicely. Somewhat strangely, the 'dual' clause

$$x \vDash A \vee B \text{ iff } x \vDash A \text{ of } x \vDash B \qquad (\vee)$$

causes trouble. It is analogous to having the rule of $\vee$-Elimination $N\mathbf{R}$

[24]Though unfortunately verification of this last does not depend purely on Idempotence, but rather on $(xy)y = xy$, which of course is equivalent to Idempotence given Associativity and Identity. The verification of the formula $A \wedge (A \rightarrow B) \rightarrow B$ 'exactly' uses Idempotence, but of course this is hardly a formula of the *implicational* fragment.

read:

$$\begin{array}{ll} A \vee B, x & \\ \quad A, x & \text{Hyp.} \\ \quad \vdots & \\ \quad c, x \cup y & \\ \quad B, x & \text{Hyp.} \\ \quad \vdots & \\ \quad C, x \cup y & \\ \hline C, x \cup y. & \end{array}$$

With this rule we can prove

$$(\sharp) \quad (A \to B \vee C) \wedge (B \to C) \to (A \to C),$$

which is not a theorem of **R** (see [Urquhart, 1972c]—the observation is Meyer's and Dunn's.)[25] And of course one can analogously verify that it is valid in the semi-lattice semantics.

Note that the condition ($\vee$) is not nearly as intuitive as the condition ($\wedge$). The condition ($\wedge$) is plausible for any piece of information x, at least if the relation $x \vDash c$ does not require that C be *explicitly* contained in x. On the other hand the condition ($\vee$) is much less than natural. Does not it happen all the time that a piece of information x determines $A \vee B$ to hold, without saying which? Is not this one of the whole points of disjunctions? Pieces of information x that satisfy ($\vee$) might be called 'prime' (in analogy with this epithet applied to theories of Section 2.4), and they have a kind of completeness or effeminateness that is rare in ordinary pieces of information. This by itself counts as no criticism of the semantics, since it is quite usual in semantical treatments to work with such idealised notions.

The condition ($\vee$) is not really as 'dual' to the condition ($\wedge$) as one might think. Thus the formula

$$(\sharp\text{d}) \quad (B \wedge C \to A) \wedge (C \to B) \to (C \to A),$$

which is the dual of ($\sharp$) is easily seen not to be valid in the semantics. This seems to be connected with another feature (problem?) of the semantics, to wit, no one has ever figured out how to add a natural semantical treatment of classical negation to the semantics (although it is straightforward to add a species of constructive negation—see [Urquhart, 1972c]).[26] The point of the

[25] It would be with $C \to C$ as an additional conjunct in the antecedent.

[26] Charlewood and Daniels have investigated a combination of the semi-lattice semantics for the positive connectives and a four-valued treatment of negation in the style of [Dunn, 1976a]. they avoid the problem just described by in effect building into their definition of a model that it must satisfy classical contraposition. This does not seem to be natural.

connection is that (♯d) would follow from (♯) given classical contraposition principles, and yet the first is valid and the second one invalid in the positive semantics. So something about the positive semantics would have to be changed as well to accommodate negation.

The semi-lattice semantics has been extensively investigated in Charlewood [1978, 1981]. He fits it out with (two) natural deduction systems one with subscripts and one without. This last is in fact the (positive) system of Prawitz [1965], which Prawitz wrongly conjectured to be the same as Anderson and Belnap's. Charlewood proves normalisation theorems (something that was anticipated by Prawitz for his system—incidentally the problem of normalisation for the Anderson–Belnap **R** seems still open). Incidentally, one advantage of these natural deduction systems is that, unlike the Anderson–Belnap one for their system **R** (cf. Section 1.5), they allow for a proof of distribution.

Charlewood also carries out in detail the engineering needed to implement K. Fine's axiomatisation of the semi-lattice semantics. What is needed is to add to the Anderson–Belnap's $\mathbf{R}^+$ the following rule:

R1: From $B_0 \wedge ((A_1 \wedge q_1, \ldots, q_n \wedge A_n) \to X) \to ((B_1 \wedge q_1, \ldots, B_n \wedge q_n) \to E)$ for $X = B, C$, and $n \geq 0$ infer the same thing with $B \vee C$ put in place of the displayed X, provided that he q_i are distinct and occur only where shown.

We forbear taking cheap shots at such an ungainly rule, the true elegance of which is hidden in the details of the completeness proof that we shall not be looking into. Obviously Anderson and Belnap's **R** is to be preferred when the issue is simplicity of Hilbert-style axiomatisations.[27]

3.7 The Relational Semantics (Routley and Meyer)

As was indicated in the last section, Routley too had the basic idea of the operational semantics at about the same time as Urquhart. Priority would be very hard to assess. At any rate Dunn first got details concerning both their work in early 1971, although J. Garson told him of Urquhart's work in December of 1970 and he has seen references made to a typescript of Routley's with a 1970 date on it (in [Charlewood, 1978]).

Meyer and Dunn were colleagues at the time, and Routley sent Meyer a somewhat incomplete draft of his ideas in early 1971. This was a courageous and open communication in response to our keen interest in the topic (instead he might have sat on it until it was perfected). The draft favoured the operational semantics, indeed the semi-lattice semantics, and was not

[27]However, the semi-lattice semantics has been taken up and generalised in the field of substructural logics in the work of [Došen, 1988, Došen, 1989] and [Wansing, 1993].

clear that this was not the way to go to get Anderson and Belnap's **R**. But the draft started with a more general point of view suggesting the use of a 3-placed accessibility relation $Rxyz$ (of course a 2-placed operation like $\cup$ is a 3-placed relation, but not always conversely), with the following valuation clause for $\rightarrow$:

$$(\rightarrow) \quad x \vDash A \rightarrow B \text{ iff } \forall y, z \in K \text{ (if } Rxyz \text{ and } y \vDash A, \text{ then } z \vDash B).$$

Forgetting negation for the moment, the clauses for $\wedge$ and $\vee$ are 'truth functional', just as for the operational semantics.

Meyer, having observed with Dunn the lack of fit between the semi-lattice semantics and **R**, was all primed to make important contributions to Routley's suggestion. In particular he saw that the more general 3-placed relation approach could be made to work for all of **R**. In interpreting $Rxyz$ perhaps the best reading is to say that the combination of the pieces of information x and y (not necessarily the union) is a piece of information in z (in bastard symbols, $x \circ y \leq z$). Routley himself called the x, y, etc. 'set-ups', and conceived of them as being something like possible worlds except that they were allowed to be inconsistent and incomplete (but always prime). On this reading $Rxyz$ can be regarded as saying that x and y are compatible according to z, or some such thing.

Before going on we want to advertise some work that we are not going to discuss in any detail at all because of space limitations. The work of Fine [1974] independently covers some of the same ground as the Routley-Meyer papers, with great virtuosity making clear how to vary the central ideas for various purposes. The book of Gabbay [1976, see chapter 15] is also deserving of mention.

We now set out in more formal detail a version of the Routley–Meyer semantics for $\mathbf{R}^+$ (negation will be reserved for the next section). The techniques are novel and the completeness proof quite complicated, so we shall be reasonably explicit about details. The presentation here is very much indebted to work (some unpublished) of Routley, Meyer and Belnap.

By an ($\mathbf{R}^+$) frame (or model structure) is meant a structure $(K, R, 0)$, where K is a non-empty set (the elements of which are called *set-ups*), R is a 3-placed relation on $K, 0 \in K$, all subject to some conditions we shall state after a few definitions. We define for $a, b \in K, a \leq b$ (Routley and Meyer used $>$) iff $R0ab$, and R^2abcd iff $\exists x$ ($Rabx$ and $Rxcd$). We also write this last as $R^2(ab)cd$ and distinguish it from $R^2a(bc)d =_{\text{df}} \exists x(Raxd \wedge Rbcx)$. The variables a, b, etc. will be understood as ranging over the elements of some K fixed by the content of discussion.

Transcribing the conditions on the semi-lattice semantics as closely as we can into this framework we get the requirements

1. (Identity) $R0aa$,

2. (Commutativity) $Rabc \Rightarrow Rbac$,

3. (Associativity) $R^2(ab)cd \Rightarrow R^2a(bc)d$,[28]

4. (Idempotence) $Raaa$.

It should be remarked that these conditions fail to pick up the whole strength of the corresponding semi-lattice conditions. Thus, e.g. Identity here only picks up $0{\cdot}a \leq a$ and not conversely, and similarly for Idempotence (also of course Commutativity and Associativity do not require any identity, but this is a slightly different point). We need for technical reasons one more condition:

5. (Monotony) $Rabc$ and $a' \leq a \Rightarrow Ra'bc$.

By a model we mean a structure $M = (K, R, 0, \vDash)$, where $(K, R, 0)$ is a frame and $\vDash$ is a relation from K to sentences of $\mathbf{R}^+$ satisfying the following conditions:

(1) (Atomic Hereditary Condition). For a propositional variable p, if $a \vDash p$ and $a \leq b$, then $b \vDash p$.

(2) (Valuational Clauses). For formulas A, B

($\rightarrow$) $a \vDash A \rightarrow B$ iff $\forall b, c \in K$ (if $Rabc$ and $b \vDash A$, then $c \vDash B$);

($\wedge$) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$;

($\vee$) $a \vDash A \vee B$ iff $a \vDash$ or $a \vDash B$.

We shall say that A is *verified* on M if $0 \vDash A$, and that A *entails* B on M if $\forall a \in K$ (if $a \vDash A$, then $a \vDash B$). We say that A is *valid* if A is verified on all models.

It is easy to prove by an induction on A, the following (note how Monotony enters in):

HEREDITARY CONDITION. For an arbitrary formula A, if $a \vDash A$ and $a \leq b$, then $b \vDash A$.

VERIFICATION LEMMA. *If in a given model* $(K, R, 0, \vDash)A$ entails B *in the sense that for every* $a \in K, a \vDash A$ *only if* $a \vDash B$, *then* $A \rightarrow B$ *is verified in the model, i.e.* $0 \vDash A \rightarrow B$.

Proof. suppose that $R0ab$ and $a \vDash A$. By the hypothesis of the Lemma, $a \vDash B$, and by the Hereditary Condition, $b \vDash B$, as is required for $0 \vDash A \rightarrow B$. ■

[28] In the original equivalent conditions of Routley and Meyer [1973] this was instead 'Pasch's Law': $R^2abcd \Rightarrow R^2acbd$. Also Monotony (condition (5) below) was misprinted there.

We are now in a position to prove the

SOUNDNESS THEOREM. *If* $\vdash_{\mathbf{R}} A$*, then* A *is valid.*

Proof. Most of this will be left to the reader. We first show that the axioms of $\mathbf{R}^+$ are valid. Since they are all of the form $A \to B$ we can simplify matters a little by using the Verification Lemma. As an illustration we verify Assertion (the reader may wish to compare this to the corresponding verification *vis à vis* the semi-lattice semantics of the last section).

To show $A \to [(A \to B) \to B]$ is valid, it suffices by the Verification Lemma to assume $a \vDash A$ and show $a \vDash A \to B \to B$. For this last we assume $Rabc$ and $b \vDash A \to B$, and show $c \vDash B$. By Commutativity, $Rabc$. By ($\to$) since we have $b \vDash A \to B$ and $a \vDash A$, we get $c \vDash B$ as desired.

The verification of the implicational axioms of Self-Implication and Prefixing are equally routine, falling right out of the Verification Lemma and Associativity for the relation R. Unfortunately the verification of Contraction is a bit contrived (cf. note 24 above), so we give it here.

To verify Contraction, we assume that (1) $a \vDash A \to .A \to B$ and show $a \vDash A \to B$. To show this last we assume that (2) $Rabc$ and (3) $b \vDash A$, and show $c \vDash B$. From (2) we get, by Commutativity, $Rbac$. But $Rbbb$ holds by Idempotence. so we have $R^2(bb)ac$. By Associativity we get $R^2b(ba)c$, i.e. for some x, both (4) $Rbxc$ and (5) $Rbax$. by Commutativity, from (5) we get $Rabx$. Using ($\to$), we obtain from this, (1), and(3) hat (6) $x \vDash A \to B$. by Commutativity from (4) we get $Rxbc$, and from this, (6), and (3) we at last get the desired $c \vDash B$.

Verification of the conjunction and disjunction axioms is routine and is safely left to the reader.

It only remains to be shown then that the rules *modus ponens* and adjunction preserve validity. Actually something stronger holds. It is easy to se that for any $a \in K$ (not just 0), if $a \vDash A \to B$ and $a \vDash A$, then $a \vDash B$ (by virtue of $Raaa$), and of course it follows immediately from ($\wedge$) that if $a \vDash A$ and $a \vDash B$, then $a \vDash A \wedge B$. ■

We next go about the business of establishing the

COMPLETENESS THEOREM. *If* A *is valid, then* $\vDash_{\mathbf{R}^+} A$*.*

The main idea of the proof is similar to that of the by now well-known Henkin-style completeness proofs for modal logic. We suppose that no $\vDash_{\mathbf{R}^+}$ A and construct a so-called 'canonical model', the set-us of which are certain prime theories (playing the role of the maximal theories of modal logic). The base set-up 0 is constructed as a regular theory (for the terminology 'regular', 'prime', etc. consult Section 2.4; of course everything is relativised to $\mathbf{R}^+$). From this point on for simplicity we shall assume that we are dealing with $\mathbf{R}^+$ outfitted with the optional extra fusion connective $\circ$ and the propositional constant t (recall these can be conservatively added — cf. Section 1.3). We then define *Rabc* to hold precisely when for all formulas A and B, whenever $A \in a$ and $B \in b$, then $A \circ B \in c$.[29]

Let us look now at the details. Pick 0 as some prime regular theory T with $A \notin T$. We can derive that at least one such exists using the Belnap Extension Lemma (it was stated in Section 2.5 for **RQ**, but it clearly holds for $\mathbf{R}^+$ as well). thus set $\Delta = \mathbf{R}^+$ and $\theta = \{A\}$.

Define K = set of prime theories,[30] and define the accessibility relation R canonically as above.

THEOREM 1 *The canonically defined structure* $(K, 0, R)$ *is an* $\mathbf{R}^+$ *frame.*

LEMMA 2 *The relation R defined canonically above satisfies Identity, Commutativity, Idempotence, and Associativity.*

PROOF.
ad Identity. We need to show that $R0aa$, i.e. if $X \in 0$ and $A \in a$, then $X \circ A \in a$. By virtue of the **R** -theorem $A \to t \circ A$, we have $t \circ A \in a$. But using the **R**-theorem $X \to .t \to x$, we have $t \to X \in 0$. By Monotony we have $X \circ A \in a$ as desired.

ad Commutativity. Suppose *Rabc*. We need show *Rbac*, i.e. if $B \in b$ and $A \in a$, then $B \circ A \in c$. From *Rabc*, it follows that $A \circ B \in c$. But by virtue of the **R**-theorem $A \circ B \to B \circ A$ (commutativity of $\circ$) we have $B \circ A \in C$, as desired.

ad Idempotence. We need show *Raaa*, i.e. if $A \in a$ and $B \in a$, then $A \circ B \in a$. This follows from the **R**-theorem $A \wedge B \to A \circ B$, which follows ultimately from the square increasingness of $\circ, (X \to X \circ X)$, as the proof sketch below makes clear.

1. $A \wedge B \to A$ Axiom
2. $A \wedge B \to B$ Axiom
3. $(A \wedge B) \circ (A \wedge B) \to A \circ B$ 1, 2, Monotony
4. $A \wedge B \to A \circ B$ 3, square increasingness

[29] The use of $\circ$ and t is a luxury to make things prettier at least at the level of description. Thus, e.g. as we shall see, the associativity of **R** follows from the associativity of $\circ$, and other mnemonically pleasant things happen. We could avoid its use by defining *Rabc* to hold whenever if $A \in a$ and $A \to B \in b$, then $B \in c$. Incidentally, the valuational clause for fusion is : $x \vDash A \circ B$ iff for some a, b such that $Rabx, a \vDash A$ and $b \vDash B$. The valuational clause for t is $x \vDash t$ iff $0 \leq x$.

[30] One actually has a choice here. We have required of theories that they be closed under implications provable in 0, i.e. require of T that whenever $A \in T$ and $A \to B \in 0$, then $B \in T$. The latter is a stronger requirement and leads to the 'smaller' *reduced* models of [Routley *et al.*, 1982], which are useful for various purposes.

ad Associativity. This is by far the least trivial property. Let us then assume that $R^2(ab)cd$, i.e. $\exists x(Rabx$ and $Rxcd)$. We need then show that there is a prime theory y such that $Rayd$ and $Rbcy$, i.e. $R^2a(bc)d$.

Set $y_0 = \{Y : \exists B \in b, C \in c :\vdash_{\mathbf{R}} B \circ C \to Y\}$. (This is sometimes referred to as $b \circ c$). Clearly the definition of y_0 assures that $Rbcy_0$.

Observe that y_0 is a theory.[31] Thus it is clear that y_0 is closed under provable **R**-implication, since this is just transitivity. We show it is also closed under adjunction. Thus suppose for some $B, B' \in b, C, C' \in c, \vdash_{\mathbf{R}} B \circ C \to Y$ and $\vdash_{\mathbf{R}} B' \circ C' \to Y'$. Then $\vdash_{\mathbf{R}} (B \circ C) \wedge (B' \circ C') \to Y \wedge Y'$ using easy properties of conjunction. But we have the **R**-theorem $(B \wedge B') \circ (C \wedge C') \to (B \circ C) \wedge (B' \circ C')$ (which follows basically from the one-way distribution of $\circ$ over $\wedge, X \circ (Y \wedge Z) \to (X \circ Y) \wedge (X \circ Z)$, which follows basically from Monotony, $Y_1 \to Y_2 \to X \circ Y_1 \to X \circ Y_2$, which is easy). So by transitivity we get $\vdash_{\mathbf{R}} (B \wedge B') \circ (C \wedge C') \to Y \wedge Y'$, from which it follows that $Y \wedge Y' \in y_0$ as promised ($B \wedge B' \in b, C \wedge C' \in c$ of course, since b, c are closed under adjunction).

We next verify that Ray_0d. Suppose that $A \in a$ and $Y \in y_0$. Then for some $B \in b, C \in c, \vdash_{\mathbf{R}} B \circ C \to Y$. Since $Rabx, A \circ B \in x$. And since $Rxcd(A \circ B) \circ C \in d$. By the associativity of $\circ$ (since d is a theory), then $A \circ (B \circ C) \in d$. but by Monotony, since $\vdash_{\mathbf{R}} B \circ C \to Y$, we have $\vdash_{\mathbf{R}} A \circ (B \circ C) \to A \circ Y$. Hence $A \circ Y \in d$, as needed.

The reader is excused if he has lost the thread a bit and thinks that we are now finished verifying the associativity of R. We wanted some prime theory y which fills in the blanks

1. Ra_d and
2. $Rbc\_$,

and we have just finished verifying that y_0 is a theory that does fill in the blanks. The kicker is that y_0 need not be prime. So we work next at pumping up y_0 to make it prime while continuing to fill in the blanks.

It clearly suffices to prove

THE SQUEEZE LEMMA. *Let a_0 and y_0 be theories that need not be prime, and let d be a prime theory. If Ra_0y_0d, then there exists a prime theory y such that (i) $y_0 \subseteq y$ and (ii) Ra_0yd.*

This can be accomplished by a Lindenbaum-style construction like that of Section 2.3 (or alternatively Zorn's Lemma may be used as in Routley

[31] The presentation of Routley–Meyer [1973] is more elegant than ours, developing as they do properties of what they call the calculus of 'intensional **R**-theories', showing that it is a partially ordered (under inclusion) commutative monoid ($\circ$ as defined above) with identity 0. Further $\circ$ is monotonous with respect to $\leq$, i.e. if $a \leq b$ then $c \circ a \leq c \circ b$, and $\circ$ is square increasing, i.e. $a \leq a \circ a$. Then defining $Rabc$ to mean $a \circ b \leq c$, the requisite properties of **R** fall right out.

and Meyer [1973]). The idea is to define y as the union of a sequence of sets of formulas y_n, where (relative to some fixed enumeration of the formulas) y_{n+1} is defined inductively as $y_n \cup \{A_{n+1}\}$ if $Ra(y_n \cup \{A_{n+1}\})d$, and otherwise y_{n+1} is just y_n.

But it is instructive to crank the existence of the given y out of the Belnap Extension Lemma for **R**.

Thus set $\Delta = y_0$ and $\theta = \{A : \exists B(A \to B) \in a \text{ and } B \notin d\}$. We need check that (Δ, θ) is exclusive.

We observe first that θ is closed under disjunction. Thus suppose $A_1, A_2 \in \theta$. Then for some $B_1, B)2, A_1 \to B_1, A_2 \to B_2 \in a$, and yet $B_1, B_2 \notin d$. Then (since d is prime) $B_1 \vee B_2 \notin d$. but since a is a theory, then $A_1 \vee A_2 \to B_1 \vee B_2$ by an appropriate theorem of **R** in the proximity of the disjunction axioms. So $A_1 \vee A_2 \in \theta$ as desired. Since Δ is closed dually under adjunction (that was the point of observing above that y_0 is a theory), this means that if the pair (Δ, θ) fails to be exclusive, then for some $X \in \Delta, A \in \theta, \vdash_{\mathbf{R}} X \to A$. So for some $B, A \to B \in a$ and $B \notin d$. But since a is a theory, by transitivity we derive that $X \to B \in a$. But since $Raxd$ and $X \in x$, we get $(X \to B) \circ X \in d$. But since $\vdash_{\mathbf{R}} X \circ (X \to B) \to B$, we have $B \in d$, contrary to the choice of B.

Now that we know (Δ, Θ) is an exclusive pair we apply the Belnap Extension Lemma to get a pair (y, y') with $y_0 = \Delta \subseteq y$ and y a prime theory, completing the proof of the Squeeze Lemma, which actually does complete the proof that the relation R is Associativity.

ad Monotony. (Yes, we still have something left to do.) Let us suppose that $R0a'a$ and $Rabc$, and show $Ra'bc$. Note that it follows from $R0a'a$ that $a' \leq a$,[32] from which it follows at once from $Rabc$ and $Ra'bc$. Thus if $X \in a'$ then since $X \to X \in 0$, then $(X \to X) \circ X \in a$. But since $\vdash_{\mathbf{R}^+} (X \to X) \circ X \to X$, then $X \in a$.

Having now finally verified that the canonical $(K, 0, R)$ has all the properties of an $\mathbf{R}^+$-frame, we need now to define an appropriate relation $\vDash$ on it. The natural definition is $a \vDash A$ iff $A \in a$, but we need now to verify that this has the properties (1) and (2) required of $\vDash$ above.

THEOREM 2. *The canonically defined* $(K, 0, R, \vDash)$ *is indeed an* **R**-*model.*

Proof. *ad* (1) (the Hereditary Condition). Suppose $a \leq b$, i.e. $R0ab$. We show that $a \leq b$, from which the Hereditary Condition immediately follows. Suppose then that $A \in a$. Since $t \in 0, t \circ A \in b$. But *via* the **R**-theorem $t \circ A \to A$, we have $A \in b$ as desired.

ad (2) (the valuation of clauses). The clauses $(\wedge)$ and $(\vee)$ are more or less immediate (primeness is of course needed for half of $(\vee)$). The clause

[32]In the 'reduced models (cf. note 46) one can show that $R0a'a$ iff $a' \leq a$.

of interest is ($\rightarrow$). Applying the canonical definition of $\vDash$, this amounts to

$$(\rightarrow_c) \quad A \rightarrow B \in a \text{ iff } \forall b, c(\text{if } Rabc \text{ and } A \in b, \text{ then } B \in c).$$

Left-to-right is argued as follows. Suppose $A \rightarrow B \in a, Rabc, A \in b$, and show $B \in c$. $Rabc$ of course means canonically that whenever $X \in a$ and $Y \in b$, then $X \circ Y \in c$. Setting $X = A \rightarrow B$ and $Y = A$, we get $A \circ (A \rightarrow B) \rightarrow C$. Then using the $\mathbf{R}^+$- theorem

$$A \circ (A \rightarrow B) \rightarrow B, \text{ we obtain } B \in c.$$

Right-to-left is harder, and in fact involves the third (and last) application of the Belnap Extension Lemma in the proof of Completeness. Thus suppose contrapositively that $A \rightarrow B \notin a$. We need to construct prime theories b and c, with $A \in b$ and $B \notin c$. We let $\Delta_b = \text{Th}(\{A\})$ and set $\Delta_c = a \circ \Delta_b$, i.e. $\{Z : \exists X \in a, \exists Y \in \Delta_b \vdash_{\mathbf{R}^+} X \circ Y \rightarrow Z\}$. This is the same as $\{Z : \exists X \in a \vdash_{\mathbf{R}^+} X \circ A \rightarrow Z\}$. We set $\theta_c = \{B\}$. Clearly (Δ_c, θ_c) is an exclusive pair, for otherwise $\vdash_{\mathbf{R}^+} X \circ A \rightarrow B$, i.e. $\vdash_{\mathbf{R}^+} X \rightarrow (A \rightarrow B)$ for some $X \in a$, and so $A \rightarrow B \in a$ contrary to our supposition. We apply Belnap's Extension Lemma to get an exclusive pair (c, c') with $\Delta_c \subseteq c$ and c prime theory. Note that by definition of Δ_b and $\Delta_c, Ra\Delta_b\Delta_c$, and so $Ra\Delta_b c$. We are now in a position to apply the Squeeze Lemma getting a prime theory $b \supseteq \Delta_b$ such that $Rabc$. Clearly $A \in b$, but also $B \notin c$ since $B \in \theta_c \subseteq c'$ (c and c' are exclusive).

This at last completes the proof of the Completeness Theorem for $\mathbf{R}^+$. ■

REMARK. It is fashionable these days to always prove *strong* completeness. This could have been done. Thus define A to be a *logical consequence* of a set of formulas Γ iff for every $\mathbf{R}^+$-model M, if $0 \vDash B$ for every $B \in \Gamma$, then $0 \vDash A$. This is a kind of classical notion and should not be confused with some kind of relevant consequence. Thus, e.g. where B is a theorem of $\mathbf{R}^+$, since always $0 \vDash B$, B will be a logical consequence of any set Γ. Define B to be deducible from Γ (again in a neo-classical sense) to mean $B \in \text{Th}(\Gamma \cup \mathbf{R}^+)$. Appropriate modifications of the work above will show that logical consequence is equivalent to deducibility.

3.8 Adding Negation to $\mathbf{R}^+$

We now discuss the Routley–Meyer semantics for the whole system $\mathbf{R}$. The idea is simply to add the Routley's treatment of negation using the $*$-operator (discussed in Section 3.4). (This is not difficult and there is very little reason to segregate it off into this separate section, except that we thought that the treatment of $\mathbf{R}^+$ was complicated enough.)

Thus an **R**-*frame* is a structure, $(K, R, 0, *)$ where $(K, R, 0)$ is an $\mathbf{R}^+$-frame and K is closed under the unary operation $*$ satisfying:

(Period two) $A^{**} = a$,
(Inversion) $Rabc \Rightarrow Rac^*b^*$

For an **R**-model the valuational clauses for the positive connectives are as for an **R**-model, and we of course add

$$(\neg) \quad a \vDash \neg A \text{ iff } a^* \nvDash A.$$

The soundness and completeness results are relatively easy modifications of those for $\mathbf{R}^+$. That $*$ is of period two naturally is used n the verification of Double Negation and Inversion is central to the verification of Contraposition. For completeness, a^* is defined canonically as $\{A : \neg A \notin a\}$ (cf. the definition of the analogue $g[P]$ in the proof of Białynicki–Birula and Rasiowa's representation of de Morgan lattices in Section 3.4), and one of course has to show that a^* is a prime theory when a is. One also has to show that canonical $*$ is of period two and satisfies (Inversion), and that canonical $\vDash$ satisfies $(\neg)$ above, i.e. $A \in a \Leftrightarrow \neg A \notin a^*$, i.e. $\neg A \notin \{B : \neg B \notin a\}$, i.e. $\neg\neg A \in a$, which of course just uses Double Negation.

It is worth remarking that since the canonical 0 is a prime regular theory, then since $\vdash_{\mathbf{R}} A \vee \neg A$, then 0 is complete (but not necessarily consistent—this is relevant to the development in Section 3.9). For your garden variety Routley–Meyer model (not necessarily canonical) notice also that $0 \vDash A$ or $0 \vDash \neg A$. This follows ultimately from $0^* \leq 0$, i.e. $R00^*0$, proven below.

1. $R0^*0^*0^*$
2. $R0^*00$ 1, (Inversion), (Period two)
3. $R00^*0$ 2, (Commutation).

Now $0^* \leq 0$ means by the Hereditary Condition that if $0 \vDash A$ then $0^* \vDash A$, i.e. $0 \vDash \neg A$ as desired.

It should be said that although either the four-valued treatment or the $*$-operator treatment of negation work equally well for *first-degree* relevant implications (at least from a technical point of view), the $*$-operator treatment seems to win hands down in the context of all of **R**. Meyer [1979a] has succeeded in giving a four-valued treatment of all of **R**, but at the price of great technical complexity (e.g. the accessibility relation has to be made four-valued as well, and that is just for starters). Further, as Meyer points out, one's models still have to be closed under $*$, so it still can be said to sneak in the back door.

3.9 Routley-Meyer Semantics for **E** *and other Neighbours of* **R**

Once one sets down a set of conditions on an accessibility relation, they can be played with n various ways so as to produce semantics for a wide variety of systems as the experience with modal logic has taught us. Also other features of the frames can be generalised.

We can here only give the flavour of a whole range of possible and actual results. In all the results below $\vDash$ will satisfy the same conditions as for $\mathbf{R}^+$ (or $\mathbf{R}$) models (as appropriate). To begin with we follow Routley and Meyer [1973] with the description of a series of conditions on positive frames and corresponding axioms for propositional logic. They begin by requiring of a $\mathbf{B}^+$-*frame* $(K, R, 0)$

B1. $a \leq a$

B2. $a \leq b$ and $b \leq x \Rightarrow a \leq c$

B3. $a' \leq a$ and $Rabc \Rightarrow Ra'bc$.

B1–B2 of course say that $\leq$ is a quasi-order, and B3 says something like that it is monotone.

'**B**' appears to be for 'Basic', for they regard the above postulates as a natural minimal set on their approach.[33] Gabbay [1976] investigates even weaker logics where no conditions at all are placed on the frame, but these have no theorems and are characterised only by rules of deducibility (unless Boolean negation and/or the Boolean material conditional is present, options which he does explore).

The sense in which the above postulates are minimal goes something like this. B3 is needed in proving the Hereditary Condition for implications, and the Hereditary Condition is needed in turn for verifying $0 \vDash A \to A$ (indeed anything) so we have at least some minimal theorems. The Hereditary Condition is used in showing the equivalence of the verification of an implication in a model and entailment in that mode, i.e. $0 \vDash A \to B$ iff $\forall x \in K(x \vDash A \Rightarrow x \vDash B)$ (cf. Section 3.7 to see how these conditions were used to establish these facts about $\mathbf{R}^+$-models). What about B2? We think it is just a 'freebie'. It seems to play no role in verifying axioms or rules, but the completeness proof can be made to yield canonical ('reduced') models (cf. note 3.7) that satisfy it, so why not have it? This seems to be what Routley *et. al.* [1982] say. It appears that B1 is even more a freebie.

It may be shown that A is a theorem of the system $\mathbf{B}^+$ (formulated in Section 1.3) iff A is valid in all $\mathbf{B}^+$ models.

[33] However, some notational confusion is possible, with Fine's use of '**B**' as another basic relevance logic differing slightly from Routley and Meyer's usage [Fine, 1974]. For Fine, **B** includes the law of the excluded middle, and for Routley and Meyer, it does not.

Routley and Meyer establish the following correspondence between conditions on the accessibility relation R and axioms:

(1)	$Raaa$	$A \wedge (A \to B) \to B$
(2)	$Rabc \Rightarrow R^2a(ab)c$	$(A \to B) \wedge (B \to C) \to (A \to C)$
(3)	$R^2abcd \Rightarrow R^2a(bc)d$	$A \to B \to ([B \to C] \to [A \to C])$
(4)	$R^2abcd \Rightarrow R^2b(ac)d$	$A \to B \to ([C \to A] \to [C \to B])$
(5)	$Rabc \Rightarrow R^2abbc$	$(A \to [A \to B]) \to (A \to B)$
(6)	$Ra0a$	$([A \to A] \to B) \to B$
(7)	$Rabc \Rightarrow Rbac$	$A \to ([A \to B] \to B)$
(8)	$0 \leq a$	$A \to (B \to B)$
(9)	$Rabc \Rightarrow b \leq c$	$A \to (B \to A)$.

Routley and Meyer connect these conditions on accessibility relations to axioms extending the basic logic **B**. The correspondence is more perspicuous when you consider the *structural rules* corresponding to each axiom or condition. We can express these as conditions on fusion:

(1)	$Raaa$	$A \vdash A \circ A$
(2)	$Rabc \Rightarrow R^2a(ab)c$	$A \circ B \vdash A \circ (A \circ B)$
(3)	$R^2abcd \Rightarrow R^2a(bc)d$	$(A \circ B) \circ C \vdash A \circ (B \circ C)$
(4)	$R^2abcd \Rightarrow R^2b(ac)d$	$(A \circ B) \circ C \vdash B \circ (A \circ C)$
(5)	$Rabc \Rightarrow R^2abbc$	$A \circ B \vdash (A \circ B) \circ B$
(6)	$Ra0a$	$A \circ t \vdash A$
(7)	$Rabc \Rightarrow Rbac$	$A \circ B \vdash B \circ A$
(8)	$0 \leq a$	$B \circ A \vdash B$ (or $A \vdash t$)
(9)	$Rabc \Rightarrow b \leq c$	$A \circ B \vdash B$.

General recipes for translating between structural rules and conditions on accessibility relations are to be found in Restall [1998, 2000].

If one wants to add to $\mathbf{B}^+$ any of the axioms on the right to get a sentential logic **X**, one merely adds the corresponding conditions to those for a $\mathbf{B}^+$ model to get the appropriate notion of an **X**-model, with a resultant sound and complete semantics.

Some logics of particular interest arising in this way are (nomenclature as in [Anderson and Belnap, 1975]) (note well that **T** has nothing to do with Feys' **t** of modal logic fame):

$$\begin{array}{ll}
\mathbf{TW}^+: & \mathbf{B}^+ + (3,4) \\
\mathbf{T}^+: & \mathbf{TW}^+ + (5) \\
\mathbf{E}^+: & \mathbf{T}^+ + (6) \\
\mathbf{R}^+: & \mathbf{E}^+ + (7) \\
\mathbf{H}^+: & \mathbf{R}^+ + (8) \\
\mathbf{S4}^+: & \mathbf{E}^+ + (8).
\end{array}$$

These are far from the most elegant formulations from a postulational point of view, being highly redundant (in particular the Prefixing and Suffixing *rules* of $\mathbf{B}^+$ are supplanted already in $\mathbf{TW}^+$ by the corresponding *axioms*. further the rule of Necessitation ($A \vdash (A \to A) \to A$) is also redundant already in $\mathbf{TW}^+$ (this is not so obvious—proof is by browsing through [Anderson and Belnap, 1975]).

What minimal conditions should be imposed on the $*$-operator when it is added to a $\mathbf{B}^+$-frame so as to give a **B**-frame? Routley *et. al.* [1982] choose

B4. $a^{**} = a$, and

B5. $a \le b \Rightarrow b^* \le a^*$.

The minimality of B5 can be defended in terms of its being needed for showing that negations satisfy the Hereditary Condition. B4 would seem to have little place in a *minimal* system except for the fact that the dominant trend in relevance logic has been to keep classical double negation.[34]

One can get semantics for the full systems **TW**, **T**, etc. simply by adding the appropriate postulates to the conditions on a **B**-model.

We could go on, but will instead refer the reader to Routley *et al.* [1982], Fine [1974] and Gabbay [1976] for a variety of variations producing systems in the neighbourhood of **R**.

Some find the conditions on the "base point" 0 on frames rather puzzling or unintuitive. Why should the basic conditions on frames include conditions such as the fact that $a \le b$ defined as $R0ab$ generate a partial order? Some recent work by Priest and Sylvan and extended by Restall has shown that these conditions can be done away with and the frames given an interpretation rather reminiscent of that of non-normal modal logics [Priest and Sylvan, 1992, Restall, 1993]. The idea is as follows. We have two sorts of set-ups in a frame — normal ones and non-normal ones. Then we split the treatment of implication along this division. Normal points are given an **S5**-like interpretation.

- $x \vDash A \to B$ iff for every y if $y \vDash A$ then $y \vDash B$

and non-normal points are given the condition which appeals to the ternary relation R

- $x \vDash A \to B$ iff for every y and z where $Rxyz$ if $y \vDash A$ then $z \vDash B$

The other connectives are treated in just the same way as in the original relational semantics. To prove soundness and completeness for this semantics, it is simplest to go through the original semantics — for it is not too difficult to show that this account is merely a notational variant, where we have set $Rxyz$ iff $y = z$ when x is a normal set-up. This satisfies all of the conditions in the original semantics, for we have set $a \le b$ to be simply $a = b$.

We turn now to one such system **RM** deserving of special treatment.

[34] In fact, B5 is too strong for a purely *minimal* logic of negation. See Section 5.1 for more discussion on this.

3.10 Algebraic and Set-theoretical Semantics for **RM**

RM has been described by Meyer as 'the laboratory of relevance logic'. It plays a role somewhat like **S5** among modal logics, being a place where conjectures can be tested relatively easily (e.g. the admissibility of γ was first shown for **RM**). This then could be a very long section because **RM** is by far the best understood of the Anderson–Belnap style systems. We shall try to keep it short by being dogmatic. The interested reader can verify the results claimed by consulting Meyer's Section 29.3 and Section 29.4 of [Anderson and Belnap, 1975] (see also [Dunn, 1970, Tokarz, 1980]).

In the first place the appropriate algebras for **RM** are the *idempotent* de Morgan monoids (strengthening $a \leq a \circ a$ to $a = a \circ a$). The subdirectly irreducible ones are all chains with de Morgan complement where $a \circ b = a \wedge b$ if $a \leq \neg b$, and $a \circ b = a \vee b$ otherwise. The designated elements are all elements a such that $\neg a \leq a$, and of course these must have a greatest lower bound to serve as the identity e. (This is just another description with $\circ$ as primitive instead of $\rightarrow$ of the 'Sugihara matrices' described in the publications cited above.) Meyer showed that if $\vdash_{\mathbf{RM}} A$, then A is valid in all the finite Sugihara matrices, establishing the finite model property for **RM**.

Dunn showed that every extension of **RM** closed under substitution and the rules of **R** has some finite Sugihara matrix as a characteristic matrix (**RM** is 'pretabular'). A similar result was shown by Scroggs to hold for the modal logic **S5**, and researchers (particularly Maksimova) have obtained results characterising all such pretabular extensions of **S4** and of the intuitionistic logic. Curiously enough there are only finitely many, and it is an interesting open problem to find some similar results for **R**. **RM** corresponds to the super-system of the intuitionistic propositional calculus **LC** (indeed **LC** can be translated into **RM**; see [Dunn and Meyer, 1971]). Much study has been done of the 'superintuitionistic' calculi (with an emphasis on the decision problem), and it would be good to see some of the ideas of this carried over to the 'super-relevant' calculi. A small start was begun in [Dunn, 1979a].

Routley and Meyer [1973] add the postulate

$$0 \leq a \text{ or } 0 \leq a^*$$

to the requirement on an **R**-frame to get an **RM**-frame. Dunn [1979a] instead adds the requirement

$$Rac \Rightarrow a \leq c \text{ or } b \leq c,$$

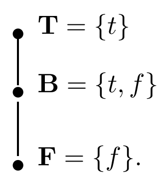
which neatly generalised to give a family of postulates yielding set-theoretical semantics for a denumerable family of weakenings of **RM** which are algebraised by adding various weakenings of idempotence ($a^{n+1} = a^n$). It is an open problem whether **R** itself is the intersection of this family and whether they all have the finite model property (if so, **R** is decidable). Since **R** is undecidable, one of these must be false. However, it is unknown at the time of writing which one fails.

Proof Sketch

1.	$\neg X \to (\neg X \to \neg X)$	Mingle Axiom, Subst.
2.	$X \to (\neg X \to X)$	1, Permutation and Contraposition
3.	$(A \vee \neg A) \wedge (B \vee \neg B) \to$ $\neg((A \vee \neg A) \wedge (B \vee \neg B)) \to$ $((A \vee \neg A) \wedge (B \vee \neg B))$	2, Subst.
4.	$\neg(A \vee \neg A) \vee \neg(B \vee \neg B) \to$ $(A \vee \neg A) \wedge (B \vee \neg B)$	3, MP, de M
5.	$A \wedge \neg A \to B \vee \neg B$	4, $\vee I, \wedge E$, de M.

Kalman [1958] especially investigated de Morgan lattices with the property $a \wedge \neg a \leq b \vee \neg b$. We will call these *Kalman* lattices. he showed that every Kalman lattice is isomorphic to a subdirect product of the de Morgan lattice 3. This implies a three- valued Homomorphism Separation Property for Kalman lattices (which also can be proven by modifying the proof of its four-valued analogue, noting that each 'side' of 4 is just a copy of 3). The representation in terms of polarities uses polarities $X = (X_1, X_2)$ where $X_1 \cup X_2 = U$, i.e. X_1 and X_2 are exhaustive.

This means informally that X always receives at least one of the values true and false. This leads to a semantics using ambivaluations into the left-hand side of 4:



This idea leads to a simpler Kripke-style semantics for **RM** using an ordinary binary accessibility relation instead of the Routley–Meyer ternary one (actually this semantics antedates the Routley–Meyer one, the results having been presented in [Dunn, 1969]—cf. [Dunn, 1976b] for a full presentation. No details will be supplied here. This semantics has been generalised to first-order **RM** with a constant domain semantics [Dunn, 1976c]). The analogous question with Routley–Meyer semantics is has now been closed in the negative in the work of [Fine, 1989], which we consider in Section 3.12.

Meyer [1980] has used this 'binary semantics' to give a proof of an appropriate Interpolation Lemma for **RM**. (Unfortunately, interpolation fails for **E** and **R** [Urquhart, 1993].)

3.11 Spin Offs from the Routley–Meyer Semantics

The Routley–Meyer semantical techniques can be used to prove a variety of results concerning the system **R** and related logics which were either more complicated using other methods (usually algebraic or Gentzen methods), or even impossible. Thus (cf. [Routley and Meyer, 1973]), it is possible to give a variety of conservative extension results (being careful in constructing the canonical model to use only connectives and sentential constant available in the fragment being extended). Also it is possible to give a proof of the admissibility of γ (see [Routley and Meyer, 1973]) that is easier than the original algebraic proof (though not as easy as Meyer's latest proof using metavaluations—cf. Section 2.4). Admissibility of γ amounts to showing that if A is refutable in a given **R**-model $(K, R, 0, \vDash)$ then A is refutable in a *normal* **R**-model $(K', R', 0', \vDash')$ (one where $0'^* = 0'$) gotten by adding a new 'zero' and redefining R' and $\vDash'$ in a certain way from R and $\vDash$.

Perhaps the most interesting new property to emerge this way is 'Halldén completeness', i.e. if $\vdash_{\mathbf{R}} A \vee B$ and A and B share no propositional variables in common, then $\vdash_{\mathbf{R}} A$ or $\vdash_{\mathbf{R}} B$ ([Routley and Meyer, 1973, Section 2.3]).

Another direction that the Routley–Meyer semantics has taken quickly ends up in heresy: classical (Boolean) negation $\sim$ can be added to **R** with horrible theorems resulting like $A \wedge \sim A \to B$, and yet **R** does not collapse to classical logic. Indeed no new theorems emerge in the original vocabulary of **R**. The idea is to take a normal **R**-model $(K, R, 0, *, \vDash)$ and turn it in for a new **R**-model $(K', R', 0', *', \vDash')$, whose $0'$ is a new element $K' = K \cup \{0'\}$, $*'$ is like $*$ but with $0\prime{*}' = 0'$, and R' is like R with the additional features:

1. $R'0'ab$ iff $R; a0'b$ iff $a = b$,

2. $R'ab0'$ iff $a = b^*$.

Also $\vDash'$ is just like $\vDash$ but with $0' \vDash A$ if $0 \vDash A$.

The whole point of this exercise is to provide refuting **R**-models for all non-**R**-theorems that have the property

$$a \leq b \text{ (i.e. } R0'ab) \ \Rightarrow a = b.$$

These are called 'classical **R**-models' (first studied in Meyer and Routley [1973a, 1973b]) and upon them one can define

$$a \vDash {\sim}A \Leftrightarrow \text{ not } a \vDash A.$$

One could not do this on ordinary **R**-models without things coming apart at the seams, because in order to have the theorem $\sim p \to \sim p$ valid, one would need the Hereditary Condition to hold for $\sim p$, i.e. if $a \leq b$, then if $a \vDash \sim p$ then $b \vDash \sim p$, i.e. if $a \vDash p$ then $b \vDash p$. But one has no reason to think that this is the case, since all one has is the converse coming from the fact that the Hereditary condition holds for p. The inductive proof the Hereditary condition breaks down in the presence of Boolean negation, but of course with classical **R**-models the Hereditary Condition becomes vacuous and there is no need for a proof.

This leads to certain technical simplicities, e.g. it is possible to give Gödel–Lemmon style axiomatisations of relevance logics like the familiar ones for modal logics, where one takes among one's axioms all classical tautologies (using $\sim$)—cf. [Meyer, 1974].

But it also leads to certain philosophical perplexities. For example, what was all the fuss Anderson and Belnap made against contradictions implying everything and disjunctive syllogism? Boolean negation trivially satisfies them, so what is the interest in de Morgan negation failing to satisfy them. Will the real negation please stand up?

A certain schism developed in relevance logic over just how Boolean negation should be regarded. See [Belnap and Dunn, 1981, Restall, 1999] for the 'con' side and [Meyer, 1978] for the 'pro' side.

Belnap and Dunn [1981] point out that although Meyer's axiomatisations of **R** with Boolean negation do not lead to any new *theorems* in the standard vocabulary of **R**, they do lead to new derivable rules, e.g. $A \wedge \neg A \vdash B$ and $\neg A \wedge (A \vee B) \vdash B$ (note well that the negation here is de Morgan negation). This can be seen quite readily if one recognises that the semantic correlate of $X \vdash Y$ is that $0 \vDash X \Rightarrow 0 \vDash Y$ in all classical **R**-models, and that since all such are normal, $\neg$ behaves at 0 in these just like classical negation. We both think this point counts against enriching **R** with Boolean negation, but Meyer [1978, note 21] thinks otherwise.

3.12 Semantics for **RQ**

The question of how to extend these techniques to handle quantified relevance logics was open for a long time. The first significant results were by Routley, who showed that the obvious constant domain semantics were sufficient to capture **BQ**, the natural first-order extension of **B** [Routley, 1980b]. However, extending the result to deal with systems involving transitivity postulates in the semantics (such as $Rabc \wedge Rcde \Rightarrow R^2abde$) proved difficult. To verify that the frame of prime theories on some constant domain actually satisfies this condition (given that the logic satisfies a corresponding condition, here the prefixing axiom) requires constructing a new prime theory x such that $Rabx$ and $Rxde$. And there seems to be no general way to show that such a theory can be constructed using the domain shared by the other theories. This is not a problem for logics like **BQ**, in which the frame conditions do not have conditions which, to be verified in the completeness proof, require the construction of new theories.

Fine showed that this is not merely a problem with our proof techniques. Logics like **RQ**, **EQ**, **TQ** and even **TWQ** are incomplete with respect to the constant domain semantics on the frames for the propositional logics [Fine, 1989]. He has given a technical argument for this, constructing a formula in the language of **RQ** which is true in all constant domain models, but which is not provable. The argument is too detailed to give here. It consists of a simple part, which shows that the formula

$$\begin{aligned}&\big((p \to \exists x Ex) \wedge \forall x((p \to Fx) \vee (Gx \to Hx))\big)\\&\qquad \to \big(\forall x(Ex \wedge Fx \to q) \wedge \forall x((Ex \to q) \vee Gx) \to \exists x Hx \vee (p \to q)\big)\end{aligned}$$

is valid in the constant domain semantics. This is merely a tedious verification that there is no counterexample. The subtle part of his argument is the construction of a countermodel. Clearly the countermodel cannot be a constant domain frame. Instead, he constructs a frame with variable domains, in which each of the axioms of **RQ** is valid (and in which the rules preserve validity) but the offending formula fails. This is quite a tricky argument, for variable domain semantics tend not to verify **RQ**'s analogue to the Barcan formula

$$\forall x(p \to Fx) \to (p \to \forall x Fx)$$

But Fine constructs his example in such a way that this formula is valid, despite the variable domains.

Despite this problem, Fine has found a semantics with respect to which the logic **RQ** is sound and complete. This semantics rests on a different view of the quantifiers. For Fine's account, a statement of the form $\forall x A(x)$ is true at a set-up not only when $A(c)$ is true for each individual c in the domain of the set-up, but instead, when $A(c)$ is true for an *arbitrary* individual c. In symbols,

$$a \vDash \forall x A(x) \text{ iff } (\exists a{\uparrow})(\exists c \in D_{a\uparrow} - D_a)(a{\uparrow} \vDash A(c)).$$

That is, for every set-up a there are *expansions* of the form $a{\uparrow}$ where we add new elements to the domain, but these are totally arbitrary. The frames Fine defines are rather complex, needing not only the $\uparrow$ operator but also a corresponding $\downarrow$ operator which cuts down the domain of a set-up, and an across operator $\leftarrow$ which identifies points in setups $(\to (a, \{c, d\})$ is the minimal extension of the set-up a in which the individuals c and d are identified. Instead of discussing the details of Fine's semantics, we refer the reader to his paper which introduced them [Fine, 1988]. Fine's work has received some attention, from Mares, who considers options for the semantics of identity [Mares, 1992]. However, it must be said that while the semantic structure pins down the behaviour of **RQ** and related systems exactly, it is not altogether clear whether the rich and complex structure of Fine's semantics is necessary to give a semantics for quantified relevance logics.

Whatever one's thoughts about the theoretical adequacy of Fine's semantics, they do raise some important issues for anyone who would give a semantic structure for quantified relevance logics. There are a number of issues to be faced and a number of options to be weighed up. One option is to give complete primacy to the frames for the propositional logics, and to use the constant domain semantics on these frames. The task then is to axiomatise this extension. The task is also to give some interpretation of what the points in these semantic structures might be. For if they are theories (or prime theories) then the evaluation clauses for the quantifiers do not make a great deal of sense without further explanation. No-one thinks that a claim of the form $\exists x A(x)$ can be a member of a theory only if there is an object in the language of the theory which satisfies A according to that theory. Nor are we so readily inclined to think that all theories need share the same domain of quantification.

If, on the other hand, we take the set-ups in frames to be quite like (some class of) theories, then we must face the issue of the relationships between these theories. No doubt, if $\forall x A(x)$ is in some theory, then $A(c)$ will be in that theory for any constant c in the language of the theory. However, the converse need not be the case.

Anyway, it is clear that there is a lot of work to be done in the semantics of relevance logics with quantifiers. One area which hasn't been explored at any depth, but which looks like it could bring some light is the semantics of positive quantified relevance logics. Without the distribution of the universal quantifier over disjunction, these systems are subsystems of intuitionistic logic.

4 THE DECISION PROBLEM

4.1 Background

When the original of this Handbook article was published back in 1985, without a doubt the outstanding open problem in relevance logics was the question as to whether there exists a decision procedure for determining whether formulas are theorems of the system **E** or **R**. Anderson [1963] listed it second among his now historic open problems (the first was the admissibility of Ackermann's rule γ discussed in Section 2). Through the work of

Urquhart [1984], we now know that there is no such decision procedure.

Harrop [1965] lends interest to the decision problem with his remark that 'all "philosophically interesting" propositional calculi for which the decision problem has been solved have been found to be decidable ...'.[35] We now have a very good counterexample to Harrop's claim.

In this section we shall examine Urquhart's proof, but before we get there we shall also consider various fragments and subsystems of **R** for which there are decision procedures. **R** will be our paradigm throughout this discussion, though we will make clear how things apply to related systems.

4.2 Zero-degree Formulas

These are formulas containing only $\wedge, \vee$, and $\neg$. As was explained in Section 1.7, the zero-degree theorems of **R** (or **E**) are precisely the same as those of the classical propositional calculus, so of course the usual two valued truth tables yield a decision procedure.

4.3 First-degree Entailments

Two different (though related) 'syntactical' decision procedures were described for these in Section 1.7 (the method of 'tautological entailments' and the method of 'coupled trees'). A 'semantical' decision procedure using a certain four element matrix **4** is described in Section 3.3. The story thus told leaves out the historically (and otherwise) very important role of a certain eight element matrix M_0 (cf. [Anderson and Belnap, 1975, Section 22.1.3]). This matrix is essential for the study of first-degree formulas and higher (see Section 4.4 below), in so much as it is impossible to define an implication operation on **4** and pick out a proper subset of designated elements so as to satisfy the axioms of **E** (*a fortiori* **R**). Indeed M_0 was used in [Anderson and Belnap Jr., 1962b] and [Belnap, 1960b] to isolate the first-degree entailments of **R**, and the formulation of Section 1.7 presupposes this use.

4.4 First-degree Formulas

These are 'truth functions' of first-degree entailments and/or formulas containing no $\rightarrow$ at all (the 'zero-degree formulas'). Belnap [1967a] gave a decision procedure using certain finite 'products' of M_0. No one such product is characteristic for $\mathbf{D}_{\mathrm{fdf}}$, but every non-theorem of $\mathbf{E}_{\mathrm{fdf}}$ is refutable in some such products M_0^n (where n may in fact be computed as the largest number of first-degree entailments occurring in a disjunction once

[35] He continues somewhat more technically '... and none is known for which it has been proved that it does not possess the finite model property with recursive bound.'

the candidate theorem has been put in conjunctive normal form). Hence $\mathbf{E}_{\mathrm{fdf}}$ has the finite model property which suffices of course for decidability (cf. [Harrop, 1965]). This is frankly one of the most difficult proofs to follow in the whole literature of relevance logics. A sketch may be found in [Anderson and Belnap, 1975, Section 19].

4.5 'Career Induction'

This is what Belnap has labelled the approach, exemplified in Sections 4.1–4.3 above of extending the positive solution to the decision problem 'a degree at a time'. The last published word on the Belnap approach is to be found in his [1967b] where he examines entailments between conjunctions of first-degree entailments and first degree entailments.

Meyer [1979c], by an amazingly general and simple proof, shows that a positive answer to the decision problem for 'second-degree formulas' (no $\rightarrow$ within the scope of an arrow within the scope of an $\rightarrow$) is equivalent to finding a decision procedure for all of **R**.

4.6 Implication Fragment

We now start another tack. Rather than looking at fragments of the whole system **R** delimited by complexity of formulas, we instead consider fragments delimited by the connectives which they contain. The earliest result of this kind is due to [Kripke, 1959b], who gave a Gentzen system for the implicational fragments of **E** and **R**, and showed them decidable. We shall here examine the implicational fragment of **R** ($\mathbf{R}_{\rightarrow}$) in some detail as a kind of paradigm for this style of argument.[36]

The appropriate Gentzen calculus[37] $L\mathbf{R}_{\rightarrow}$ is the same as that given by Gentzen [1934] except for two trivial differences and one profound difference. The first trivial difference is the obvious one that we take only the operational rules for implication, and the second trivial difference consequent on this (with negation it would have to be otherwise) is that we can restrict our sequents to those with a single formula in the consequent. The profound difference is that we drop the structural rule variously called 'thinning' or 'weakening'. This leaves:

Axioms.

$$A \vdash A.$$

[36] Actually this and various other results discussed below using Gentzen calculi presupposes 'separation theorems' due to Meyer, showing, e.g. as is relevant to this case, that all of the theorems containing only $\rightarrow$ are provable from the axioms containing only $\rightarrow$.

[37] We do not follow Anderson and Belnap [1975] in calling Gentzen systems 'consecution calculi', much as their usage has to recommend it.

Structural Rules.

$$\text{Permutation}\ \frac{\alpha, A, B, \beta, \vdash C}{\alpha, B, A, \beta, \vdash C} \qquad \text{Contraction}\ \frac{\alpha, a, A \vdash B}{\alpha, A \vdash B}$$

Operational Rules.

$$(\vdash\rightarrow)\quad \frac{\alpha, A \vdash B}{\alpha \vdash A \rightarrow B} \qquad (\rightarrow\vdash)\quad \frac{\alpha \vdash A \quad \beta, B \vdash C}{\alpha, \beta, A \rightarrow B \vdash C}.$$

It is easy to see why thinning would be a disaster for relevant implication. Thus:

$$\cfrac{\cfrac{\cfrac{A \vdash A}{A, B \vdash A}\ \text{Thinning}}{A \vdash B \rightarrow A}\ (\vdash\rightarrow)}{\vdash A \rightarrow (B \rightarrow A)}\ (\vdash\rightarrow)$$

It is desirable to prove 'The Elimination Theorem', which says that the following rule would be redundant (could be eliminated).

$$(\text{Cut})\quad \frac{\alpha \vdash A \quad \beta, A \vdash B}{\alpha, \beta \vdash B}.$$

This is needed to show the equivalence of $L\mathbf{R}_{\rightarrow}$ to its usual Hilbert-style (axiomatic system '$H\mathbf{R}_{\rightarrow}$' $\mathbf{R}_{\rightarrow}$ one of the formulations of Section 1.3). We will not pause on details here, but the principal question regarding the equivalence is whether *modus ponens* (The sole rule for $H\mathbf{R}_{\rightarrow}$) is admissible in the sense that whenever $\vdash A$ and $\vdash A \rightarrow B$ are both derivable in $L\mathbf{R}_{\rightarrow}$, so is $\vdash B$ (let α and β be empty).

The strategy of the proof of the Elimination Theorem can essentially be that of Gentzen with one important but essentially minor modification. Thus, Gentzen actually proved something stronger than Cut elimination, namely,

$$(\text{Mix})\quad \frac{\alpha \vdash A \quad \beta \vdash B}{\alpha, [\beta - A] \vdash B},$$

where $[\beta - A]$ is the result of deleting *all* occurrences of A from β. This is useful in the induction, but sometimes it takes out too many occurrences of A. In Gentzen's framework these could always be thinned back in, but of course this is not available with $L\mathbf{R}_{\rightarrow}$. We thus instead generalise Cut to the rule

$$(\text{Fusion})\quad \frac{\alpha \vdash A \quad \beta \vdash B}{\alpha, (\beta - A) \vdash B},$$

where β contains some occurrences of A and $(\beta - A)$ is the result of deleting as many of those occurrences as one wishes (but at least one).

The main strategy of the decision procedure for $L\mathbf{R}_{\rightarrow}$ is to limit applications of the contraction rule so as to prevent a proof search from running on forever in the following manner: 'Is $p \vdash q$ derivable? Well it is if $p, p \vdash q$ is derivable. Is $p, p \vdash q$ derivable? Well it is if $p, p, p \vdash q$ is, etc.'.

We need one simple notion before strategy can be achieved. We shall say that the sequent of $\alpha' \vdash A$ is *a contraction* of sequent $\alpha \vdash A$ just in case $\alpha' \vdash A$ can be derived from $\alpha \vdash A$ by (repeated) applications of the rules Contraction and Permutation (with respect to this last it is helpful not even to distinguish two sequents that are mere permutations of one another). The idea that we now want to put in effect is to drop the rule Contraction, replacing it by building into the operational rules a limited amount of contraction (in the generalised sense just explained).

More precisely, the idea is to allow a contraction of the conclusion of an operational rule only in so far as the same result could not be obtained by first contracting the premises. A little thought shows that this means no change for the rule ($\vdash\rightarrow$), and that the following will suffice for

$$(\rightarrow\vdash') \quad \frac{\alpha \vdash A \quad \beta, B \vdash C}{[\alpha, \beta, A \rightarrow B] \vdash C}$$

where $[\alpha, \beta, A \rightarrow B]$ is any contraction of $\alpha, \beta, A \rightarrow B$ such that :

1. $A \rightarrow B$ occurs only 0, 1, or 2 times fewer than in $\alpha, \beta, A \rightarrow B$;
2. Any formula other than $A \rightarrow B$ occurs only 0 or 1 time fewer.

It is clear that after modifying $L\mathbf{R}_{\rightarrow}$ by building some limited contraction into ($\rightarrow\vdash$) in the manner just discussed, the following is provable by an induction on length of derivations:

CURRY'S LEMMA.[38] *If a sequent Γ' is a contraction of a sequent Γ and Γ has a derivation of length n, then Γ' has a derivation of length $\leq n$.*

[38] This is named (following [Anderson and Belnap, 1975]) after an analogous lemma in [Curry, 1950] in relation to classical (and intuitionistic) Gentzen systems. There, with free thinning available, Curry proves his lemma with ($\rightarrow\vdash$) (in its singular version) stated as:

$$\frac{\Gamma, A \rightarrow B \vdash A \quad \Gamma, A \rightarrow B, B \vdash C}{\Gamma, A \rightarrow B \vdash C}.$$

This in effect *requires* the maximum contraction *permitted* in our statement of ($\rightarrow\vdash$) above, but this is OK since items contracted 'too much' can always be thinned back in. Incidentally, our statement of ($\rightarrow\vdash$) also differs somewhat from the statement of Anderson and Belnap [1975] or Belnap and Wallace [1961], in that we build in just the minimal amount of contraction needed to do the job.

Clearly this lemma shows that the modification of $L\mathbf{R}_{\rightarrow}$ leaves the same sequents derivable (since the lemma says the effect of contraction is retained). So henceforth we shall by $L\mathbf{R}_{\rightarrow}$ always mean the modified version.

Besides the use just adverted to, Curry's Lemma clearly shows that every derivable sequent has an *irredundant* derivation in the following sense: one containing no branch with a sequent Γ' below a sequent Γ of which it is a contraction.

We are finally ready to begin explicit talk about the decision procedure. Given a sequent Γ, one begins the test for derivability as follows (building a 'complete proof search tree'): one places above Γ all possible premises or pairs of premises from which Γ follows by one of the rules. Note well that even with the little bit of contraction built into ($\rightarrow\vdash$) this will still be only a finite number of sequents. Incidentally, one draws lines from those premises to Γ. One continues in this way getting a tree. It is reasonably clear that if a derivation exists at all, then it will be formed as a subtree of this 'complete proof search there', by the paragraph just above, the is complete proof search tree can be constructed to be irredundant. But the problem is that the complete proof search tree may be infinite, which would tend to louse up the decision procedure. There is a well-known lemma which begins to come to the rescue:

KÖNIG'S LEMMA. *A tree is finite iff both (1) there are only finitely many points connected directly by lines to a given point ('finite fork property') and (2) each branch is finite ('finite branch property').*

By the 'note well' in the paragraph above, we have (1). The question remaining then is (2), and this is where an extremely ingenious lemma of Kripke's plays a role. To state it we first need a notion from Kleene. Two sequents $\alpha \vdash A$ and $\alpha' \vdash A$ are *cognate* just when exactly the same formulas (not counting multiplicity) occur in α as in α'. Thus, e.g. all of the following are cognate to each other:

(1) $X, Y \vdash A$

(2) $X, X, Y \vdash A$

(3) $X, Y, Y \vdash A$

(4) $X, X, Y, Y \vdash A$

(5) $X, X, X, Y, Y \vdash A$.

We call the class of all sequents cognate to a given sequent a *cognation class*.

KRIPKE'S LEMMA. *Suppose a sequence of cognate sequents $\Gamma_0, \Gamma_1, \ldots,$ is* irredundant *in the sense that for no Γ_i, Γ_j with $i < j$, is Γ_i a contraction of Γ_j. Then the sequence is finite.*

We postpone elaboration of Kripke's Lemma until we see what use it is to the decision procedure. First we remark an obvious property of $L\mathbf{R}_{\to}$ that is typical of Gentzen systems (that lack Cut as a primitive rule):

SUBFORMULA PROPERTY. *If* Γ *is a derivable sequent of* $L\mathbf{R}_{\to}$*, then any formula occurring in any sequent in the derivation is a subformula of some formula occurring in* Γ*.*

This means that the number of cognation classes occurring in any derivation (and hence in each branch) is finite. But Kripke's Lemma further shows that only a finite number of members of each cognation class occur in a branch (this is because we have constructed the complete proof search tree to be irredundant). So every branch is finite, and so both conditions of König's lemma hold. Hence the complete proof search tree is finite and so there is a decision procedure.

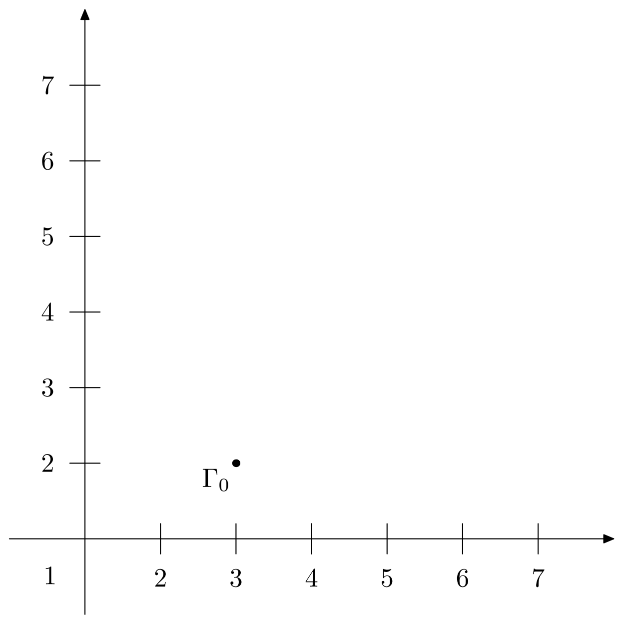


Figure 1. Sequents in the Plane

Returning now to Kripke's Lemma, we shall not present a proof (for which see [Belnap Jr. and Wallace, 1961] or [Anderson and Belnap, 1975]). Instead we describe how it can be geometrically visualised. For simplicity we consider sequents cognate to $X, Y \vdash A$ ((1), (2), (3), etc. above). Each such sequent can be represented as a point in the upper right-hand quadrant of the co-ordinate plane (where origin is labelled with 1 rather than 0 since (1) is the minimal sequent in the cognation class). See Figure 1. Thus, e.g. (5) gets represented as '3 X units' and '2 Y units'.

Now given any sequent, say

$$(\Gamma_0) \quad X, X, X, Y, Y \vdash A$$

as a starting point one might try to build an irredundant sequence by first building up the number of Y's tremendously (for purposes of keeping on the page we let this be to six rather than say a million). But in so doing one has to reduce the number of X's (say, to be strategic, by one). The graph now looks like 2 for the first two members of the sequence Γ_0, Γ_1.

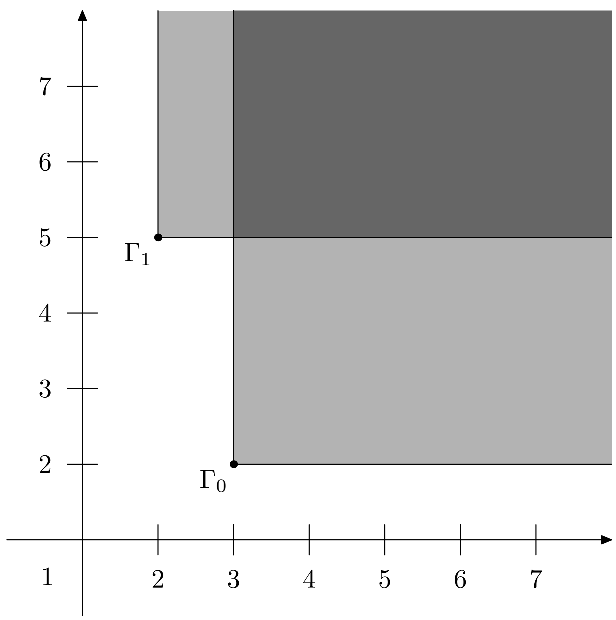


Figure 2. Descending Regions

The purpose of the intersecting lines at each point is to mark off areas (shaded in the diagram) into which no further points of the sequence may be placed. Thus if Γ_2 were placed as indicated at the point (6, 5), it would reduce to Γ_0. What this means is that each new point must march either one unit closer to the X axis or one unit closer to the Y axis. Clearly after a finite number of points one or the other of the two axes must be 'bumped', and then after a short while the other must be bumped as well. When this happens there is no space left to play without the sequence becoming redundant.

The generalisation to the case of n formulas in the antecedent to Euclidean n-space is clear (this is with n finite—with n infinite no axis need ever be bumped).

Incidentally, Kripke's Lemma (as Meyer discovered) is equivalent to a theorem of Dickson about prime numbers: Let M be a set of natural numbers all of which are composed out of the first m primes. Then every $n \in M$ is of the form $P_1^{n_1} \cdot P_2^{n_2} \cdot \ldots P_k^{n_k}$, and hence (by unique decomposition) can be regarded as a sequence of the P_i's in which each P_i is repeated n_i times. Divisibility corresponds then to contraction (at least neglecting the case $n_i = 0$). Dickson's theorem says that if no member of M has a proper divisor in M, then M is finite.

Before going on to consider how the addition of connectives changes the complexity, let us call the reader's attention to a major open problem: It is still unknown whether the implication fragment of **T** is decidable.

4.7 Implication–Negation Fragment

The idea of $L\mathbf{R}_{\rightarrow}^{\neg}$ is to accommodate the classical negation principles presenting **R** in the same way that Gentzen [1934] accommodated them for classical logic: provide multiple right-hand sides for the sequents. this means that a sequent is of the form $\alpha \vdash \beta$, where α and β are (possible empty) finite sequences of formulas. One adds structural rules for Permutation and Contraction on the right-hand side, reformulates $(\vdash\rightarrow)$ and $(\rightarrow\vdash)$ as follows

$$(\vdash\rightarrow)\quad \frac{\alpha, A \vdash B, \beta}{\alpha \vdash A \rightarrow B, \beta} \qquad (\rightarrow\vdash)\quad \frac{\alpha \vdash A, \gamma \quad \beta, B \vdash \delta}{\alpha, \beta, A \rightarrow B \vdash \gamma, \delta},$$

and adds 'flip and flop' rules for negation:

$$(\vdash \neg)\quad \frac{\alpha, A \vdash \beta}{\alpha \vdash \neg A, \beta} \qquad (\neg \vdash)\quad \frac{\alpha \vdash A, \beta}{\alpha, \neg A \vdash \beta}.$$

$L\mathbf{E}_{\rightarrow}^{\neg}$ is the same except that in the rule $(\vdash\rightarrow)\beta$ must be empty and α must consist only of formulas whose main connective is $\rightarrow$. The decision procedure for $L\mathbf{E}_{\rightarrow}^{\neg}$ was worked out by Belnap and Wallace [1961] along basically the lines of the argument of Kripke just reported in the last section, and is clearly reported in [Anderson and Belnap, 1975, Section 13]. the modification to $L\mathbf{R}_{\rightarrow}^{\neg}$ is straightforward (indeed $L\mathbf{R}_{\rightarrow}^{\neg}$ is easier because one need not prove the theorem of p. 128 of [Anderson and Belnap, 1975], and so one can avoid all the apparatus there of 'squeezes'). McRobbie and Belnap [1979] have provided a nice reformulation of $L\mathbf{R}_{\rightarrow}^{\neg}$ in an analytic tableau style, and Meyer has extended this to give analytic tableau for linear logic and other systems in the vicinity of **R** [Meyer *et al.*, 1995].

4.8 Implication–Conjunction Fragment, and **R** *Without Distribution*

This work is to be found in [Meyer, 1966]. The idea is to add to $L\mathbf{R}_{\rightarrow}$ the Gentzen rules:

$$(\wedge\vdash)\quad \frac{\alpha, A \vdash C}{\alpha, A\wedge B \vdash C}\quad \frac{\alpha, \beta \vdash C}{\alpha, A\wedge B \vdash C}\qquad (\vdash\wedge)\quad \frac{\alpha \vdash A \quad \alpha \vdash B}{\alpha \vdash A\wedge B}.$$

Again the argument for decidability is a simple modification of Kripke's.

Note that it is important that the rule $(\wedge\vdash)$ is stated in two parts, and not as one 'Ketonen form' rule:

$$(K\wedge\vdash)\quad \frac{\alpha, A, B \vdash C}{\alpha, A\wedge B \vdash C}.$$

The reason is that without thinning it is impossible to derive the rule(s) $(\wedge\vdash)$ from $(K\wedge\vdash)$.

Early on it was recognised that the distribution axiom

$$A\wedge(B\vee C)\rightarrow(A\vee B)\vee C$$

was difficult to derive from Gentzen-style rules for **E** and **R**. Thus Anderson [1963] saw this as the sticking point for developing Gentzen formulations, and Belnap [1960b, page 72]) says with respect to $L\mathbf{E}_{\rightarrow}$ that 'the standard rules for conjunction and disjunction could be added ... the Elimination Theorem (suitably modified) remaining provable. However, [since distribution would not be derivable], the game does not seem worth the candle'. Meyer [1966] carried out such an addition to $L\mathbf{R}^{-}_{\rightarrow}$, getting a system he called $L\mathbf{R}-$, whose Hilbert-style version is precisely **R** without the distribution axiom. He showed using a Kripke-style argument that this system is decidable. This system is now called **LR**, for "lattice **R**".

Meyer [1966] also showed how **LR** can be translated into $\mathbf{R}_{\rightarrow,\wedge}$ rather simply. Given a formula A in the language of $\mathbf{LR}^{+}$, let V be the set of variables in A, and let two atomic propositions p_t and p_f not in V. Set $\neg A$

for the moment to be $A \to p_f$, to define a translation A' of A as follows.

$$\begin{aligned} p' &= p \\ t' &= p_t \\ (A \to B)' &= A' \to B' \\ (A \wedge B)' &= A' \wedge B' \\ (A \vee B)' &= \neg(\neg A' \wedge \neg B') \\ (A \circ B)' &= \neg(A' \to \neg B') \end{aligned}$$

then setting $t(A) = \bigwedge\{p_t \to (p \to p) : p \in V \cup \{p_t, p_f\}\}$ and $f(A) = \bigwedge\{\neg\neg p \to p : p \in V \cup \{p_t, p_f\}\}$, we get the following theorem:

TRANSLATION THEOREM (Meyer). *If A is a formula in* $\mathbf{LR}^+$ *then A is provable in* $\mathbf{LR}^+$ *if and only if* $(t(A) \wedge f(A) \wedge p_t) \to A'$ *is provable in* $\mathbf{R}_{\to,\wedge}$.

The proof is given in detail in [Urquhart, 1997], and we will not present it here.

Some recent work of Alasdair Urquhart has shown that although $\mathbf{R}_{\to,\wedge}$ is decidable, it is *only just* decidable [Urquhart, 1990, Urquhart, 1997].

More formally, Urquhart has shown that given any particular formula in the language of $\mathbf{R}_{\to,\wedge}$, there is no primitive recursive bound on either the time or the space taken by a computation of whether or not that formula is a theorem. Presenting the proof here would take us too far away from the *logic* to be worthwhile, however we can give the reader the kernel of the idea behind Urquhart's result.

Urquhart follows work of [Lincoln *et al.*, 1992] by using a propositional logic to encode the behaviour of a *branching counter machines*. A counter machine has a finite number of *registers* (say, r_i for suitable i) which each hold one non-negative integer, and some finite set of possible *states* (say, q_j for suitable j). Machines are coded with a list of instructions, which enable you to *increment* or *decrement* registers, and test for registers' being zero. A *branching* counter machine dispenses with the test instructions and allows instead for machines to take multiple execution paths, by way of *forking* instructions. The instruction $q_i + r_j q_k$ means "when in q_i, add 1 to register r_j and enter stage q_k," and $q_i - r_j q_k$ means "when in q_i, subtract 1 to register r_j (if it is non-empty) and enter stage q_k," and $q_i f q_j q_k$ is "when in q_i, fork into two paths, one taking state q_j and the other taking q_k."

A machine configuration is a state, together with the values of each register. Urquhart uses the logic **LR** to simulate the behaviour of a machine. For each register r_i, choose a distinct variable R_i, for each state q_j choose a distinct variable Q_j. The configuration $\langle q_i; n_1, \ldots, n_l\rangle$, where n_i is the value of r_i is the formula

$$Q_i \circ R_1^{n_1} \circ \cdots \circ R_l^{n_l}$$

and the instructions are modelled by sequents in the Gentzen system, as follows:

Instruction	Sequent
$q_i + r_j q_k$	$Q_i \vdash Q_k \circ R_j$
$q_i - r_j q_k$	$Q_i, R_j \vdash Q_k$
$q_i f q_j q_k$	$Q_i \vdash Q_j \vee Q_k$

Given a machine program (a set of instructions) we can consider what is provable from the sequents which code up those instructions. This set of sequents we can call the *theory* of the machine. $Q_i \circ R_1^{n_1} \circ \cdots \circ R_l^{n_l} \vdash Q_j \circ R_1^{m_1} \circ \cdots \circ R_l^{m_l}$ is intended to mean that from state configuration $\langle q_i; n_1, \ldots, n_l \rangle$ all paths will go through configuration $\langle q_j; m_1, \ldots, m_l \rangle$ after some number of steps.

A branching counter machine *accepts* an initial configuration if when run on that configuration, all branches terminate at the final state q_f, with all registers taking the value zero. The corresponding condition in **LR** will be the provability of

$$Q_i \circ R_1^{n_1} \circ \cdots \circ R_l^{n_l} \vdash Q_m$$

This will *nearly* do to simulate branching counter machines, except for the fact that in **LR** we have $A \vdash A \circ A$. This means that each of our registers can be incremented as much as you like, provided that they are non-zero to start with. This means that each of our machines need to be equipped with every instruction of the form $q_i{>}0 + r_j q_i$, meaning "if in state q_i, add 1 to r_j, provided that it is already nonzero, and remain in state q_i."

Given these definitions, Urquhart is able to prove that a configuration is accepted in branching counter machine, if and only if the corresponding sequent is provable from the theory of that machine. But this is equivalent to a formula

$$\bigwedge \text{Theory}(M) \wedge t \to (Q_1 \to Q_m)$$

in the language of **LR**. It is then a short step to our complexity result, given the fact that there is no primitive recursive bound on determining acceptability for these machines. Once this is done, the translation of **LR** into $\mathbf{R}_{\to\wedge}$ gives us our complexity result.

It is still unknown if $\mathbf{R}_{\to}$ has similar complexity or whether it is a more tractable system.

Despite this complexity result, Kripke's algorithm can be implemented with quite some success. The theorem prover `Kripke`, written by McRobbie, Thistlewaite and Meyer, implements Kripke's decision procedure, together with some quite intelligent proof-search pruning, by means of finite models. If a branch is satisfiable in **RM3**, for example, there is no need to extend it to give a contradiction. This implementation *works* in many cases [Thistlewaite *et al.*, 1988]. Clearly, work must be done to see whether the horrific complexity of this problem in general can be transferred to results about *average case* complexity.

Finally, before moving to add distribution, we should mention that Linear Logic (see Section 5.5) also lacks distribution, and the techniques used in the theorem prover `Kripke` have application in that field also.

4.9 Positive **R**

In this section we will examine extensions of the Gentzen technique to cover all of positive relevance logic. We know (see Section 4.12) that this will not provide decidability. However, they provide another angle on **R** and cousins. Dunn and Minc independently developed a Gentzen-style calculus (with some novel features) for **R** without negation ($L\mathbf{R}^+$).[39] Belnap [1960b] had already suggested the idea of a Gentzen system in which antecedents were sequences of sequences of formulas, rather than just the usual sequences of formulas (in this section 'sequence' always means *finite* sequence). The problem was that the Elimination Theorem was not provable. $L\mathbf{R}^+$ goes a step 'or two' further, allowing an antecedent of a sequent instead to be a sequence of sequence of ...sequences of formulas. More formally, we somehow distinguish two kinds of sequences, 'intensional sequences' and 'extensional sequences' (say prefix them with an 'I' or an 'E'). an antecedent can then be an intensional sequence of formulas, an extensional sequence of the last mentioned, etc. or the same thing but with 'intensional' and 'extensional' interchanged. (We do not allow things to 'pile up', with, e.g. *intensional* sequences of *intensional* sequences—there must be alternation).[40] Extensional sequences are to be interpreted using ordinary 'extensional' conjunction $\wedge$, whereas intensional sequences are to be interpreted using 'intensional conjunction' $\circ$, which may be defined in the full system **R** as $A \circ B = \neg(A \to \neg B)$, but here it is taken as primitive—see below).

[39]Dunn's result was presented by title at a meeting of the Association for Symbolic Logic, December, 1969 (see [Dunn, 1974]), and the full account is to be found in [Anderson and Belnap, 1975, Section 28.5]). Minc [1972, earliest presentation said to there to be February 24] obtained essentially the same results (but for the system with a necessity operator). See also [Belnap Jr. *et al.*, 1980].

[40]This differs from the presentation of [Anderson and Belnap, 1975] which allows such 'pile ups', and then adds additional structural rules to eliminate them. Belnap felt this was a clearer, more explicit way of handling things and he is undoubtedly right, but Dunn has not been able to read his own section since he rewrote it, and so return to the simpler, more sloppy form here.

We state the rules, using commas for extensional sequences, semicolons for intentional sequences, and asterisks ambiguously for either; we also employ an obvious substitution notation.[41]

$$\text{Permutation}\quad \frac{\alpha[\beta * \gamma] \vdash A}{\alpha[\gamma * \beta] \vdash A} \qquad \text{Contraction}\quad \frac{\alpha[\beta * \beta] \vdash A}{\alpha[\beta] \vdash A}$$

$$\text{Thinning}\quad \frac{\alpha[\beta] \vdash A}{\alpha[\beta, \gamma] \vdash A}, \quad \text{provided } \beta \text{ is non-empty.}$$

$$\frac{\alpha; A \vdash B}{\alpha \vdash A \to B}(\vdash\to) \qquad \frac{\alpha \vdash A \quad \beta[B] \vdash C}{\beta[\alpha; A \to B] \vdash C}(\to\vdash)$$

$$\frac{\alpha \vdash A \quad \alpha \vdash B}{\alpha \vdash A \wedge B}(\vdash \wedge) \qquad \frac{\alpha[A, B] \vdash C}{\alpha[A \wedge B] \vdash C}(\wedge \vdash)$$

$$\frac{\alpha \vdash A}{\alpha \vdash A \vee B}(\vdash \vee) \quad \frac{\alpha \vdash B}{\alpha \vdash A \vee B}(\vdash \vee) \qquad \frac{\alpha[a] \vdash C \quad \alpha[B] \vdash C}{\alpha[A \vee B] \vdash C}(\vee \vdash)$$

$$\frac{\alpha \vdash A \quad \beta \vdash B}{\alpha; \beta \vdash A \circ B}(\vdash \circ) \qquad \frac{\alpha[a; B] \vdash C}{\alpha \circ B] \vdash C}(\circ \vdash)$$

For technical reasons (see below) we add the sentential constant t with the axiom $\vdash t$ and the rule:

$$\frac{\alpha[B] \vdash A}{\alpha[\beta; t] \vdash A}(t \vdash)$$

The point of the two kind of sequences can now be made clear. Let us examine the classically (and intuitionistically) valid derivation:

(1)	$\underline{A \vdash A}$	Axiom
(2)	$\underline{A, B \vdash A}$	Thinning
(3)	$A \vdash B \to A$	$(\vdash\to)$.

It is indifferent whether (2) is interpreted as

$(2\wedge)$	$(A \wedge B) \to A$, or
$(2\to)$	$A \to (B \to A)$,

[41]With the understanding that substitutions do not produce 'pile ups'. Thus, e.g. a 'substitution' of an intensional sequence for an item in an intensional sequence does not produce an intensional sequence with an element that is an intensional sequence formed by juxtaposition. Again this differs from the presentation of [Anderson and Belnap, 1975, cf. note 28].

because of the principles of exportation and importation. In $L\mathbf{R}^+$ however we may regard (2) as ambiguous between

$$
\begin{array}{lll}
(2,) & A, B \vdash A & \text{(extensional), and} \\
(2;) & A; B \vdash A & \text{(intensional).}
\end{array}
$$

(2,) continues to be interpreted as ($2\wedge$), but (2;) is interpreted as

$$(2\circ) \quad (A \circ B) \to A.$$

Now in $\mathbf{R}$, exportation holds for $\circ$ but not for $\wedge$ (importation holds for both). Thus the move from (2;) to (3) is valid, but not from (2,) to (3). On the other hand, in $\mathbf{R}$, the inference from $A \to C$ to $(A \wedge B) \to C$ is valid, whereas the inference to $(A \circ B) \to C$ is not. Thus the move from (1) to (2,) is valid, but not the move from (1) to (2;). the whole point of $L\mathbf{R}^+$ is to allow some thinning, but only in extensional sequences.

This allows the usual classical derivation of the distribution axiom to go through, since clearly

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee C$$

can be derived with no need of any but the usual extensional sequence. The following sketch of a derivation of distribution in the consequent is even more illustrative of the powers of $L\mathbf{R}^+$ (permutations are left implicit; also the top half is left to the reader);

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X \vdash X \qquad \dfrac{X \vdash X \qquad \dfrac{A, B \vdash (A \wedge B) \vee C \qquad A, C \vdash (A \wedge B) \vee C}{A, B \vee C \vdash (A \wedge B) \vee C}\,(\vee\vdash)}{A, (X; X \to B \vee C) \vdash (A \wedge B) \vee C}\,(\to\vdash)}{(X; X \to A), (X; X \to A, X \to B \vee C) \vdash (A \wedge B) \vee C}\,(\to\vdash)}{X \to A, X \to B \vee C; X \vdash (A \wedge B) \vee C}}{(X \to A) \wedge (X \to B \vee C); X \vdash (A \wedge B) \vee C}}{\vdash (X \to A) \wedge (X \to B \vee C) \to [X \to (A \wedge B) \vee C]}}{}
$$

$L\mathbf{R}^+$ is equivalent to $\mathbf{R}^+$ in the sense that for any negation-free sentence A of $\mathbf{R}, \vdash A$ is derivable in $L\mathbf{R}^+$ iff A is a theorem of $\mathbf{R}$. The proofs of both halves of the equivalence are complicated by technical details. Right-to-left (the interpretation theorem) requires the addition of intensional conjunction as primitive, and then a lemma, due to R. K. Meyer, to the effect that this is harmless (a conservative extension). Left-to-right (the Elimination

Theorem) is what requires the addition of the constant true sentence t. This is because the 'Cut' rule is stated as:

$$\frac{\alpha \vdash A \quad \beta(A) \vdash B}{\beta(\alpha) \vdash B},$$

where $\beta(\alpha)$ is the result of replacing arbitrarily many occurrences of A in $\beta(A)$ by α if α is non-empty, and otherwise by t.[42] Without this emendation of the Cut rule one could derive $B \vdash A$ whenever $\vdash A$ is derivable (for arbitrary B, relevant or not) as follows

$$\frac{\vdash A \quad \dfrac{A \vdash A}{A, B \vdash A}\text{ Thinning}}{B \vdash A}\text{ Cut}$$

Discussing decidability a bit, one problem seems to be that Kripke's Lemma (appropriately modified) is just plain false. The following is a sequence of cognate sequents in just the two propositional variables X and Y which is irredundant in the sense that structural rules will not get you from a later member to an earlier member:

$$X; Y \vdash X \quad (X; Y), X \vdash X \quad ((X; Y), X); Y \vdash X \ldots$$ [43]

4.10 Systems Without Contraction

Gentzen systems without the contraction rule tend to be more amenable to decision procedures than those with it. Clearly, all of the work in Kripke's Lemma is in keeping contraction under control. So it comes as no surprise that if we consider systems without contraction for intensional structure, decision procedures are forthcoming. If we remove the contraction rule from **LR** we get the system which has been known as **LRW** (**R** without **W** without distribution), and which is equivalent to the additive and multiplicative fragment of Girard's linear logic [Girard, 1987]. It is well known that this

[42] Considerations about the eliminability of occurrences of t are then needed to show the admissibility of *modus ponens*. This was at least the plan of [Dunn, 1974]. A different plan is to be found in [Anderson and Belnap, 1975, Section 28.5], where things are arranged so that sequents are never allowed to have empty left-hand sides (they have t there instead).

[43] Further, this is not just caused by a paucity of structural rules. Interpreting the sequents of formulas of $\mathbf{R}^+$ ($\wedge$ for comma, $\circ$ for semicolon, $\rightarrow$ for $\vdash$) no later formula provably implies an earlier formula. Incidentally, one does need at least two variables (cf. R. K. Meyer [1970b]).

system is decidable. In the Gentzen system, define the *complexity* of a sequent to be the number of connectives and commas which appear in it. It is trivial to show that complexity never increases in a proof and that as a result, from any given sequent there are only a finite number of sequents which could appear in a proof of the original sequent (if there is one). This gives rise to a simple decision procedure for the logic. (Once the work has already been done in showing that Cut is eliminable.)

If we add the extensional structure which appears in the proof theories of traditional relevance logics then the situation becomes more difficult. However, work by Giambrone has shown that the Gentzen systems for positive relevance logics without contraction do in fact yield decision procedures [Giambrone, 1985]. In these systems we do have extensional contraction, so such a simple minded measure of complexity as we had before will not yield a result. In the rest of this section we will sketch Giambrone's ideas, and consider some more recent extensions of them to include negation. For details, the reader should consult his paper. The results are also in the second volume of *Entailment* [Anderson *et al.*, 1992].

Two sequents are *equivalent* just when you can get from one to the other by means of the invertible structural rules (intensional commutativity, extensional commutativity, and so on). A sequent is *super-reduced* if no equivalent sequent can be the premise of a rule of extensional contraction. A sequent is *reduced* if for any equivalent sequents which are the premise of a rule of extensional contraction, the conclusion of that rule is super-reduced. So, intuitively, a super-reduced sequent has no duplication in it, and a reduced sequent can have one part of it 'duplicated', but no more. Clearly any sequent is equivalent to a super-reduced sequent. The crucial lemma is that any super-reduced sequent has a proof in which every sequent appearing is reduced. This is clear, for given any proof you can transform it into one in which every sequent is reduced without too much fuss.

As a result, we have gained as much control over extensional contraction as we need. Giambrone is able to show that only finitely many reduced sequent can appear in the proof of a given sequent, and as a result, the size of the proof-search tree is bounded, and we have decidability. This technique does not work for intensional contraction, as we do not have the result that every sequent is equivalent to an intensionally super-reduced sequent, in the absence of the mingle rule. While $A \wedge A \vdash B$ is equivalent to $A \vdash B$, we do not have the equivalence of $A \circ A \vdash B$ and $A \vdash B$, without mingle.

These methods can be extended to deal with negation. Brady [1991] constructs out of *signed* formulae TA and FA instead of formulae alone, and this is enough to include negation without spoiling the decidability property. Restall [1998] uses the techniques of Belnap's *Display Logic* (see

Section 5.2) to provide an alternate way of modelling negation in sequent systems. These techniques show that the decidability of systems without intensional contraction are decidable, to a large extent independently of the other properties of the intensional structure.

4.11 Various Methods Used to Attack the Decision Problem

Decision procedures can basically be subdivided into two types: syntactic (proof-theoretic) and semantic (model-theoretical). A paradigm of the first type would be the use of Gentzen systems, and a paradigm of the second would be the development of the finite model property. It seems fair to say, looking over the previous sections, that syntactic methods have dominated the scene when nested implications have been afoot, and that semantical methods have dominated when the issue has been first-degree implications and first-degree formulas.[44]

There are two well-known model-theoretic decision procedures used for such non-classical logics as the intuitionistic and modal logics. One is due to McKinsey and Tarski and is appropriate to algebraic models (matrices) (cf. [Lemmon, 1966, p. 56 ff.]), and the other (often called 'filtration') is due to Lemmon and Scott and is appropriate to Kripke-style models (cf. [Lemmon, 1966, p. 208 ff]). Actually these two methods are closely connected (equivalent?) in the familiar situation where algebraic model sand Kripke models are duals. The problem is that neither seems to work with **E** and **R**. The difficulty is most clearly stated with **R** as paradigm. For the algebraic models the problem is given a de Morgan monoid $(M, \wedge, \vee, ^-, \circ)$ and a finite de Morgan sublattice $(D, \wedge', \vee, ^{-'})$, how to define a new multiplicative operation $\circ'$ on D so as to make it a de Morgan monoid and so for $x, y \in D$, if $x \circ y \in D$ then $x \circ y = x \circ' y$. the chief difficult is in satisfying the associative law. For the Kripke-style models (say the Routley–Meyer variety) the problem is more difficult to state (especially if the reader has skipped Section 3.7) but the basic difficulty is in the satisfying of certain requirements on the three-placed accessibility relation once set-ups have been identified into a finite number of equivalence classes by 'filtration'. Thus, e.g. the requirement corresponding to the algebraic requirement of associativity is $Rayx$ & $Rbcy \Rightarrow \exists y(Raby$ & $Rycx)$[45] the problem in a nutshell is that after filtration one does not know that there exists the appropriate equivalence class $\bar{y}$ needed to feed such an existentially hungry postulate.

The McKinsey-Tarski method has been used successfully by Maksimova [1967] with respect to a subsystem of **R**, which differs essentially only in

[44] As something like 'the exception that proves the rule' it should be noted that Belnap's [1967a] work on first-degree formulas and slightly more complex formulas has actually been a subtle blend of model-theoretic (algebraic) and proof-theoretic methods.

[45] This is suggestively written (following Meyer) as $Ra(bc)x \Rightarrow R(ab)cx$.

that it replaces the nested form of the transitivity axiom

$$(A \to B) \to [(B \to C) \to (A \to C)]$$

by the 'conjoined' form

$$(A \to B) \wedge (B \to C) \to (A \to C).^{46}$$

Perhaps the most striking positive solution to the decision problem for a relevance logic is that provided for **RM** by Meyer (see [Anderson and Belnap, 1975, Section 29.3], although the result was first obtained by Meyer [1968].[47] Meyer showed that a formula containing n propositional variables is a theorem of **RM** iff it is valid in the 'Sugihara matrix' defined on the non-zero integers from $-n$ to $+n$. this result was extended by [Dunn, 1970] to show that every 'normal' extension of **RM** has some finite Sugihara matrix (with possibly 0 as an element) as a characteristic matrix. So clearly **RM** and its extensions have at least the finite model property. Cf. Section 3.10 for further information about **RM**.

Meyer [private communication] has thought that the fact that the decidability of **R** is equivalent to the solvability of the word problem for de Morgan monoids suggests that **R** might be shown to be undecidable by some suitable modification of the proof that the word problem for monoids is unsolvable. It turns out that this is technique is the one which pays off — although the proof is very complex. The complexity arises because there is an important disanalogy between monoids and de Morgan monoids in that in the latter the multiplicative operation is necessarily commutative (and the word problem for commutative monoids *is* solvable).[48] Still it has occurred to both Meyer and Dunn that it might be possible to define a new multiplication operation $\times$ for both $\circ$ and $\wedge$ in such a way as to embed the free monoid into the free de Morgan monoid. This suspicion has turned out to be right, as we shall see in the next section.

[46] Dunn remembers R. Routley communicating some such result to in say the late 1960s, but he now finds no record of it. Also both Meyer and Dunn experimented with applying McKinsey-Tarksi methods to weak relevance logics at about this time.

[47] In fact neither McKinsey-Tarski methods nor filtration was used in this proof. We are no clearer now that they could not be used, and we think the place to start would be to try to apply filtration to the Kripke-style semantics for **RM** of [Dunn, 1976b], which uses a *binary* accessibility relation and seems to avoid the problems caused by 'existentially hungry axioms' for the ternary accessibility relation.

[48] In this connection two things should be mentioned. First, Meyer [unpublished typescript, 1973] has shown that not all finitely generated de Morgan monoids are finitely presentable. Second, Meyer and Routley [1973c] have constructed a positive relevance logic $\mathbf{Q}^+$ (the algebraic semantics for which dispenses with commutativity) and shown it undecidable.

4.12 **R**, **E** *and Related Systems*

As is quite well known by now, the principal systems of relevance logic, **R**, **E** and others, are undecidable. Alasdair Urquhart proved this in his ground breaking papers [Urquhart, 1983, Urquhart, 1984]. We have recounted earlier attempts to come to a conclusion on the decidability question. The insights that helped decide the issue came from an unexpected quarter — projective geometry. To see why projective geometry gave the necessary insights, we will first consider a simple case, the undecidability of the system **KR**. **KR** is given by adding $A \wedge \neg A \rightarrow B$ to **R**. A **KR** frame is one satisfying the following conditions (given by adding the clause that $a = a^*$ to the conditions for an **R** frame).

$R0ab$ iff $a = b$	$Rabc$ iff $Rbac$ iff $Racb$ (total permutation)
$Raaa$ for each a	R^2abcd only if R^2acbd

The clauses for the connectives are standard, with the proviso that $a \vDash \neg A$ iff $a \nvDash A$, since $a = a^*$.

Urquhart's first important insight was that **KR** frames are quite like projective spaces. A *projective space* $\mathcal{P}$ is a set P of points, and a collection L of subsets of P called *lines*, such that any two distinct points are on exactly one line, and any two distinct lines intersect in exactly one point. But we can define projective spaces instead through the ternary relation of collinearity. Given a projective space $\mathcal{P}$, its collinearity relation C is a ternary relation satisfying the condition:

> $Cabc$ iff $a = b = c$, or a, b and c are distinct and they lie on a common line.

If $\mathcal{P}$ is a projective space, then its collinearity relation C satisfies the following conditions,

> $Caaa$ for each a. $Cabc$ iff $Cbac$ iff $Cacb$. C^2abcd only if C^2acbd.

provided that every line has at least four points (this last requirement is necessary to verify the last condition). Conversely, if we have a set with a ternary relation C satisfying these conditions, then the space defined with the original set as points and the sets $l_{ab} = \{c : Cabc\} \cup \{a, b\}$ where $a \neq b$ as lines is a projective space.

Now the similarity with **KR** frames becomes obvious. If $\mathcal{P}$ is a projective space, the frame $\mathcal{F}(\mathcal{P})$ generated by P is given by adjoining a new point 0, adding the conditions $C0aa$, $Ca0a$, and $Caa0$, and by taking the extended relation C to be the accessibility relation of the frame.

Projective spaces have a naturally associated undecidable problem. The problem arises when considering the *linear subspaces* of projective spaces.

A subspace of a projective space is a subset which is also a projective space under its inherited collinearity relation. Given any two linear subspaces $\mathcal{X}$ and $\mathcal{Y}$, the subspace $\mathcal{X}+\mathcal{Y}$ is the set of all points on lines through points in $\mathcal{X}$ and points in $\mathcal{Y}$.

In **KR** frames there are propositions which play the role of linear subspaces in projective spaces. We need a convention to deal with the extra point 0, and we simply decree that 0 should be *in* every "subspace." Then linear subspaces are equivalent to the *positive idempotents* in a frame. That is, they are the propositions X which are *positive* (so $0 \in X$) and *idempotent* (so $X = X \circ X$). Clearly for any sentence A and any **KR** model $\mathcal{M}$, the extension of A, $||A||$ in $\mathcal{M}$ is a positive idempotent iff $0 \vDash A \wedge (A \leftrightarrow A \circ A)$. It is then not too difficult to show that if A and B are positive idempotents, so are $A \circ B$ and $A \wedge B$, and that t and $\top$ are positive idempotents.

Given a projective space $\mathcal{P}$, the lattice algebra $\langle \mathcal{L}, \cap, + \rangle$ of all linear subspaces of the projective space, under intersection and + is a modular geometric lattice. That is, it is a complete lattice, satisfying these conditions:

Modularity $a \geq c \Rightarrow (\forall b)\big(a \cap (b+c) \leq (a \cap b) + c\big)$

Geometricity Every lattice element is a join of atoms, and if a is an atom and X is a set where $a \leq \Sigma X$ then there's some finite $Y \subseteq X$, where $a \leq \Sigma Y$.

The lattice of linear subspaces of a projective space satisfies these conditions, and that in fact, any modular geometric lattice is isomorphic to the lattice of linear subspaces of some projective space. Furthermore the lattice of positive idempotents of any **KR** frame is also a modular geometric lattice.

The undecidable problem which Urquhart uses to prove the undecidability of **KR** is now simple to state. Hutchinson [1973] and Lipshitz [1974] proved that

> The word problem for any class of modular lattices which includes the subspace lattice of an infinite dimensional projective space is undecidable.

Given an infinite dimensional projective space in which every line includes at least four points $\mathcal{P}$, the logic of the frame $(\mathcal{P})$ is said to be a *strong* logic. Our undecidability theorem then goes like this:

> Any logic between **KR** and a strong logic is undecidable.

The proof is not too difficult. Consider a modular lattice problem

$$\text{If } v_1 = w_1 \ \ldots \ v_n = w_n \text{ then } v = w$$

stated in a language with variables x_i $(i = 1, 2, \ldots)$ constants 1 and 0, and the lattice connectives $\cap$ and $+$. Fix a map into the language of **KR** by setting $x_i^t = p_i$ for variables, $0^t = t$, $1^t = \top$, $(v \cap w)^t = v^t \wedge w^t$ and $(v + w)^t = v^t \circ w^t$. The translation of our modular lattice problem is then the **KR** sentence

$$\big(B \wedge (v_1^t \leftrightarrow w_1^t) \wedge \cdots \wedge (v_n^t \leftrightarrow w_n^t) \wedge t\big) \rightarrow (v^t \leftrightarrow w^t)$$

where the sentence B is the conjunction of all sentences $p_i \wedge (p_i \leftrightarrow p_i \circ p_i)$ for each p_i appearing in the formulae v_j^t or w_j^t.

We will show that given a particular infinite dimensional projective space (with every line containing at least four points) $\mathcal{P}$, then the word problem is valid in the lattice of linear subspaces of $\mathcal{P}$ if and only if its translation is provable in L, for any logic L intermediate between **KR** and the logic of the frame $\mathcal{F}(\mathcal{P})$.

If the translation of the word problem is valid in L, then it holds in the frame $\mathcal{F}(\mathcal{P})$. Consider the word problem. If it were invalid, then there would be linear subspaces $x_1, x_2, \ldots$ in the space $\mathcal{P}$ such that each $v_i = w_i$ would be true while $v \neq w$. Construct a model on the frame $\mathcal{F}(\mathcal{P})$ as follows. Let the extension of p_i be the space x_i together with the point 0. It is then simple to show that $0 \vDash B$, as each p_i is a positive idempotent. In addition, $0 \vDash t$, and $0 \vDash v_i^t \leftrightarrow w_i^t$, for the extension of each v_i^t and w_i^t will be the spaces picked out by v_i and w_i (both with the obligatory 0 added). However, we would have $0 \nvDash v^t \leftrightarrow w^t$, since the extensions of v^t and w^t were picked out to differ. This would amount to a counterexample to the translation of the word problem, which we said was valid. As a result, the word problem is valid in the space $\mathcal{P}$. The converse reasoning is similar.

Unfortunately, these techniques do not work for systems weaker than **KR**. The proof that positive idempotents are modular uses essentially the special properties of **KR**. Not every positive idempotent in **R** need be modular. But nonetheless, the techniques of the proof can be extended to apply to a much wider range of systems. Urquhart examined the structure of the of the modular lattice undecidability result, and he showed that you could make do with much less. You do not need to restrict your attention to modular lattices to construct an undecidable word problem. But to do that, you need to examine Lipshitz and Hutchinson's proof more carefully. In the rest of this section, we will sketch the structure of Urquhart's undecidability proof. The techniques are quite involved, so we do not have the space to go into detail. For that, the reader is referred to Urquhart [1984].

Lipshitz and Hutchinson proved that the word problem for modular lattices was undecidable by embedding into that problem the already known undecidable word problem for semigroups. It is enough to show that a structure can define a "free associative binary operation", for then you will have the tools for representing arbitrary semigroup problems. (A semigroup is a set with an associative binary operation. An operation is a "free associative" operation if it satisfies those conditions satisfied by any associative operation but no more.) We will sketch how this can be done without resorting to a modular lattice.

The required structure is what is called a 0-structure, and a modular 4-frame defined within a 0-structure. An 0-structure is a set equipped with the following structure

- A semilattice with respect to $\sqcap$.
- With a binary operator $+$ which is associative and commutative.
- And $x \leq y \;\Rightarrow\; x + z \leq y + z$.
- $0 + x = x$.
- $y \geq 0 \;\Rightarrow\; x \sqcap (x + y) = x$.

A 4-frame in a 0-structure is a set $\{a_1, a_2, a_3, a_4\} \cup \{c_{ij} : i \neq j,\, i, j = 1, \ldots, 4\}$ such that

- The a_is are *independent.* If $G, H \subseteq \{a_1, \ldots, a_4\}$ then $(\Sigma G) \sqcap (\Sigma H) = \Sigma(G \cap H)$ (where $\Sigma\emptyset = 0$)
- If $G \subseteq \{a_1, \ldots, a_4\}$ then ΣG is modular
- $a_i + a_i = a_i$
- $c_{ij} = c_{ji}$
- $a_i + a_j = a_i + c_{ik}$; $c_{ij} \sqcap a_j = 0$, if $i \neq j$
- $(a_i + a_k) \sqcap (c_{ij} + c_{jk}) = c_{ik}$ for distinct i, j, k

Now, we are nearly at the point where we can define a semigroup structure. First, for each distinct i, j, we define the set L_{ij} to be $\{x : x + a_j = a_i + a_j \text{ and } x \sqcap a_j = 0\}$. Then if $b \in L_{ij}$ and $d \in L_{jk}$ where i, j, k are distinct, then we set $b \otimes d = (b + d) \sqcap (a_i + a_k)$, and it is not difficult to show that $b \otimes d \in L_{ik}$. Then, we can define a semigroup operation '.' on L_{12} by setting

$$x.y = (x \otimes c_{23}) \otimes (c_{31} \otimes y)$$

Now it is quite an involved operation to show that this is in fact an associative operation, but it can be done. And in fact, in certain circumstances, the operation is a free associative operation. Given a countably infinite-dimensional vector space $\mathcal{V}$, its lattice of subspaces is a 0-structure, and it is possible to define a modular 4-frame in this lattice of subspaces, such that any countable semigroup is isomorphic to a subsemigroup of L_{12} under the defined associative operation. (Urquhart gives the complete proof of this result [Urquhart, 1984].)

The rest of the work of the undecidability proof involves showing that this construction can be modelled in a logic. Perhaps surprisingly, it can all be done in a weak logic like $\mathbf{TW}[\wedge, \vee, \rightarrow, \top, \bot]$. We can do without negation by picking out a distinguished propositional atom f, and by defining $-A$ to be $A \rightarrow f$, t to be $-f$, and $A\,{:}\,B$ to be $-(A \rightarrow -B)$. A is a *regular* proposition iff $--A \leftrightarrow A$ is provable. The regular propositions form an 0-structure, under the assumption of the formula $\Theta = \{R(t,f,\top,\bot), N(t,f,\top,\bot), -\top \leftrightarrow \bot\}$. where $R(A)$ is $- - A \leftrightarrow A$, $N(A)$ is $(t \rightarrow A) \rightarrow A$, and $R(A,B,\ldots)$ is $R(A) \wedge R(B) \wedge \cdots$ and similarly for N. In other words, we can show that the conditions for an 0-structure hold in the regular propositions, assuming Θ as an extra premise. To interpret the 0-structure conditions we interpret $\sqcap$ by $\wedge$, $+$ by $:$ and 0 by t.

Now we need to model a 4-frame in the 0-structure. This can be done as we get just the modularity we need from another condition which is simple to state. Define $K(A)$ to be $R(A) \wedge (A \wedge -A \leftrightarrow \bot) \wedge (A \vee -A \leftrightarrow \top) \wedge (A{:} - A \leftrightarrow -A) \wedge (A \leftrightarrow A\,{:}\,A)$. Then we can show the following

$$K(A), R(B,C), C \rightarrow A \;\vdash\; A \wedge (B\,{:}\,C) \leftrightarrow (A \wedge B)\,{:}\,C$$

In other words, if $K(A)$, then A is modular in the class of regular propositions. Then the conditions for a 4-frame are simple to state. We pick out our atomic propositions $A_1, \ldots, A_4$ and $C_{12}, \ldots, C_{34}$ which will do duty for $a_1, \ldots, a_4$ and $c_{12}, \ldots, c_{34}$. Then, for example, one independence axiom is

$$(A_1\,{:}\,A_2\,{:}\,A_3) \wedge (A_2\,{:}\,A_3\,{:}\,A_4) \leftrightarrow (A_2\,{:}\,A_3)$$

and one modularity condition is

$$K(A_1\,{:}\,A_3\,{:}\,A_4)$$

We will let Π be the conjunction of the statements that express that the propositions A_i and C_{ij} form a 4-frame in the 0-structure of regular propositions. So, $\Theta \cup \Pi$ is a finite (but complex) set of propositions. In any algebra in which $\Theta \cup \Pi$ is true, the lattice of regular propositions is a 0-structure, and the denotations of the propositions A_i and C_{ij} form a 4-frame. Finally, when coding up a semigroup problem with variables $x_1, x_2, \ldots, x_m$, we will need formulae in the language which do duty for these variables. Thus we need a condition which picks out the fact that p_i (standing for x_i) is in L_{12}. We define $L(p)$ to be $(p\,{:}\,A_2 \leftrightarrow A_1\,{:}\,A_2) \wedge (p \wedge A_2 \leftrightarrow t)$. Then the semigroup operation on elements of L_{12} can be defined in terms of $\wedge$ and $:$ and the formulae A_i and C_{ij}. We assume that done, and we will simply take it that there is an operation $\cdot$ on formulae which picks out the algebraic operation on L_{12}. This is enough for us to sketch the undecidability argument.

The deducibility problem for any logic between $\mathbf{TW}[\wedge, \vee, \rightarrow, \top, \bot]$ and $\mathbf{KR}$ is undecidable.

Take a semigroup problem which is known to be undecidable. It may be presented in the following way

$$\text{If } v_1 = w_1 \ \ldots \ v_n = w_n \text{ then } v = w$$

where each term v_i, w_i is a term in the language of semigroups, constructed out of the variables $x_1, x_2, \ldots, x_m$ for some m. The translation of that problem into the language of $\mathbf{TW}[\wedge, \vee, \rightarrow, \top, \bot]$ is the deducibility problem

$$\Theta, \Pi, L(p_1, \ldots, p_m), v_1^t \leftrightarrow w_1^t, \ldots, v_n^t \leftrightarrow w_n^t \vdash v^t \leftrightarrow w^t$$

where each the translation u^t of each term u is defined recursively by setting x_i^t to be p_i, and $(u_1.u_2)^t$ to be $u_1^t \cdot u_2^t$.

Now the undecidability result will be immediate once we show that for any logic between $\mathbf{TW}$ and $\mathbf{KR}$ the word problem in semigroups is valid if and only if its translation is valid in that logic.

For left to right, if the word problem is valid in the theory of semigroups, its translation must be valid, for given the truth of Θ and Π and $L(p_1, \ldots, p_m)$, the operator $\cdot$ is provably a semigroup operation on the propositions in L_{12} in the algebra of the logic, and the terms v_i and w_i satisfy the semigroup conditions. As a result, we must have v^t and w^t picking out the same propositions, hence we have a proof of $v^t \leftrightarrow w^t$.

Conversely, if the word problem is invalid, then it has an interpretation in the semigroup $\mathcal{S}$ defined on L_{12} in the lattice of subspaces of an infinite dimensional vector space. The lattice of subspaces of this vector space is the 0-structure in our countermodel. However, we need a countermodel for our — the 0-structure is not a model of the whole of the logic, since it just models the regular propositions. How can we construct this? Consider the argument for $\mathbf{KR}$. There, the subspaces were the positive idempotents in the frame. The other propositions in the frame were *arbitrary* subsets of points. Something similar can work here. On the vector space, consider the subsets of points which are closed under multiplication (that is, if $x \in \alpha$, so is kx, where k is taken from the field of the vector space). This is a De Morgan algebra, defining conjunction and disjunction by means of intersection and union as is usual. Negation is modelled by set difference. The fusion $\alpha \circ \beta$ of two sets of points is the set $\{x + y : x \in \alpha \text{ and } y \in \beta\}$. It is not too difficult to show that this is commutative and associative, and square increasing, when the vector space is in a field of characteristic other than 2, since if $x \in \alpha$ then $x = \frac{1}{2}x + \frac{1}{2}x \in \alpha \circ \alpha$. Then $\alpha \rightarrow \beta$ is simply $-(\alpha \circ -\beta)$. It is not too difficult to show that this is an algebraic model for $\mathbf{KR}$, and that the regular propositions in this model are exactly the subspaces of the vector space. It follows that our counterexample in the 0-structure is a counterexample in a model of **KR** to the translation of the word problem. As a result, the translation is not provable in **KR** or in any weaker logic.

This result applies to systems between **TW** and **KR**, and it shows that the deducibility problem is undecidable for any of these systems. In the presence of the *modus ponens* axiom $A \wedge (A \to B) \wedge t \to B$, this immediately yields the undecidability of the *theoremhood* problem, as the deducibility problem can be rewritten as a single formula.

$$\big(\Theta \wedge \Pi \wedge L(p_1, \ldots, p_m) \wedge (v_1^t \leftrightarrow w_1^t) \wedge \cdots \wedge (v_n^t \leftrightarrow w_n^t) \wedge t\big) \to (v^t \leftrightarrow w^t)$$

As a result, the theoremhood problem for logics between **T** and **KR** is undecidable. In particular, **R**, **E** and **T** are all undecidable.

The restriction to **TW** is necessary in the theorem. Without the prefixing and suffixing axioms, you cannot show that the lattice of regular propositions is closed under the 'fusion-like' connective ' : '.

Before moving on to our next section, let us mention that these geometric methods have been useful not only in proving the undecidability of logics, but also in showing that interpolation fails in **R** and related logics [Urquhart, 1993].

5 LOOKING ABOUT

A lot of the work in relevance logics taking place in the late 1980's and in the 1990's has not focussed on Anderson's core problems. Now that these have been more or less resolved, work has proceeded apace in other directions. In this section we will give an undeniably indiosyncratic and personal overview of what we think are some of the strategic directions of this recent research. The first two sections in this part deal with generalisations — first of semantics, and second of proof theory — which situate relevance logic into a wider setting. The next sections deal with neighbouring formal theories, and we end with one philosophical application of the machinery of relevance logics.

5.1 Gaggle Theory

The fusion connective $\circ$ has played an important part in the study of relevance logics. This is because fusion and implication are tied together by the *residuation condition*

$$a \leq b \to c \text{ iff } a \circ b \leq c$$

In addition, in the frame semantics, fusion and implication are tied to the same ternary relation R, implication with the universal condition and fusion with the existential condition.

This is an instance of a generalised Galois connection. Galois studied connections between functions on partially ordered sets. A Galois connection between two partial orders $\leq$ on A and $\leq'$ on B is a pair of functions $f : A \to B$ and $g : B \to A$ such that

$$b \leq' f(a) \text{ iff } a \leq g(b)$$

The condition tying together fusion and implication is akin to that tying together f and g for Galois. So, gaggle theory (for 'ggl': *generalised Galois logic*) studies these connections in their generality, and it turns out that relevance logics like **R**, **E** and **T** are a part of a general structure which not only includes other relevance logics, but also traditional modal logics, Jónsson and Tarski's Boolean algebras with operators [Jónsson and Tarski, 1951] and many other formal systems. Dunn has shown that if a logic has a family of n-ary connectives which are tied together with a generalised galois connection, then the logic has a frame semantics in which those connectives are modelled using the one $n+1$-ary relation, in the way that fusion and implication are modelled by the same ternary relation in relevance logics [Dunn, 1991, Dunn, 1993a, Dunn, 1994].

In general, an n-ary connective f has a *trace* $(\tau_1, \ldots, \tau_n) \mapsto +$ if

- $f(c_1, \ldots, \mathbf{1}, \ldots, c_n) = \mathbf{1}$, if $\tau_i = +$ (where the $\mathbf{1}$ is in position i).
- $f(c_1, \ldots, \mathbf{0}, \ldots, c_n) = \mathbf{1}$, if $\tau_i = -$ (where the $\mathbf{0}$ is in position i).
- If $a \leq b$, and if $\tau_i = +$ then $f(c_1, \ldots, a, \ldots, c_n) \leq f(c_1, \ldots, b, \ldots, c_n)$.
- If $a \leq b$, and if $\tau_i = -$ then $f(c_1, \ldots, b, \ldots, c_n) \leq f(c_1, \ldots, a, \ldots, c_n)$.

We write this as $T(f) = (\tau_1, \ldots, \tau_n) \mapsto +$. On the other hand, the connective f has *trace* $(\tau_1, \ldots, \tau_n) \mapsto -$ if

- $f(c_1, \ldots, \mathbf{1}, \ldots, c_n) = \mathbf{0}$, if $\tau_i = +$ (where the $\mathbf{0}$ is in position i).
- $f(c_1, \ldots, \mathbf{0}, \ldots, c_n) = \mathbf{0}$, if $\tau_i = -$ (where the $\mathbf{1}$ is in position i).
- If $a \leq b$, and if $\tau_i = +$ then $f(c_1, \ldots, b, \ldots, c_n) \leq f(c_1, \ldots, a, \ldots, c_n)$.
- If $a \leq b$, and if $\tau_i = -$ then $f(c_1, \ldots, a, \ldots, c_n) \leq f(c_1, \ldots, b, \ldots, c_n)$.

We write this as $T(c) = (\tau_1, \ldots, \tau_n) \mapsto -$. Here are a few examples of traces of connectives. Conjunction-like connectives tend to be $(-, -) \mapsto -$, disjunction-like connectives tend to be $(+, +) \mapsto +$, necessity-like connectives tend to be $+ \mapsto +$, possibility-like connectives tend to be $- \mapsto -$, and negations can be either $+ \mapsto -$ or $- \mapsto +$ (and in many cases they are both).

Now we are nearly able to state the abstract law of residuation. First, we define $S(f, a_1, \ldots, a_n, b)$ as follows. If $T(f) = (\cdots) \mapsto +$, then $S(f, a_1, \ldots, a_n, b)$ is the condition $f(a_1, \ldots, a_n) \leq b$. If, on the other hand, $T(f) = (\cdots) \mapsto -$, then $S(f, a_1, \ldots, a_n, b)$ is $b \leq f(a_1, \ldots, a_n)$. Then, two connectives f and g are *contrapositives in place* j iff, if $T(f) = (\tau_1, \ldots, \tau_j, \ldots, \tau_n) \mapsto \tau$, then $T(g) = (\tau_1, \ldots, -\tau, \ldots, \tau_n) \mapsto -\tau_j$. (Where we define $-+$ as $-$ and $--$ as $+$.) Two operators f and g satisfy the *abstract law of residuation* iff f and g are contrapositives in place j, and $S(f, a_1, \ldots, a_j, \ldots, a_n, b)$ iff $S(g, a_1, \ldots, b, \ldots, a_n, a_j)$.

A collection of connectives in which there is some connective f such that every element of the collection satisfies the abstract law of residuation with f, is called a *founded family* of connectives. Dunn's major result is that if you have an algebra in which every connective is in a founded family, then the algebra is isomorphic to a subalgebra of the collection of propositions in a model in which each founded family of n-ary connectives shares an $n+1$-ary relation. The soundness and completeness of the Routley–Meyer ternary relational semantics is for the implication-fusion fragment of relevance logics is an instance of this more general result.

The gaggle theoretic account of negation in relevance logics is interesting. We do not automatically get negation modelled by the Routley star — instead, being a unary connective, negation is modelled with a *binary* relation. One way negation can be modelled along gaggle theoretic lines is as follows. The De Morgan negation connective has trace $- \mapsto +$, so the gaggle theoretic result is that there is a binary relation C between set-ups such that

- $x \vDash \neg A$ iff for each y where xCy, $y \not\vDash A$

This is the general semantic structure which models negation connectives with trace $- \mapsto +$. Given a relation C, which we may read as 'compatibility', we can define another negation connective $\sim$, using C's converse:

- $x \vDash {\sim} A$ iff for each y where yCx, $y \not\vDash A$

Then it follows that $A \vdash {\sim} B$ iff $B \vdash \neg A$. For the De Morgan negation of relevance logics, $\sim$ and $\neg$ are the same, for the compatibility relation C is symmetric. But in more general settings, this need not hold.

The general perspective of gaggle theory not only opens up new formal systems to study — it also helps with interpreting the semantics. The condition for $\neg$ above can be read as follows: $\neg A$ is true at x iff for each y *compatible with* x, A is not true at y. This certainly sounds like a more palatable condition for negation than that using Routley star. We have an understanding of what it is for two set-ups (theories, worlds or situations) to be compatible, and the notion of compatibility is tied naturally to that of negation. Furthermore, the Routley star condition is an instance of this more general 'compatibility' condition. For any set-up a, a^* can be seen as the set-up which 'wraps up' all set-ups compatible with a. We can argue whether there is such an all-encompassing set-up, but if there is, then the semantics for negation in terms of the compatibility relation is equivalent to that of the Routley star. And in addition, we have another means of explaining it.

Furthermore, once we have this generalised position from which to view negation, we can tinker with the binary accessibility relation in just the same way that modal logics are studied. Clearly if Boolean negation (written '$-$') is present, then $\neg A$ is simply $\Box - A$ for the positive modal operator $\Box$ which uses C as its accessibility relation; and the study of these negation is dealt with using the techniques of modal logic. However, in relevance logics and other related systems, boolean negation is not present. And in this case the theory of negations arising from compatibility clauses like the one we have seen is a young and interesting subject in its own right. This perspective is pursued in Dunn [1994], and Restall [1999] develops a philosophical interpretation of the semantics.

5.2 Display Logic

Nuel Belnap has developed proof theoretical techniques which are quite similar to those from gaggle theory. Consider the general problem of providing a sequent calculus for logics like **R** and others. We have the choice of how to formulae sequents. If they are of the form $X \vdash A$, where X is a structured collection of formulae, and A is a formula, then we have the problem of how to state the introduction and elimination of negation rules in such a way as to make $\neg\neg A$ equivalent to A. It is unclear how to do this while maintaining that the succedent of every sequent is a single formula. On the other hand, if we allow that sequents are of the form $X \vdash Y$, where now both X and Y are structured complexes of formulae, it is unclear how to state a cut rule which is both valid and admits of a cut-elimination proof in the style of Gentzen. If we are restricted to single formulae in the succedent position the rule is easy to state:

$$\frac{X \vdash A \quad Y(A) \vdash B}{Y(X) \vdash B}$$

but in the presence of multiple succedents it is unclear how to state the rule generally enough to be eliminable yet strictly enough to be valid under interpretation. If there is only one sort of structuring in the consequent this might be possible, in the way used in the proof theories of classical or linear logic, for example:

$$\frac{X \vdash A, Y \quad X', A \vdash Y'}{X, X' \vdash Y, Y'}$$

But if we have $X \vdash Y(A)$ and $X'(A) \vdash Y'$ where the indicated instances of A are buried under multiple sorts of structure, then what is the appropriate conclusion of a cut rule? $X'(X) \vdash Y(Y')$ will not do in general, for it is invalid in many instances. For example, in **R** if $Y(A)$ is $A \wedge B$ and $X'(A)$ is $A \circ D$, then we have $A \wedge B \vdash A \wedge B$ and $A \circ D \vdash A \circ D$, but we don't have $(A \wedge B) \circ D \vdash (A \circ D) \wedge B$ in general. (Consider the case where $B = A$. $A \circ D$ needn't imply A.)

The alternative examined by Belnap is to make do with Cut where the cut formula is "displayed" in both premises of the rule.

$$\frac{X \vdash A \quad A \vdash Y}{X \vdash Y}$$

In order to get away with this, a system needs to be such that whenever you need to use a cut you can. The way Belnap does this is by requiring what he calls the "display condition". The display condition is satisfied iff for every formula, every sequent including that formula is equivalent (using invertible rules) to one in which that formula is either the entire antecedent or the entire succedent of the sequent. For Belnap's original formulation, this is achieved by having a binary structuring connective $\circ$ (not to be confused with the sentential connective $\circ$) and a unary connective $*$. The display rules were as follows:

$$X \circ Y \vdash Z \Longleftrightarrow X \vdash *Y \circ Z$$
$$X \vdash Y \circ Z \Longleftrightarrow X \circ *Y \vdash Z \Longleftrightarrow X \vdash Z \circ Y$$
$$X \vdash Y \Longleftrightarrow *Y \vdash *X \Longleftrightarrow **X \vdash Y$$

A structure is in antecedent position if it is in the left under an even number of stars, or in the right under an odd number of stars. If it is not in antecedent position, it is in succedent position. The star is read as negation, and the circle is read as conjunction in antecedent position, and disjunction in succedent position. The display postulates are a reworking of conditions like the residuation condition for fusion and implication. Here we have the conditions that $a \circ b \leq c$ iff $a \leq {\sim}b + c$ (where $x + y$ is the *fission* of x and y).

Belnap's system allows that different families of structural connectives can be used for different families of connectives in the language. For example, when $\circ$ and $*$ are read intensionally, we can have the following rules for implication:

$$\frac{X \circ A \vdash B}{X \vdash A \to B} \qquad \frac{X \vdash A \quad B \vdash Y}{A \to B \vdash *X \circ Y}$$

If the properties of $\circ$ vary, so do the properties of the connective $\to$. We can give $\circ$ properties of extensional conjunction in order to get a material conditional. Or conditions can be tightened, to give $\to$ modal properties. It is clear that the family of structural connectives (here $\circ$ and $*$) act in analogously to accessibility relations on frames. However, the connections with gaggle theory run deeper, however. It can be shown a connective introduced in with rules without side conditions, and in a way which 'mimics' structural connectives (just as here $A \to B$ mimics $*X \circ Y$ in consequent position) must have a definable trace. Any implication satisfying those rules will have trace $(-,+) \mapsto +$, for example. For more details of this connection and a general argument, see Restall's paper [Restall, 1995a].

Display logic gives these systems a natural cut-free proof theory, for Belnap has shown that under a broad set of conditions, any proof theory with this structure will satisfy cut-elimination. So again, just as with gaggle theory, we have an example of the way that the study of relevance logics like **R** and **E** have opened up into a more general theory of logics with similar structures.

5.3 Paraconsistency

Relevance logics are paraconsistent, in that argument forms such as $A \wedge \neg A \vdash B$ are taken to be invalid. As a result, relevance logics have been seen to be important for the study of paraconsistent theories. [[See Priest's article in this volume]]. Relevance logics are suited to applications for which a paraconsistent notion of consequence is needed however, not all logics are equal in this regard. For example, paraconsistentists have often considered the topic of naïve theories of sets and of truth (any predicate yields the set of things satisfying that predicate, the proposition p is true if and only if p). With a relevance logic at hand, you can avoid the inference to triviality from contradictions such as that arising from the liar

> This proposition is not true.

(from which you can deduce that it is true, and hence that it isn't) and Russell's paradox ($\{x : x \notin x\}$ both is and is not a member of itself). However, the *Curried* forms of these paradoxes

> If this proposition is true then there is a Santa Claus.

and $\{x : (x \in x) \to P\}$ are more difficult to deal with. These yield arguments for the existence of Santa Claus and the truth of P (which was arbitrary) in logics like **R**, or any others with theorems related to the rule of contraction. The theoremhood of propositions such as $\big(A \to (A \to B)\big) \to (A \to B)$ and $A \wedge (A \to B) \to B$ rule out a logic for service in the cause of paraconsistent theories like these [Meyer *et al.*, 1979].

However, this has not deterred some hardier souls in considering weaker relevance logics which do not allow one to deduce triviality in these theories. Some work has been done to show that in some logics these theories are consistent, and in others, though inconsistent, not everything is a theorem [Brady, 1989].

Another direction of paraconsistency in which techniques of relevance logics have borne fruit is in the more computational area of reasoning with inconsistent information. The techniques of first degree entailment have found a home in the study of "bilattices" by Melvin Fitting and others, who seen in them a suitable framework for reasoning under the possibility of inconsistent information [Fitting, 1989].

5.4 Semantic Neighbours

Another area in which research has grown in the recent years has been toward connections with other fields. It has turned out that seemingly completely unrelated fields have studied structures remarkably like those studied in relevance logics. These neighbours are helpful, not only for giving independent evidence for the fact that relevance logicians have been studying *something* worthwhile, but also because of the different insights they can bring to bear on theorising. In this section we will see just three of the neighbours which can shed light on work in relevance logics.

The first connection comes with Barwise and Perry's situation semantics [1983]. For Barwise and Perry, utterances classify situations (parts of the world) which may be incomplete with regard to their semantic 'content'. Consider the claim that Max saw Queensland win the Sheffield Shield". How is this to be understood? For the Barwise and Perryof *Situations and Attitudes* [Barwise and Perry, 1983], this was to be parsed as expressing a relationship between Max and a *situation*, where a situation is simply a restricted part of the world. Situations are parts of the world and they support information. Max saw a situation and in this situation, Queensland won the Sheffield Shield. If, in this very situation, Queensland beat South Australia, then Max saw Queensland beat South Australia.

This shows why for this account situations have to be (in general) *restricted* bits of the world. The situation Max saw had better not be one in which Paul Keating lost the 1996 Federal Election, lest it follow from the fact that Max witnessed Queensland's victory that he also witnessed Keating's defeat, and surely *that* would be an untoward conclusion. Let's denote this relationship between situations an the information they support as follows. We'll abbreviate the claim that the situation s supports the information that A by writing '$s \vDash A$', and we'll write its negation, that s doesn't support the information that A by writing '$s \nvDash A$'. This is standard in the situation theoretic literature. The information carried by these situations has, according to Barwise and Perry, a kind of logical coherence. For them, infons are closed under conjunction and disjunction, and $s \vDash A \wedge B$ if and only if $s \vDash A$ and $s \vDash B$, and $s \vDash A \vee B$ if and only if $s \vDash A$ or $s \vDash B$. However, negation is a different story — clearly situations don't support the traditional equivalence between $s \vDash \neg A$ and $s \nvDash A$ (where $\neg A$ is the negation of A), for our situation witnessed by Max supports neither the infon "Keating won the 1996 election" nor its negation.

What to do? Well, Barwise and Perry suggest that negation interacts with conjunction and disjunction in the familiar ways — $\neg(A \vee B)$ is (equivalent to) $\neg A \wedge \neg B$, and $\neg(A \wedge B)$ is (equivalent to) $\neg A \vee \neg B$. And similarly, $\neg\neg A$ is (equivalent to) A. This gives us a logic of sorts of negation — it is *first degree entailment*. Now for Barwise and Perry, there are no *actual* situations in which $s \vDash A \wedge \neg A$ (the world is not self-contradictory). However, they agree that it is helpful to consider *abstract* situations which allow this sort of inconsistency. So, Barwise and Perry have an independent motivation for a semantic account of first-degree entailment. (More work has gone on to consider other connections between situation theory and relevance logics [Mares, 1997, Restall, 1994, Restall, 1995b].)

Another connection with a parallel field has come from completely different areas of research. The semantic structures of relevance logics have close cousins in the models for the Lambek Calculus and in Relation algebras. Let's consider relation algebras first.

A relation algebra is a Boolean algebra with some extra operations, a binary operation which denotes composition of relation, a unary operation $^{\smile}$,for the converse of a relation, and a constant 1 for the identity relation. There is a widely accepted axiomatisation of the variety **RA** of relation algebras. A relation algebra is set R with operations $\wedge, \vee, -, 1, \circ, ^{\smile}$ such that

- $\langle R, \wedge, \vee, - \rangle$ is a boolean algebra.
- $^{\smile}$ is an automorphism on the algebra, satisfying $a^{\smile\smile} = a$, $(a \wedge b)^{\smile} = a^{\smile} \wedge b^{\smile}$, $-(a^{\smile}) = (-a)^{\smile}$.
- $\circ$ is associative, with a left and right identity 1, satisfying $(a \vee b) \circ c = (a \circ c) \vee (b \circ c)$, $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$.
- $^{\smile}$ and $\circ$ are connected by setting $(a \circ b)^{\smile} = b^{\smile} \circ a^{\smile}$.

These conditions are satisfied by the class of relations on any base set (that is, by any *concrete* relation algebra). However, not every algebra satisfying these equations is isomorphic to a subalgebra of a concrete relation algebra.

These algebras are quite similar to de Morgan monoids. If we define $\neg A$ to be $-(a)^{\smile}$ or $(-a)^{\smile}$ then the conjunction, disjunction, $\neg$, 1 fragment is that of first degree entailment. We do not have $a \leq b \vee \neg b$, and nor do we have $a \wedge \neg a \leq b$. Consider the relation a:

a	x	y
x	1	1
y	0	1

Then $\neg a$ is the following relation

a	x	y
x	0	1
y	0	0

So we don't have $b \leq a \vee \neg a$ for every b, and nor do we have $a \wedge \neg a \vee b$. (However, we do have $1 \leq a \vee \neg a$.)

The class of relation algebras have a natural form of implication to go along with the fusionlike connective $\circ$. If we define $a \to b$ to be $\neg(\neg b \circ a)$, then we have the residuation condition $a \circ b \leq c$ iff $a \leq b \to c$. However, that is not the only implication-like connective we may define. If we set $b \leftarrow a$ to be $\neg(a \circ \neg b)$, then $a \circ b \leq c$ iff $b \leq c \leftarrow a$. Since $\circ$ is not, in general, commutative, we have two residuals.

In logics like **R** this is not possible, for the left and the right residuals of fusion are the same connective. However, in systems in the vicinity of **E**, these implication operations come apart. This is mirrored by the behaviour on frames, since we can define $B \leftarrow A$ by setting $x \vDash B \leftarrow A$ iff for each y, z where $Ryxz$ if $y \vDash A$ then $z \vDash B$. This will be another residual for fusion, and it will not agree with $\to$ in the absence of commutativity of R (if $Rxyz$ then $Ryxz$).[49]

It was hoped for some time that relation algebras would give an interesting model for logics like **R**. However, there does not seem to be a natural class of relations for which composition is commutative and square increasing. (The class of symmetric relations will not do. Even if $a = a^{\smile}$ and $b = b^{\smile}$, it does not follow that $a \circ b = b \circ a$. You merely get that $a \circ b = a^{\smile} \circ b^{\smile} = (b \circ a)^{\smile}$.) Considered as a logic, **RA** is a sublogic of **R** (ignoring boolean negation for the moment). It is not a sublogic of **E**, since in **RA**, $a = 1 \to a$. Another difference between **RA** and typical relevance logics is the behaviour of contraposition. We do not have $a \to b = \neg b \to \neg a$. Instead, $a \to b = \neg a \leftarrow \neg b$.

A final connection between **RA** and relevance logics is in the issue of semantics. As we stated earlier, not all relation algebras are representable as subalgebras of concrete relation algebras. However, Dunn has shown

[49] We should flag here that in the relevance logic literature, [Meyer and Routley, 1972] seems to have been the first to consider both left- and right-residuals for fusion.

that all relation algebras *are* representable by algebras of propositions on a particular class of Routley–Meyer frames [Dunn, 1993b]. This is the first representation theorem for **RA**, and it shows that the semantical techniques of relevance logics have a wider scope than applications to **R**, **E** and their immediate neighbours.

In a similar vein, Dunn and Meyer [1997] have provided a Routley–Meyer style frame semantics for combinatory logic. The key idea here is that the ternary relation R satisfies *no* special conditions, but these properties are encoded by *combinators*, which are modelled by special propositions on frames.

Lambek's *categorial grammar* is also similar to relevance logics, though this time it is introduced with frames, not algebras [Lambek, 1958, Lambek, 1961]. Here, the points in frames are pieces of syntax, and the 'propositions' are syntactic classifications of various kinds. For example, the classifications into noun phrases, verbs, and sentences. The interest comes with the way in which these classifications can be combined. For example $A \circ B$ can be defined, where we say $x \vDash A \circ B$ iff x is a concatenation of two strings y and z, where $y \vDash A$ and $z \vDash B$. We can also define 'slicing' operations, setting $x \vDash A\backslash B$ iff for each y where $y \vDash A$, $yx \vDash B$; and $x \vDash B/A$ iff for each y where $y \vDash A$, $xy \vDash B$. These are obviously analogues for $\circ$ and $\to$ in relevance logics, and again, we have a 'left' and 'right' residuals for fusion. In these frames $Rxyz$ iff $xy = z$. So the Lambek calculus gives us an independently motivated interpretation of a class of Routley–Meyer frames. This connection has been explored by Kurtonina [1995], which is a helpful sourcebook of some recent work on ternary frames in connection with the Lambek calculus and related logics.

If you like, you can enrich the logic of strings with conjunction and disjunction, and if you do it in the obvious way (using the same clauses as in relevance logics) you get a formal logic quite like **RA** [Restall, 1994]. But more importantly, the conditions for conjunction and disjunction may be independently motivated. A string is of type $A \vee B$ just when it is of type A or of type B. A string is of type $A \wedge B$ just when it is of type A and of type B. The resulting logic is clearly interpretable, but it was a number of years before a proof theory was found for it. Here the techniques for the Gentzenisation for positive relevance logics are appropriate, and the proof theory can be found by utilising the proof theory for $\mathbf{R}^+$, and removing the commutativity and contraction of the intensional bunching operation. The resulting proof theory captures exactly the Lambek calculus enriched with conjunction and disjunction. In addition, the techniques of Giambrone show that the resulting logic is decidable [Restall, 1994].

5.5 Linear Logic

The burgeoning phenomenon of linear logic is one which has a number of formal similarities to relevance logics [Girard, 1987, Troelstra, 1992]. Linear logic is the study of systems in the vicinity of **LRW** (**R** without contraction, without distribution). This is proof-theoretically a very stable system. It is simple to show that it is decidable. Girard's innovation, however, is to extend the proof theory with a modal operator ! which allows intuitionistic logic to be modelled inside linear logic. This operation in given as follows, in single-succedent Gentzen systems.

$$\frac{X \vdash B}{X, !A \vdash B} \quad \frac{X; A \vdash B}{X; !A \vdash B} \quad \frac{!X \vdash B}{!X \vdash !B} \quad \frac{X, !A, !A \vdash B}{X, !A \vdash B}$$

Given this proof theory it is possible to show that $A \Rightarrow B$ defined as $!A \rightarrow B$ is an intuitionistic implication. This is similar to Meyer's result that $A \wedge t \rightarrow B$ is an intuitionistic implication in **R** (indeed, $!A$ defined as $A \wedge t$ satisfies each of the conditions for ! above in **R**, but not in systems without contraction). However, nothing like it holds in relevance logics without contraction.

Linear logic also brings with it many new algebraic structures and models in category theory. None of these models have been mined to see if they can bring any 'relevant' insight. However, some transfer has gone on in the other direction — Allwein and Dunn [1993] have shown that the multiplicative and additive fragment of linear logic can be given a Routley–Meyer style semantics. This is not a simple job, as the absence of the distribution of (additive) conjunction over disjunction means that at least one of these connectives (in this case, disjunction) must take a non-standard interpretation.

5.6 Relevant Predication

There has been one major way in which relevance logics have been used in application to philosophical issues, and this application makes a good topic to end this article. The topic is Dunn's work on relevant predication [Dunn, 1987].[50] The guiding idea is that a theory of relevant implication will give you some way of marking out the distinction between the way that Socrates' wisdom is a property of Socrates, in the way that Socrates' wisdom is not a property of Bill Clinton.

[50] The reference [Dunn, 1987] is of course "Relevant Predication": Of course all work has precursors, in this instance (largely unpublished) thoughts in the 1970's by N. Belnap, J. Freeman, and most importantly R. K. Meyer and A. Urquhart (and Dunn). Cf. Sec. 9 of [Dunn, 1987] for some history.

Classical first order logic is not good at marking out such a distinction, for if Wx stands for 'x is wise', and s stands for Socrates, and c stands for Bill Clinton, then Wx is true of x iff it is wise, and $(Ws \wedge x = x) \vee Ws$ is true of something iff Socrates is wise. Why is one a 'real' property and the other not? The guiding idea for relevant predication is the following distinction. It is true that if x is Socrates then x is wise. However, it is not true that if x is Bill Clinton then Socrates is wise. At least, it is plausible that this conditional fail, when read 'relevantly'. This can be cashed out formally as follows. F is a *relevant property of* a (written $(\rho x Fx)a$) if and only if $(\forall x)(x = a \to Fx)$.

Given this definition, if F is a relevant property of a then Fa holds (quite clearly) and if F and G are relevant properties of a then so is their conjunction, and the disjunction of any relevant property with anything at all is still a relevant property.

Furthermore, one can define what it is for a relation to truly be a relation between objects. If Hx is 'x's height is over 1 meter', and Ly is 'y is a logician' then, it is true that Greg's height is over 1 meter and Mike is a logician. However, it would be bizarre to hold that in this there is a real relation that holds between Greg and Mike because of this fact. We would have the following

$$\forall x \forall y(x = g \wedge y = m \to Hx \wedge Ly)$$

(assuming that $(\rho x Hx)g$ and $(\rho y Ly)m$) but it need not follow that

$$\forall x \forall y(x = g \to (y = m \to Hx \wedge Ly))$$

for there is no reason that Hx should follow from $y = m$, even given that $x = g$ holds. There is no connection between 'y's being m' and Hg. This latter proposition is a good candidate for expressing that there is a real relationship holding between g and m. In other words, we can define $(\rho xy Lxy)ab$ to be

$$\forall x \forall y\big(x = a \to (y = b \to Lxy)\big)$$

to express the holding of a relevant relation. For more on relevant predication, consult Dunn's series of papers [Dunn, 1987, Dunn, 1990a, Dunn, 1990b]

Relevance logics are very good at telling you what follows from what as a matter of logic — and in this case, the logical structure of relevant predication and relations. However, more work needs to be done to see in what it consists to say that a relevant implication is *true*. For that, we need a better grip on how to understand the models of relevance logics. It is our hope that this chapter will help people in this aim, ant to bring the technique of relevance logics to a still wider audience.

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REFERENCES

[Ackermann, 1956] Wilhelm Ackermann. Begründung einer strengen implikation. *Journal of Symbolic Logic*, 21:113–128, 1956.

[Akama, 1997] Seiki Akama. Relevant counterfactuals and paraconsistency. In *Proceedings of the First World Conference on Paraconsistency, Gent, Belgium.*, 1997.

[Allwein and Dunn, 1993] Gerard Allwein and J. Michael Dunn. A kripke semantics for linear logic. *Journal of Symbolic Logic*, 58(2):514–545, 1993.

[Anderson and Belnap Jr., 1959a] A. R. Anderson and N. D. Belnap Jr. Modalities in Ackermann's 'rigorous implication'. *Journal of Symbolic Logic*, 24:107–111, 1959.

[Anderson and Belnap Jr., 1959b] A. R. Anderson and N. D. Belnap Jr. A simple proof of Gödel's completeness theorem. *Journal of Symbolic Logic*, 24:320–321, 1959. (Abstract.).

[Anderson and Belnap Jr., 1961] A. R. Anderson and N. D. Belnap Jr. Enthymemes. *Journal of Philosophy*, 58:713–723, 1961.

[Anderson and Belnap Jr., 1962a] A. R. Anderson and N. D. Belnap Jr. The pure calculus of entailment. *Journal of Symbolic Logic*, 27:19–52, 1962.

[Anderson and Belnap Jr., 1962b] A. R. Anderson and N. D. Belnap Jr. Tautological entailments. *Philosophical Studies*, 13:9–24, 1962.

[Anderson and Belnap, 1975] Alan Ross Anderson and Nuel D. Belnap. *Entailment: The Logic of Relevance and Necessity*, volume 1. Princeton University Press, Princeton, 1975.

[Anderson *et al.*, 1992] Alan Ross Anderson, Nuel D. Belnap, and J. Michael Dunn. *Entailment: The Logic of Relevance and Necessity*, volume 2. Princeton University Press, Princeton, 1992.

[Anderson, 1960] A. R. Anderson. Entailment shorn of modality. *Journal of Symbolic Logic*, 25:388, 1960. (Abstract.).

[Anderson, 1963] A. R. Anderson. Some open problems concerning the system E of entailment. *Acta Philosophica Fennica*, 16:7–18, 1963.

[Avron, 1986] Arnon Avron. On purely relevant logics. *Notre Dame Journal of Formal Logic*, 27:180–194, 1986.

[Avron, 1990a] Arnon Avron. Relevance and paraconsistency — a new approach. *Journal of Symbolic Logic*, 55:707–732, 1990.

[Avron, 1990b] Arnon Avron. Relevance and paraconsistency — a new approach. part II: The formal systems. *Notre Dame Journal of Formal Logic*, 31:169–202, 1990.

[Avron, 1990c] Arnon Avron. Relevance and paraconsistency — a new approach. part III: Cut-free gentzen-type systems. *Notre Dame Journal of Formal Logic*, 32:147–160, 1990.

[Avron, 1991] Arnon Avron. Simple consequence relations. *Information and Computation*, 92:105–139, 1991.

[Avron, 1992] Arnon Avron. Whither relevance logic? *Journal of Philosophical Logic*, 21:243–281, 1992.

[Barcan Marcus, 1946] R. C. Barcan Marcus. The deduction theorem in a functional calculus of first-order based on strict implication. *Journal of Symbolic Logic*, 11:115–118, 1946.

[Barwise and Perry, 1983] Jon Barwise and John Perry. *Situations and Attitudes*. MIT Press, Bradford Books, 1983.

[Belnap and Dunn, 1981] Nuel D. Belnap and J. Michael Dunn. Entailment and the disjunctive syllogism. In F. Fløistad and G. H. von Wright, editors, *Philosophy of Language / Philosophical Logic*, pages 337–366. Martinus Nijhoff, The Hague, 1981. Reprinted as Section 80 in *Entailment* Volume 2, [Anderson *et al.*, 1992].

[Belnap and Spencer, 1966] Nuel D. Belnap and J. H. Spencer. Intensionally complemented distributive lattices. *Portugaliae Mathematica*, 25:99–104, 1966.

[Belnap Jr. and Wallace, 1961] N. D. Belnap Jr. and J. R. Wallace. A decision procedure for the system $E_{\bar{i}}$ of entailment with negation. Technical Report 11, Contract No. SAR/609 (16), Office of Naval Research, New Haven, 1961. Also published as [Belnap Jr. and Wallace, 1965].

[Belnap Jr. and Wallace, 1965] N. D. Belnap Jr. and J. R. Wallace. A decision procedure for the system $E_{\bar{i}}$ of entailment with negation. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 11:261–277, 1965.

[Belnap Jr. *et al.*, 1980] N. D. Belnap Jr., A. Gupta, and J. Michael Dunn. A consecution calculus for positive relevant implication with necessity. *Journal of Philosophical Logic*, 9:343–362, 1980.

[Belnap, 1960a] Nuel D. Belnap. EQ and the first-order functional calculus. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 6:217–218, 1960.

[Belnap, 1960b] Nuel D. Belnap. A formal analysis of entailment. Technical Report Contract No. SAR/Nonr. 609(16), Office of Naval Research, New Haven, 1960.

[Belnap, 1962] Nuel D. Belnap. Tonk, plonk and plink. *Analysis*, 22:130–134, 1962.

[Belnap, 1967a] Nuel D. Belnap. Intensional models for first degree formulas. *Journal of Symbolic Logic*, 32:1–22, 1967.

[Belnap, 1967b] Nuel D. Belnap. Special cases of the decision problem of relevant implication. *Journal of Symbolic Logic*, 32:431–432, 1967. (Abstract.).

[Belnap, 1977a] Nuel D. Belnap. How a computer should think. In G. Ryle, editor, *Contemporary Aspects of Philosophy*. Oriel Press, 1977.

[Belnap, 1977b] Nuel D. Belnap. A useful four-valued logic. In J. Michael Dunn and George Epstein, editors, *Modern Uses of Multiple-Valued Logics*, pages 8–37. Reidel, Dordrecht, 1977.

[Białynicki-Birula and Rasiowa, 1957] A. Białynicki-Birula and H. Rasiowa. On the representation of quasi-boolean algebras. *Bulletin de L'académie Polonaise des Sciences*, 5:259–261, 1957.

[Brady, 1989] Ross T. Brady. The non-triviality of dialectical set theory. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic: Essays on the Inconsistent*, pages 437–470. Philosophia Verlag, 1989.

[Brady, 1991] Ross T. Brady. Gentzenization and decidability of some contraction-less relevant logics. *Journal of Philosophical Logic*, 20:97–117, 1991.

[Brady, 1994] Ross T. Brady. Rules in relevant logic — I: Semantic classification. *Journal of Philosophical Logic*, 23:111–137, 1994.

[Brady, 1996] Ross T. Brady. Relevant implication and the case for a weaker logic. *Journal of Philosophical Logic*, 25:151–183, 1996.

[Burgess, 1981] J. P. Burgess. Relevance: A fallacy? *Notre Dame Journal of Formal Logic*, 22:97–104, 1981.

[Castañeda, 1975] H. N. Castañeda. *Thinking and Doing: The Philosophical Foundations of Institutions*. Reidel, Dordrecht, 1975.

[Charlewood, 1978] G. W. Charlewood. *Representations of Semilattice Relevance Logic*. PhD thesis, University of Toronto, 1978.

[Charlewood, 1981] G. W. Charlewood. An axiomatic version of positive semi-lattice relevance logic. *Journal of Symbolic Logic*, 46:233–239, 1981.

[Church, 1951a] Alonzo Church. The weak positive implication calculus. *Journal of Symbolic Logic*, 16:238, 1951. Abstract of "The Weak Theory of Implication" [Church, 1951b].

[Church, 1951b] Alonzo Church. The weak theory of implication. In A. Menne, A. Wilhelmy, and H. Angsil, editors, *Kontroliertes Denken: Untersuchungen zum Logikkalkül und zur Logik der Einzelwissenschaften*, pages 22–37. Kommissions-Verlag Karl Alber, Munich, 1951. Abstracted in "The Weak Positive Implication Calculus" [Church, 1951a].

[Clark, 1980] M. Clark. The equivalence of tautological and 'strict' entailment: proof of an amended conjecture of lewy's. *Journal of Symbolic Logic*, 9:9–15, 1980.

[Copeland, 1979] B. J. Copeland. On when a semantics is not a semantics: some reasons for disliking the Routley-Meyer semantics for relevance logic. *Journal of Philosophical Logic*, 8:399–413, 1979.

[Curry, 1950] Haskell B. Curry. *A Theory of Formal Deducibility*, volume 6 of *Notre Dame Mathematical Lectures*. Notre Dame University Press, 1950.

[D'Agostino and Gabbay, 1994] Marcello D'Agostino and Dov M. Gabbay. A generalization of analytic deduction via labelled deductive systems. part I: Basic substructural logics. *Journal of Automated Reasoning*, 13:243–281, 1994.

[Dipert, 1978] R. R. Dipert. *Development and Crisis in Late Boolean Logic; The Deductive Logics of Peirce, Jevons and Schröder*. PhD thesis, Indiana University, 1978.

[Došen, 1988] Kosta Došen. Sequent systems and groupoid models, part 1. *Studia Logica*, 47:353–386, 1988.

[Došen, 1989] Kosta Došen. Sequent systems and groupoid models, part 2. *Studia Logica*, 48:41–65, 1989.

[Došen, 1992] Kosta Došen. The first axiomatisation of relevant logic. *Journal of Philosophical Logic*, 21:339–356, 1992.

[Dunn and Meyer, 1971] J. Michael Dunn and Robert K. Meyer. Algebraic completeness results for Dummett's LC and its extensions. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 17:225–230, 1971.

[Dunn and Meyer, 1989] J. Michael Dunn and Robert K. Meyer. Gentzen's cut and Ackermann's γ. In Richard Sylvan and Jean Norman, editors, *Directions in Relevant Logic*. Kluwer, Dordrecht, 1989.

[Dunn and Meyer, 1997] J. Michael Dunn and Robert K. Meyer. Combinators and structurally free logic. *Logic Journal of the IGPL*, 5:505–357, 1997.

[Dunn, 1966] J. Michael Dunn. *The Algebra of Intensional Logics*. PhD thesis, University of Pittsburgh, 1966.

[Dunn, 1967] J. Michael Dunn. The effective equivalence of certain propositions about De Morgan lattices. *Journal of Symbolic Logic*, 32:433–434, 1967. (Abstract).

[Dunn, 1969] J. Michael Dunn. Natural language versus formal language. In *Joint APA-APL Symposium, New York, December 17*, 1969.

[Dunn, 1970] J. Michael Dunn. Algebraic completeness for R-mingle and its extensions. *Journal of Symbolic Logic*, 35:1–13, 1970.

[Dunn, 1971] J. Michael Dunn. An intuitive semantics for first degree relevant implications (abstract). *Journal of Symbolic Logic*, 36:363–363, 1971.

[Dunn, 1974] J. Michael Dunn. A 'Gentzen' system for positive relevant implication. *Journal of Symbolic Logic*, 38:356–357, 1974. (Abstract).

[Dunn, 1976a] J. Michael Dunn. Intuitive semantics for first-degree entailments and "coupled trees". *Philosophical Studies*, 29:149–168, 1976.

[Dunn, 1976b] J. Michael Dunn. A Kripke-style semantics for R-mingle using a binary accessibility relation. *Studia Logica*, 35:163–172, 1976.

[Dunn, 1976c] J. Michael Dunn. Quantification and RM. *Studia Logica*, 35:315–322, 1976.

[Dunn, 1976d] J. Michael Dunn. A variation on the binary semantics for RM. *Relevance Logic Newsletter*, 1:56–67, 1976.

[Dunn, 1979a] J. Michael Dunn. R-mingle and beneath: Extensions of the Routley-Meyer semantics for R. *Notre Dame Journal of Formal Logic*, 20:369–376, 1979.

[Dunn, 1979b] J. Michael Dunn. A theorem in 3-valued model theory with connections to number theory, type theory, and relevant logic. *Studia Logica*, 38:149–169, 1979.

[Dunn, 1980a] J. Michael Dunn. Relevant Robinson's arithmetic. *Studia Logica*, 38:407–418, 1980.

[Dunn, 1980b] J. Michael Dunn. A sieve for entailments. *Journal of Philosophical Logic*, 9:41–57, 1980.

[Dunn, 1987] J. Michael Dunn. Relevant predication 1: The formal theory. *Journal of Philosophical Logic*, 16:347–381, 1987.

[Dunn, 1990a] J. Michael Dunn. Relevant predication 2: Intrinsic properties and internal relations. *Philosophical Studies*, 60:177–206, 1990.

[Dunn, 1990b] J. Michael Dunn. Relevant predication 3: Essential properties. In J. Michael Dunn and Anil Gupta, editors, *Truth or Consequences*, pages 77–95. Kluwer, 1990.

[Dunn, 1991] J. Michael Dunn. Gaggle theory: An abstraction of Galois connections and residuation with applications to negation and various logical operations. In *Logics in AI, Proceedings European Workshop JELIA 1990*, volume 478 of *Lecture Notes in Computer Science*. Springer-Verlag, 1991.

[Dunn, 1993a] J. Michael Dunn. Partial-gaggles applied to logics with restricted structural rules. In Peter Schroeder-Heister and Kosta Došen, editors, *Substructural Logics*. Oxford University Press, 1993.

[Dunn, 1993b] J. Michael Dunn. A representation of relation algebras using Routley–Meyer frames. Technical report, Indiana University Logic Group Preprint Series, IULG–93–28, 1993.

[Dunn, 1994] J. Michael Dunn. Star and perp: Two treatments of negation. In James E. Tomberlin, editor, *Philosophical Perspectives*, volume 7, pages 331–357. Ridgeview Publishing Company, Atascadero, California, 1994.

[Dunn, 1996] J. Michael Dunn. Generalised ortho negation. In Heinrich Wansing, editor, *Negation: A Notion in Focus*, pages 3–26. Walter de Gruyter, Berlin, 1996.

[Feys and Ladrière, 1955] R. Feys and J. Ladrière. Notes to *recherches sur la déduction logique*, 1955. Translation of "Untersuchungen über das logische Schliessen" [Gentzen, 1934], Paris.

[Fine, 1974] Kit Fine. Models for entailment. *Journal of Philosophical Logic*, 3:347–372, 1974.

[Fine, 1988] Kit Fine. Semantics for quantified relevance logic. *Journal of Philosophical Logic*, 17:27–59, 1988.

[Fine, 1989] Kit Fine. Incompleteness for quantified relevance logics. In Jean Norman and Richard Sylvan, editors, *Directions in Relevant Logic*, pages 205–225. Kluwer Academic Publishers, Dordrecht, 1989.

[Fitting, 1989] Melvin C. Fitting. Bilattices and the semantics of logic programming. *Journal of Logic Programming*, 11(2):91–116, 1989.

[Fogelin, 1978] R. J. Fogelin. *Understanding Arguments*. Harcourt, Brace, Jovanovich, New York, 1978.

[Friedman and Meyer, 1992] Harvey Friedman and Robert K. Meyer. Whither relevant arithmetic? *Journal of Symbolic Logic*, 57:824–831, 1992.

[Gabbay, 1976] D. M. Gabbay. *Investigations in Modal and Tense Logics with Applications to Problems in Philosophy and Linguistics*. Reidel, Dordrecht, 1976.

[Gabbay, 1997] Dov M. Gabbay. *Labelled Deductive Systems*. Number 33 in Oxford Logic Guides. Oxford University Press, 1997.

[Gentzen, 1934] Gerhard Gentzen. Untersuchungen über das logische schliessen. *Math. Zeitschrift*, 39:176–210 and 405–431, 1934. Translated in *The Collected Papers of Gerhard Gentzen* [Gentzen, 1969].

[Gentzen, 1969] Gerhard Gentzen. *The Collected Papers of Gerhard Gentzen*. North Holland, 1969. Edited by M. E. Szabo.

[Giambrone, 1985] Steve Giambrone. TW_+ and RW_+ are decidable. *Journal of Philosophical Logic*, 14:235–254, 1985.

[Girard, 1987] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–101, 1987.

[Grice, 1975] H. P. Grice. Logic and conversation. In P. Cole and J. L. Morgan, editors, *Syntax and Semantics: Speech Acts*, volume 3, pages 41–58. Academic Press, New York, 1975. Reprinted in Jackson [Jackson, 1991].

[Haack, 1974] S. Haack. *Deviant Logic: Some Philosophical Issues*. Cambridge University Press, 1974. Reissued as *Deviant Logic, Fuzzy Logic: Beyond the Formalism* [Haack, 1996].

[Haack, 1996] Susan Haack. *Deviant Logic, Fuzzy Logic: Beyond the Formalism*. Cambridge University Press, 1996.

[Harrop, 1965] R. Harrop. Some structure results for propositional calculi. *Journal of Symbolic Logic*, 30:271–292, 1965.

[Hutchinson, 1973] G. Hutchinson. Recursively unsolvable word problems of modular lattices and diagram chasing. *Journal of Algebra*, pages 385–399, 1973.

[Jackson, 1991] Frank Jackson, editor. *Conditionals*. Oxford Readings in Philosophy. Oxford, 1991.

[Jónsson and Tarski, 1951] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators: Part I. *American Journal of Mathematics*, 73:891–939, 1951.

[Kalman, 1958] J. A. Kalman. Lattices with involution. *Transactions of the American Mathematical Society*, 87:485–491, 1958.

[Kripke, 1959a] Saul A. Kripke. A completeness theorem in modal logic. *Journal of Symbolic Logic*, 24:1–15, 1959.

[Kripke, 1959b] Saul A. Kripke. The problem of entailment. *Journal of Symbolic Logic*, 24:324, 1959. Abstract.

[Kripke, 1963] Saul A. Kripke. Semantical considerations on modal logic. *Acta Philosophica Fennica*, 16:83–94, 1963.

[Kripke, 1965] Saul A. Kripke. Semantical analysis of modal logic II: Non-normal modal propositional calculi. In Addison et al., editor, *The Theory of Models*, pages 206–220. North Holland Publishing Co, 1965.

[Kron, 1973] A. Kron. Deduction theorems for relevant logic. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 19:85–92, 1973.

[Kron, 1976] A. Kron. Deduction theorems for T, E and R reconsidered. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 22:261–264, 1976.

[Kurtonina, 1995] Natasha Kurtonina. *Frames and Labels: A Modal Analysis of Categorial Inference*. PhD thesis, Institute for Logic, Language and Computation, University of Utrecht, 1995.

[Lambek, 1958] Joachim Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65:154–170, 1958.

[Lambek, 1961] Joachim Lambek. On the calculus of syntactic types. In R. Jacobsen, editor, *Structure of Language and its Mathematical Aspects*, Proceedings of Symposia in Applied Mathematics, XII. American Mathematical Society, 1961.

[Lemmon, 1965] E. J. Lemmon. *Beginning Logic*. Nelson, 1965.

[Lemmon, 1966] E. J. Lemmon. Algebraic semantics for modal logics, I and II. *Journal of Symbolic Logic*, 31:46–55 and 191–218, 1966.

[Lewis and Langford, 1932] C. I. Lewis and C. H. Langford. *Symbolic Logic*. The Century Co, New York and London, 1932.

[Lincoln *et al.*, 1992] P. Lincoln, J. Mitchell, A. Scenrov, and N. Shankar. Decision problems for propositional linear logic. *Annals of Pure and Applied Logic*, 56:239–311, 1992.

[Lipshitz, 1974] L. Lipshitz. The undecidability of the word problems for projective geometries and modular lattices. *Transactions of the American Mathematical Society*, pages 171–180, 1974.

[Maksimova, 1967] L. L. Maksimova. O modéláh isčisléniá E. *Algébra i Logika, Séminar*, 6:5–20, 1967. (English title: On Models of the System E).

[Maksimova, 1971] L. L. Maksimova. An interpolation and separation theorem for the logical systems e and r. *Algebra and Logic*, 10:232–241, 1971.

[Maksimova, 1973] L. L. Maksimova. A semantics for the calculus E of entailment. *Bulletin of the section of Logic*, 2:18–21, 1973. Published by Polish Academy of Sciences, Institute of Philosophy and Sociology.

[Mares and Fuhrmann, 1995] Edwin D. Mares and André Fuhrmann. A relevant theory of conditionals. *Journal of Philosophical Logic*, 24:645–665, 1995.

[Mares, 1992] Edwin Mares. Semantics for relevance logic with identity. *Studia Logica*, 51:1–20, 1992.

[Mares, 1997] Edwin Mares. Relevant logic and the theory of information. *Synthese*, 109:345–360, 1997.

[Martin and Meyer, 1982] E. P. Martin and R. K. Meyer. Solution to the P-W problem. *Journal of Symbolic Logic*, 47:869–886, 1982.

[McKinsey and Tarski, 1948] J. C. C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *Journal of Symbolic Logic*, 13:1–15, 1948.

[McRobbie and Belnap Jr., 1979] M. A. McRobbie and N. D. Belnap Jr. Relevant analytic tableaux. *Studia Logica*, 38:187–200, 1979.

[Meyer and Dunn, 1969] Robert K. Meyer and J. Michael Dunn. E, R and γ. *Journal of Symbolic Logic*, 34:460–474, 1969.

[Meyer and Leblanc, 1970] Robert K. Meyer and H. Leblanc. A semantical completeness result for relevant quantification theories. *Journal of Symbolic Logic*, 35:181, 1970. Abstract.

[Meyer and McRobbie, 1979] Robert K. Meyer and Michael A. McRobbie. Firesets and relevant implication. Technical Report 3, Logic Group, RSSS, Australian National University, 1979.

[Meyer and Routley, 1972] Robert K. Meyer and Richard Routley. Algebraic analysis of entailment. *Logique et Analyse*, 15:407–428, 1972.

[Meyer and Routley, 1973a] Robert K. Meyer and Richard Routley. Classical relevant logics I. *Studia Logica*, 32:51–66, 1973.

[Meyer and Routley, 1973b] Robert K. Meyer and Richard Routley. Classical relevant logics II. *Studia Logica*, 33:183–194, 1973.

[Meyer and Routley, 1973c] Robert K. Meyer and Richard Routley. An undecidable relevant logic. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 19:289–397, 1973.

[Meyer and Urbas, 1986] Robert K. Meyer and Igor Urbas. Conservative extension in relevant arithmetics. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 32:45–50, 1986.

[Meyer *et al.*, 1974] Robert K. Meyer, J. Michael Dunn, and H. Leblanc. Completeness of relevant quantification theories. *Notre Dame Journal of Formal Logic*, 15:97–121, 1974.

[Meyer *et al.*, 1979] Robert K. Meyer, Richard Routley, and J. Michael Dunn. Curry's paradox. *Analysis*, 39:124–128, 1979.

[Meyer *et al.*, 1995] Robert K. Meyer, Michael A. McRobbie, and Nuel D. Belnap. Linear analytic tableaux. In *Proceedings of the Fourth Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, volume 918 of *Lecture Notes in Computer Science*, 1995.

[Meyer, 1966] Robert K. Meyer. *Topics in Modal and Many-valued Logic*. PhD thesis, University of Pittsburgh, 1966.

[Meyer, 1968] Robert K. Meyer. An undecidability result in the theory of relevant implication. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 14:255–262, 1968.

[Meyer, 1970a] Robert K. Meyer. E and S4. *Notre Dame Journal of Formal Logic*, 11:181–199, 1970.

[Meyer, 1970b] Robert K. Meyer. R_I—the bounds of finitude. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 16:385–387, 1970.

[Meyer, 1971] Robert K. Meyer. On coherence in modal logics. *Logique et Analyse*, 14:658–668, 1971.

[Meyer, 1973] Robert K. Meyer. Intuitionism, entailment, negation. In H. Leblanc, editor, *Truth, Syntax and Modality*, pages 168–198. North Holland, 1973.

[Meyer, 1974] Robert K. Meyer. New axiomatics for relevant logics — I. *Journal of Philosophical Logic*, 3:53–86, 1974.

[Meyer, 1976a] R. K. Meyer. Ackermann, Takeuti and Schnitt; γ for higher-order relevant logics. *Bulletin of the Section of Logic*, 5:138–144, 1976.

[Meyer, 1976b] Robert K. Meyer. Metacompleteness. *Notre Dame Journal of Formal Logic*, 17:501–517, 1976.

[Meyer, 1976c] Robert K. Meyer. Relevant arithmetic. *Bulletin of the Section of Logic*, 5:133–137, 1976.

[Meyer, 1978] Robert K. Meyer. Why I am not a relevantist. Technical Report 1, Logic Group, RSSS, Australian National University, 1978.

[Meyer, 1979a] R. K. Meyer. A Boolean-valued semantics for r. Technical Report 4, Logic Group, RSSS, Australian National University, 1979.

[Meyer, 1979b] Robert K. Meyer. Arithmetic formulated relevantly. Unpublished manuscript. Some of the results appear in *Entailment* Volume 2 [Anderson *et al.*, 1992], 1979.

[Meyer, 1979c] Robert K. Meyer. Career induction stops here (and here = 2). *Journal of Philosophical Logic*, 8:361–371, 1979.

[Meyer, 1980] Robert K. Meyer. Relevantly interpolating in RM. Technical Report 9, Logic Group, RSSS, Australian National University, 1980.

[Meyer, 1998] Robert K. Meyer. $\supset$E is admissible in 'true' relevant arithmetic. *Journal of Philosophical Logic*, 27:327–351, 1998.

[Minc, 1972] G. Minc. Cut-elimination theorem in relevant logics. In J. V. Matijasevic and O. A. Silenko, editors, *Isslédovaniá po konstructivnoj mathematiké i matematičeskoj logike V*, pages 90–97. Izdatél'stvo "Nauka", 1972. (English translation in "Cut-Elimination Theorem in Relevant Logics" [Minc, 1976]).

[Minc, 1976] G. Minc. Cut-elimination theorem in relevant logics. *The Journal of Soviet Mathematics*, 6:422–428, 1976. (English translation of the original article [Minc, 1972]).

[Mortensen, 1995] Chris Mortensen. *Inconsistent Mathematics*. Kluwer Academic Publishers, 1995.

[Parry, 1933] W. T. Parry. Ein axiomensystem für eine neue art von implikation (analytische implication). *Ergebnisse eines mathematischen Kolloquiums*, 4:5–6, 1933.

[Pottinger, 1979] G. Pottinger. On analysing relevance constructively. *Studia Logica*, 38:171–185, 1979.

[Prawitz, 1965] Dag Prawitz. *Natural Deduction: A Proof Theoretical Study*. Almqvist and Wiksell, Stockholm, 1965.

[Priest and Sylvan, 1992] Graham Priest and Richard Sylvan. Simplified semantics for basic relevant logics. *Journal of Philosophical Logic*, 21:217–232, 1992.

[Priest, 1987] Graham Priest. *In Contradiction: A Study of the Transconsistent*. Martinus Nijhoff, The Hague, 1987.

[Priest, 1995] Graham Priest. *Beyond the Limits of Thought*. Cambridge University Press, Cambridge, 1995.

[Quine, 1953] W. V. Quine. Three grades of modal involvement. In *Proceedings of the XIth International Congress of Philosophy*. North Holland, Amsterdam, 1953. Reprinted in *The Ways of Paradox*, Random House, New York, 1966.

[Rasiowa, 1974] H. Rasiowa. *An Algebraic Approach to Non-classical Logics*. North Holland, 1974.

[Read, 1988] Stephen Read. *Relevant Logic*. Basil Blackwell, Oxford, 1988.

[Restall, 1993] Greg Restall. Simplified semantics for relevant logics (and some of their rivals). *Journal of Philosophical Logic*, 22:481–511, 1993.

[Restall, 1994] Greg Restall. A useful substructural logic. *Bulletin of the Interest Group in Pure and Applied Logic*, 2(2):135–146, 1994.

[Restall, 1995a] Greg Restall. Display logic and gaggle theory. *Reports in Mathematical Logic*, 29:133–146, 1995.

[Restall, 1995b] Greg Restall. Information flow and relevant logics. In Jerry Seligman and Dag Westerstahl, editors, *Logic, Language and Computation: The 1994 Moraga Proceedings*, pages 463–477. CSLI Publications, 1995.

[Restall, 1998] Greg Restall. Displaying and deciding substructural logics 1: Logics with contraposition. *Journal of Philosophical Logic*, 27:179–216, 1998.

[Restall, 1999] Greg Restall. Negation in Relevant Logics: How I Stopped Worrying and Learned to Love the Routley Star. In Dov Gabbay and Heinrich Wansing, editors, *What is Negation?*, volume 13 of *Applied Logic Series*, pages 53–76. Kluwer Academic Publishers, 1999.

[Restall, 2000] Greg Restall. *An Introduction to Substructural Logics*. Routledge, 2000.

[Routley and Meyer, 1973] Richard Routley and Robert K. Meyer. Semantics of entailment. In Hugues Leblanc, editor, *Truth Syntax and Modality*, pages 194–243. North Holland, 1973. Proceedings of the Temple University Conference on Alternative Semantics.

[Routley and Meyer, 1976] Richard Routley and Robert K. Meyer. Dialectal logic, classical logic and the consistency of the world. *Studies in Soviet Thought*, 16:1–25, 1976.

[Routley and Routley, 1972] Richard Routley and Valerie Routley. Semantics of first-degree entailment. *Noûs*, 3:335–359, 1972.

[Routley *et al.*, 1982] Richard Routley, Val Plumwood, Robert K. Meyer, and Ross T. Brady. *Relevant Logics and their Rivals*. Ridgeview, 1982.

[Routley, 1977] Richard Routley. Ultralogic as universal. *Relevance Logic Newsletter*, 2:51–89, 1977. Reprinted in *Exploring Meinong's Jungle* [Routley, 1980a].

[Routley, 1980a] Richard Routley. *Exploring Meinong's Jungle and Beyond*. Philosophy Department, RSSS, Australian National University, 1980. Interim Edition, Departmental Monograph number 3.

[Routley, 1980b] Richard Routley. Problems and solutions in the semantics of quantified relevant logics — I. In A. I. Arrduda, R. Chuaqui, and N. C. A. da Costa, editors, *Proceedings of the Fourth Latin American Symposium on Mathematical Logic*, pages 305–340. North Holland, 1980.

[Schroeder-Heister and Došen, 1993] Peter Schroeder-Heister and Kosta Došen, editors. *Substructural Logics*. Oxford University Press, 1993.

[Schütte, 1956] K. Schütte. Ein system des verknüpfenden schliessens. *Archiv für Mathematische Logik*, 2:55–67, 1956.

[Scott, 1971] Dana Scott. On engendering an illusion of understanding. *Journal of Philosophy*, 68:787–807, 1971.

[Shaw-Kwei, 1950] Moh Shaw-Kwei. The deduction theorems and two new logical systems. *Methodos*, 2:56–75, 1950.

[Slaney, 1985] John K. Slaney. 3088 varieties: A solution to the Ackermann constant problem. *Journal of Symbolic Logic*, 50:487–501, 1985.

[Slaney, 1990] John K. Slaney. A general logic. *Australasian Journal of Philosophy*, 68:74–88, 1990.

[Smullyan, 1968] R. M. Smullyan. *First-Order Logic*. Springer-Verlag, Berlin, 1968. Reprinted by Dover Press, 1995.

[Stone, 1936] M. H. Stone. The theory of representation for Boolean algebras. *Transactions of the American Mathematical Society*, 40:37–111, 1936.

[Suppes, 1957] P. Suppes. *Introduction to Logic*. D. Van Nostrand Co, Princeton, 1957.

[Tennant, 1994] Neil Tennant. The transmission of truth and the transitivity of deduction. In Dov Gabbay, editor, *What is a Logical System?*, volume 4 of *Studies in Logic and Computation*, pages 161–177. Oxford University Press, Oxford, 1994.

[Thistlewaite *et al.*, 1988] Paul Thistlewaite, Michael McRobbie, and Robert K. Meyer. *Automated Theorem Proving in Non-Classical Logics*. Wiley, New York, 1988.

[Tokarz, 1980] M. Tokarz. *Essays in Matrix Semantics of Relevant Logics*. The Polish Academy of Science, Warsaw, 1980.

[Troelstra, 1992] A. S. Troelstra. *Lectures on Linear Logic*. CSLI Publications, 1992.

[Urquhart, 1972a] Alasdair Urquhart. The completeness of weak implication. *Theoria*, 37:274–282, 1972.

[Urquhart, 1972b] Alasdair Urquhart. A general theory of implication. *Journal of Symbolic Logic*, 37:443, 1972.

[Urquhart, 1972c] Alasdair Urquhart. Semantics for relevant logics. *Journal of Symbolic Logic*, 37:159–169, 1972.

[Urquhart, 1972d] Alasdair Urquhart. *The Semantics of Entailment*. PhD thesis, University of Pittsburgh, 1972.

[Urquhart, 1983] Alasdair Urquhart. Relevant implication and projective geometry. *Logique et Analyse*, 26:345–357, 1983.

[Urquhart, 1984] Alasdair Urquhart. The undecidability of entailment and relevant implication. *Journal of Symbolic Logic*, 49:1059–1073, 1984.

[Urquhart, 1990] Alasdair Urquhart. The complexity of decision procedures in relevance logic. In J. Michael Dunn and Anil Gupta, editors, *Truth or Consequences*, pages 77–95. Kluwer, 1990.

[Urquhart, 1993] Alasdair Urquhart. Failure of interpolation in relevant logics. *Journal of Philosophical Logic*, 22:449–479, 1993.

[Urquhart, 1997] Alasdair Urquhart. The complexity of decision procedures in relevance logic ii. Available from the author, University of Toronto, 1997.

[van Fraassen, 1969] Bas van Fraassen. Facts and tautological entailments. *Journal of Philosophy*, 66:477–487, 1969. Reprinted in *Entailment* Volume 1 [Anderson and Belnap, 1975].

[van Fraassen, 1973] Bas van Fraassen. Extension, intension and comprehension. In M. Munitz, editor, *Logic and Ontology*, pages 101–103. New York University Press, New York, 1973.

[Wansing, 1993] Heinrich Wansing. *The Logic of Information Structures*. Number 681 in Lecture Notes in Artificial Intelligence. Springer-Verlag, 1993.

[Wolf, 1978] R. G. Wolf. Are relevant logics deviant. *Philosophia*, 7:327–340, 1978.

INDEX