WHAT ARE WE TO ACCEPT,
AND WHAT ARE WE TO REJECT,
WHILE SAVING TRUTH FROM PARADOX?

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I'm delighted to have the opportunity to respond to Hartry Field's *Saving Truth From Paradox* [3]. This is a wonderful book: it's clear and precise, interesting and engaging, and deep and important all at once. Truth and the paradoxes comprise a very difficult field in which to work. Nonetheless, the work in that field is deep and important. Field advances the state of the art of work on truth and paradox in three distinct and linked ways. (1) It is the best available presentation of existing approaches to the paradoxes, both those that attempt to adhere to classical logic and those that feel free to abandon classical logic for non-classical logical principles. (2) He develops new results in non-classical approaches to truth, extending this work in important and fruitful ways. (3) He pushes the philosophical analyses of these matters much further. Any one of these advances would be of use to us. Any two would form the basis of an excellent book, to be highly recommended. But with *Saving Truth From Paradox*, we have been given all three. Field covers classical and non-classical approaches to the theory of truth. He looks in depth at different issues in the philosophy of logic, including the connection between truth, accepting, rejecting and logical consequence, the nature of mathematical theories, and more. Along the way come introductions to different kinds of non-classical logic and different ways to construct models for those logics. It is a rich book with many lessons to be learned by anyone interested in logic and in truth.

There are many aspects of this book to enjoy: I especially appreciate the detailed discussion of accepting and rejecting, and the way that these are to be understood by those who endorse truth-value gaps or truth-value gluts. Since Field

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comes to this work from the outside (he is not known, before the 21st Century, for work on non-classical logic), he takes care to present it in a language that the outsider can understand. While an *Australian* like me—who breathes in non-classical approaches from my early logical education—learns how to work with paraconsistent logics in a culture in which they're a regular topic of discussion, and taken seriously by all, whether paraconsistentist or completely opposed, it is important to realise that the rest of the world is not necessarily like this. Field carefully explains distinctions that proponents of non-classical logic just seem to 'get' and that those whose logical background takes in little more than classical approaches will find harder to swallow. An example: friends of non-classical logics agree that it's possible to treat the accepting a negation ($\neg p$) as different from rejecting the thing negated (p), in *dual* ways, for treatments of truth-value gaps and truth-value gluts. While Graham Priest and others of us have been making this point about the intelligibility of dialetheism for years, we work in a fairly narrow circle. Having someone with Field's background make the point—and say it so well—is a welcome addition to the discussion. The gentle handholding in the discussion about how to be a dialetheist, from page 363 onwards, is worth the price of admission alone. Thankyou Hartry.

Of course, it must be said that neither Field nor I are dialetheists. We don't think that contradictions are true. However, we both agree that dialetheism—especially the view of the semantic paradoxes taken by Graham Priest among others—is worth taking seriously, and that all views are better served if we treat other views on their merits as discussion partners.

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However, I haven't come here to praise the book merely by reiterating how *good* it is. I am also here to praise it by engaging with the argument, and pushing Field further along in the development of his view.

Here are the two ingredients in the book I will work with, and use for the raw materials in my discussion. (a) *The naïve theory of properties*, which stands to the theory of truth as a straightforward generalisation. The naïve theory of truth holds that for any sentence ϕ, the sentence $\phi \leftrightarrow T\langle\phi\rangle$ is true: snow is white if and only if it is true that snow is white. For a theory of *properties* we generalise to open sentences $\phi(x)$. Then the sentence $(\forall y)(\phi(y) \leftrightarrow y \text{ is } \langle x : \phi(x)\rangle)$ is true. An object y has the property of being something that is white if and only if y is white. Just as the naïve conception of truth is troubled with paradox, so is the naïve conception of properties. We do not even need to use diagonalisation, demonstratives or other techniques for self-reference to construct the paradox. Consider the heterological property H: $\langle x : \neg(x \text{ is } x)\rangle$ and ask whether H is H or not.

Field's account in this book, however, gives us a regimentation of the naïve theory of truth and properties in which all the fancy footwork is done, not by carving away at the theory of properties and of truth, but rather, at the logic. So,

the *logic* will be my second port of call. However, instead of attending merely to the detail of this or that formal system (and a range of different logical options, both paraconsistent and paracomplete are considered in this book), I wish to attend to (b) *The logic of accepting and rejecting*. We have already seen that Field has discussed some connections between accepting and rejecting: he highlights that for one who accepts contradictions, accepting ¬p need not be seen as incompatible with accepting p—it need not be seen as amounting to *rejecting* p. Similarly, for one who admits truth-value gaps, rejecting p need not be seen as part and parcel of accepting ¬p. We are to attend to both accepting and rejecting.

> . . . while the standard way of talking about logical systems specifies only what the system accept, it would be natural to introduce an expanded conception of logical system in which it is specified what the system *rejects*. (footnote 5 on page 371)

However, paying just this much this attention is a halfway house. Once we see that accepting and rejecting[1] are both constrained by logical norms, we are immediately confronted with a new issue. It is important not only to know what a logical system recommends (or constrains) with regard to accepting and rejecting *tout court*, but also, the way that various combinations of accepting and rejecting hang together. There is a *logical* error made when one accepts $p \vee q$ and rejects both p and q, or when one rejects $p \wedge q$ but accepts both p and q, or when one accepts $p \rightarrow q$, accepts p but rejects q. If all a logic recommends is some body of propositions to be accepted on the one hand, and another body of propositions to be rejected, then we miss out on the rich fabric of connections between accepting and rejecting.[2]

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So, let's start with (a) The Theory of Properties. For Field, the major concern is Saving *Truth* From Paradox, not saving *Properties*. Regardless, he does attend to property theory. In particular he shows how his consistency proof for a naïve theory of truth can be extended to provide consistency for a theory of properties. The theory of properties is committed to the principle (NC) of naïve comprehension. For each open sentence φ we have

$$(\exists y)(\forall x)(Property(y) \wedge (x \text{ } instantiates \text{ } y \leftrightarrow \phi(x)))$$

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[1]I prefer, in other contexts, to talk of assertion and denial, for it seems to me that logical norms apply to acts of assertion and denial whether they correlate with what I accept or reject. However, nothing important will ride on the difference here [7].

[2]This is one of the morals of Lewis Carroll's tale of Achilles, the Tortoise and *modus ponens*. The force of *modus ponens* is not merely to recommend to us a statement $((p \rightarrow q) \wedge p \rightarrow q)$ for our acceptance, but to render the combination—accept $p \rightarrow q$, accept p, reject q—as out of bounds [2].

Field shows that adding (NC) to his favoured logic is consistent.[3] The result is a consistent theory in which *every* open sentence determines a property.

In what follows, I will introduce a notation for properties. For each open sentence $\phi(x)$, we will introduce the term $\langle y : \phi(y) \rangle$ to denote a property for which x *instantiates* $\langle y : \phi(y) \rangle \leftrightarrow \phi(x)$, and for which *Property*($\langle y : \phi(y) \rangle$) This term, if you like, Skolemises the existential quantifier '($\exists y$)' in (NC). Further, we'll shorten '*instantiates*' to 'is.' The result is a simple principle

$$(\forall x)(x \text{ is } \langle y : \phi(y) \rangle \leftrightarrow \phi(x))$$

So far, so good. This theory, with terms for properties, is consistent. Field's model construction supplies entities for each term '$\langle x : \phi(x) \rangle$' may refer.

Once we attend to properties, modelled in this way, we may ask whether when two property terms pick out the one property. When is it the case that $\langle x : \phi(x) \rangle = \langle y : \psi(y) \rangle$?[4] Field's theory, which ajoins properties to a base logic using (NC) alone, tells us little to address this issue. Clearly, if x is P and $\neg(x$ is Q), then P and Q differ, but this is nothing over and above Leibniz's law. For *sets*, the identity conditions are clear: sets a and b are identical when they have the same members. Difference in membership is the only way sets can differ. For properties, this is much too coarse a condition for identity: there may be two properties instantiated by exactly the same objects, but which remain resolutely distinct. (Think 'renate' and 'cordate'.)

However, (NC) is so weak that it allows that any syntactic difference be reflected in a difference in property. Nothing rules out $\langle x : Fx \wedge Gx \rangle$ differing from $\langle x : Gx \wedge Fx \rangle$.[5] But is this an important difference? Are *properties* simply reifications of *sentences*, in all their idiosyncratic particularity? Using a model for (NC) in which these properties differ is not a problem, however, it is unsatisfactory unless we can find a way to go on to a richer and more interesting property theory on the one hand, or on the other, a defense of a syntactic criterion of identity for properties.

Perhaps one way to go on is to realise that within (NC) we may have the resources to enrich our theory of properties. We may, for example, be interested in a coarser level of individuation for properties. After all, if a is P and a is Q are *logically* eqivalent, then this seems like a more plausible condition under which P and Q may be identified as properties. If one can infer that a is Q from the supposition that a is P, and *vice versa*, then there seems to be nothing over and above instantiating P than there is in instantiating Q.

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[3]In the model construction, we add to the domain an open sentence (paired with a tuple of objects for each variable other than x free in $\phi(x)$) as the denotation of the property term. The work in the construction is taken up in fixing the interpretation of *instantiates* in such a way that (NC) is satisfied.

[4]This question is not an artefact of the notation I have introduced. It may be asked without that notation in a more long-winded manner. When do we have $(\exists y)(\exists y')(\forall x)((x$ is $y \leftrightarrow \phi(x)) \wedge (x$ is $y' \leftrightarrow \psi(x)) \wedge y = y')$? This question gets to the same issue: the identity conditions for properties.

[5]These two properties differ in the model construction Field gives, since Fx $\wedge$ Gx and Gx $\wedge$ Fx are different sentences.

Now, perhaps this handwaving justification cannot be made out. One thing which *can* be made out is the following line of reasoning. Perhaps the properties of our favoured understanding of (NC) are very finely individuated, where distinct properties may have logically equivalent possession conditions. Regardless, we can introduce a *coarser* account of properties, by bundling together all logically coextensive properties. If from a is P is it logically follows that a is Q and vice versa, we will say that the properties P and Q are logically coextensive. In this case, bearing P and bearing Q are no different when it comes to deduction. We can keep track of whether or not a has property P by keeping track of whether it has Q. Note well: This does not mean that we must identify necessarily coextensive properties. If you think that of necessity being water stands or falls with being H_2O, it does not follow that from the assumption that x is water one can validly deduce that x is H_2O. Water may be H_2O of necessity without that necessity being a matter of logic. The issue here is whether it would be flatly incoherent to assert that x is water while denying that it is H_2O.

So, take a *coarse* property to be the bundle of *finely grained* ones. Now, an object has a coarse property if and only if it has the corresponding finely grained properties (it matters not which one we choose, since bearing one is logically equivalent to bearing any of the others), and now, if $\phi(x)$ is logically equivalent to $\phi'(x)$, then the coarse property corresponding to ϕ is the same as that corresponding to ϕ'. So, even if our favoured theory of properties is not so coarse as to identify properties which are logically coextensive, 'coarse properties' may be readily introduced using the resources of a theory of fine properties. So, let's use '$\langle\langle x : \phi(x)\rangle\rangle$' for the coarse property of being ϕ. So if $\phi(a)$ entails $\psi(a)$, and vice versa (where a is an eigenvariable) then $\langle\langle x : \phi(x)\rangle\rangle = \langle\langle x : \psi(x)\rangle\rangle$. Mere syntactic difference between $\phi(x)$ and $\psi(x)$ is not enough to ground a difference in the properties they express. For there to be a difference, there must be some semantic difference, going beyond the syntax.

This reasoning seems to me to be uproblematic. If (NC) is to be admitted, so is the naïve theory of coarse properties. However, once we pay attention to the logic of accepting and rejecting, we will see that the naïve theory of coarse properties is threatened by paradox. This brings us to our second topic.

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We have agreed that a logic might give us not only guidance on what statements are to be accepted (the tautologies, or *theorems* of the logic), but also what statements are to be rejected (the anti-tautologies, or *co-theorems*). In a classical, two-valued setting, one class is definable in terms of the other. If A is a tautology, ¬A is to be rejected, and vice versa. In the presence of gaps or gluts, this connection is no longer available.

I have argued elsewhere that even when gaps or gluts are admitted, Gentzen's multiple–premise and multiple–conclusion sequents have a clear interpretation. The sequent $X \Rightarrow Y$ can be understood as marking that accepting each state-

ment in X and rejecting each statement in Y is ruled out on logical grounds [7]. *Identity* sequents

$$X, A \Rightarrow A, Y \ [\textit{Id}]$$

mark that accepting A together with rejecting A is ruled out regardless of what else we accept or reject. Similarly, Gentzen's *Cut* rule

$$\frac{X \Rightarrow A, Y \quad X, A \Rightarrow Y}{X \Rightarrow Y} \ [\textit{Cut}]$$

marks another connection between accepting and rejecting. If accepting everything in X and rejecting everything in Y is not ruled out on logical grounds, then if accepting A *is* ruled out (in combination with accepting X and rejecting Y), then it is implicitly rejected in the presence of accepting X and rejecting Y, so making that rejection explicit cannot be ruled out on logical grounds, since that rejection is implicit in what has already been accepted and rejected.[6] Given these structural rules, sequents of the form $X, A \Rightarrow B, Y$ may be used to give an account of logical consequence. If, when we accept X and reject Y, the addition of accepting A and rejecting B is ruled out on logical grounds, then if we accept A the only option with B (if we wish to take a stand on it) is to accept it, since it is undeniable—rejecting it is totally ruled out on grounds of logic alone (given that we accept X and A and reject Y). Conversely, if we reject B we must reject A, as it is totally unacceptable.[7]

So, given the background of [*Id*] and [*Cut*], we have a context in which we can restate logical and semantic principles without relying on connectives. In the context of discussions of paraconsistent and paracomplete theories of truth, this is important because there is such divergence over the behaviour of connectives such as negations or conditionals. This is especially worrisome in the discussion of 'the' conditional, as in the context of the naïve theory of truth and of properties, we must reject the validity of the inference of contraction. $A \rightarrow (A \rightarrow B)$ does not entail $A \rightarrow B$, lest the theory founder on the rocks of Curry's paradox.

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[6]A case Field discusses (see page 74)—a statement A for which we take there to be no possible evidence for or against—might be thought to be a counterexample to the rule [*Cut*]. After all, I may be committed to not accept A, and committed not to reject it, since there is no possible evidence for or against it. So, suppose X and Y collect all that I accept and all that I reject, including those commitments upon which my judgement about the evidential status of A is based. Isn't it the case that I should not accept A and I should not reject A? Aren't accepting A and rejecting A both ruled out on logical grounds, relative to accepting X and denying Y? Isn't this a counterexample to [*Cut*]?

No, they aren't. There need not be anything *incoherent* in accepting A (or in rejecting A) while accepting that there is no evidence for or against A. Accepting A is something we may (hypothetically) consider when we try adding A as an hypothesis and see what this addition does to our commitments. If that *were* inconsistent with what we had granted, this would form the basis of a reductio argument against A. Yes, there is a *tension* in accepting A while granting that there is no evidence for it. However it is readily apparent that we may hypothetically consider what would follow if A were actually the case.

[7]Remember, in these cases, we do not merely take accepting A or rejecting B to be unacceptable for going past the evidence. It is ruled out by those other claims we accept and reject.

But this means that there is a multitude of conditional connectives: each $\to_n$ supports *modus ponens* and acts just like a conditional connective, where we set $A \to_1 B$ to be $A \to B$, and $A \to_{n+1} B$ to be $A \to (A \to B)$.[8] Given such a menagerie of conditional connectives, how are we to state (NC)? On what basis should I think that $\phi(a) \leftrightarrow a \in \langle\!\langle x : \phi(x) \rangle\!\rangle$ instead of $\phi(a) \leftrightarrow_2 a \in \langle\!\langle x : \phi(x) \rangle\!\rangle$ or $\phi(a) \leftrightarrow_{3,088} a \in \langle\!\langle x : \phi(x) \rangle\!\rangle$? How am I to decide between these? Given a sequent calculus, the matter seems more straightforward. To treat $\langle\!\langle\ :\ \rangle\!\rangle$ as a property abstraction is to be committed to the following rules of inference:

$$\frac{X, \phi(t) \Rightarrow Y}{X, t \text{ is } \langle\!\langle x : \phi(x) \rangle\!\rangle \Rightarrow Y} \text{ [is } L] \qquad \frac{X \Rightarrow \phi(t), Y}{X \Rightarrow t \text{ is } \langle\!\langle x : \phi(x) \rangle\!\rangle, Y} \text{ [is } R]$$

The logical status of accepting that t is $\langle\!\langle x : \phi(x) \rangle\!\rangle$ is that of accepting that $\phi(t)$. The same stands for rejecting. To deny [is L] is to take there to be some context of accepting X and rejecting Y where accepting $\phi(t)$ is logically ruled out, but where accepting $t \in \langle\!\langle x : \phi(x) \rangle\!\rangle$ is *not* logically ruled out. (How could that be? If it is not ruled out to accept $t \in \langle\!\langle x : \phi(x) \rangle\!\rangle$ then why can we not infer $\phi(t)$? To take $\langle\!\langle\ :\ \rangle\!\rangle$ to denote properties in the naïve theory is to take [is L] to be valid.) Similarly, to deny [is R] is to take there to be some context of accepting X and rejecting Y where rejecting $\phi(t)$ is logically ruled out, but where rejecting $t \in \langle\!\langle x : \phi(x) \rangle\!\rangle$ is *not* logically ruled out.

To take $\langle\!\langle\ :\ \rangle\!\rangle$ to be *coarse* property abstraction requires commitments concerning identity. Field does not discuss the logic of identity in *Saving Truth From Paradox*, for good reason—it is rather difficult in the context of non-classical logics. I need not discuss much in the logic of identity here, but I need one general principle.

$$\frac{X, t \text{ is } T \Rightarrow Y}{X, t \text{ is } S, S = T \Rightarrow Y} \text{ [} = L]$$

If accepting that t is T is ruled out (given accepting X and rejecting Y) so is accepting that t is S and that $S = T$. To deny [$= L$] is to countenance some context in which one cannot accept that t is T but in which one does (coherently) accept that t is S and that $S = T$. To countenance *such* a context seems to be a great deal to bear, given that we wish to treat '$=$' as *identity*.[9] The point of

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[8] I am tempted to add $\to^0$ where $A \to_0 B$ is B, but I agree, *that* is not so much a conditional connective. The connectives $\to_n$ where $n \geqslant 1$ each act like a conditional in that they support *modus ponens*, and some form of the deduction theorem if that theorem is satisfied by the ground level conditional $\to_1$. If $\to_1$ satisfies the inference of *weakening* ($A \Rightarrow B \to_1 A$) then the connectives $\to_n$ are progressively weaker ($A \to_n B \Rightarrow A \to_{n+1} B$).

[9] This is a simple example of more general principles concerning identity:

$$\frac{X, \phi(a) \Rightarrow Y}{X, \phi(b), a = b \Rightarrow Y} \text{ [} = L_1] \qquad \frac{X \Rightarrow \phi(a), Y}{X, a = b \Rightarrow \phi(b), Y} \text{ [} = L_2]$$

we need only the very simple form [$= L$] so we do not need to discuss the nature of the contexts '$\phi(\ )$' in which an identity may be validly substituted.

having an identity predicate was to state what it is for '$\langle\!\langle\ :\ \rangle\!\rangle$' to pick out coarse properties. The *intentional* criterion for identity for properties is this:

$$\frac{\phi(a) \Rightarrow \psi(a) \qquad \psi(a) \Rightarrow \phi(a)}{\Rightarrow \langle\!\langle x : \phi(x)\rangle\!\rangle = \langle\!\langle x : \psi(x)\rangle\!\rangle}\ [=Int]$$

If accepting $\phi(a)$ and rejecting $\psi(a)$ is ruled out on logical grounds *independently of all other commitments*, and if accepting $\psi(a)$ and rejecting $\phi(a)$ is ruled out on logical grounds, then the coarse property of being ϕ is the coarse property of being ψ. We can state this principle as an inference rule, again with no extraneous connectives. To deny that $\langle\!\langle x : \phi(x)\rangle\!\rangle$ and $\langle\!\langle x : \psi(x)\rangle\!\rangle$ are the same coarse property is to grant that one can coherently countenance accepting $\phi(a)$ and rejecting $\psi(a)$, or the converse.

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Now we have all the raw materials for the paradoxical argument. Despite the seeming straightforwardness of coarse properties, they provide the means to derive inconsistency. The simplest proof I know of is a generalisation of a result of Roland Hinnion [4]. We start with a definition of a property, rather like the problematic hetorological property, except we will not use negation in its definition. Instead, we will use a statement which is to be rejected on the grounds of logic alone. $\perp$ is such a statement, if it satisfies the inference principle:

$$X, \perp \Rightarrow Y \quad [\perp L]$$

Given such a proposition, we may define the following problematic coarse property: $P := \langle\!\langle x : \langle\!\langle y : x \text{ is } x\rangle\!\rangle = \langle\!\langle y : \perp\rangle\!\rangle\rangle\!\rangle$. What is this property? Let's consider it piece by piece.

First, $\langle\!\langle y : \perp\rangle\!\rangle$ is a property that y has if and only if $\perp$. In other words, it is a property that *nothing* has, as a matter of logic. Here, we abstract on $\perp$, in which the variable y is not free. We could eliminate this vacuous binding by replacing '$\perp$' by '$\perp \wedge (\perp \vee y = y)$' or some such sentence in which 'y' occurs in the syntax but is logically intert (since $\perp \wedge (\perp \vee y = y)$ is logically equivalent to $\perp$). But this would needlessly complicate matters.

Similarly, $\langle\!\langle y : x \text{ is } x\rangle\!\rangle$ is a property that y has if and only if x instantiates itself. So, $\langle\!\langle x : \langle\!\langle y : x \text{ is } x\rangle\!\rangle = \langle\!\langle y : \perp\rangle\!\rangle\rangle\!\rangle$ is the property of being an x such that the property that *everything* has if x is self-instantiating is the same property that *nothing* has, as a matter of logic.

The derivation in Figure 1 shows that the presence of property P reduces the inference principles [*Id*], [*Cut*], [is *L*], [is *R*], [= *L*], [= *Int*] and [$\perp L$] to absurdity. I will not step through every inference here, but just a few, to explain how the moves go. Consider the top left fragment of the derivation, to the step 'P is P $\Rightarrow \perp$,' where we expand the abbreviation 'Q' to '$\langle\!\langle y : P \text{ is } P\rangle\!\rangle$'. The result is a derivation showing that there is a clash involved in accepting that P is P

$$\dfrac{\dfrac{\text{P is P} \Rightarrow \text{P is P}}{\text{P is P} \Rightarrow \text{a is Q}}\,[\text{is }R] \qquad \dfrac{\dfrac{\top \Rightarrow \top}{\text{a is }\langle\!\langle y:\top\rangle\!\rangle \Rightarrow \top}\,[\text{is }L]}{\text{a is Q}, Q = \langle\!\langle y:\bot\rangle\!\rangle \Rightarrow \bot}\,[=L]}{\dfrac{\dfrac{\text{P is P}, Q = \langle\!\langle y:\bot\rangle\!\rangle \Rightarrow \bot}{\text{P is P} \Rightarrow \bot}\,[\text{is }L]}{\dfrac{\text{a is Q} \Rightarrow \bot}{\text{a is Q} \Rightarrow \text{a is }\langle\!\langle y:\bot\rangle\!\rangle}\,[\text{is }R]}}\,[Cut] \qquad \dfrac{\bot \Rightarrow \text{a is Q}}{\text{a is }\langle\!\langle y:\bot\rangle\!\rangle \Rightarrow \text{a is Q}}\,[\text{is }L]}{\dfrac{\Rightarrow Q = \langle\!\langle y:\bot\rangle\!\rangle}{\Rightarrow \text{P is P}}\,[\text{is }R]}\,[=Int]$$

$$\dfrac{\dfrac{\text{P is P} \Rightarrow \text{P is P}}{\text{P is P} \Rightarrow \text{a is Q}}\,[\text{is }R] \qquad \dfrac{\dfrac{\top \Rightarrow \top}{\text{a is }\langle\!\langle y:\top\rangle\!\rangle \Rightarrow}\,[\text{is }L]}{\text{a is Q}, Q = \langle\!\langle y:\bot\rangle\!\rangle \Rightarrow}\,[=L]}{\dfrac{\text{P is P}, Q = \langle\!\langle y:\bot\rangle\!\rangle \Rightarrow}{\text{P is P} \Rightarrow}\,[\text{is }L]}\,[Cut]}{\Rightarrow}$$

ABBREVIATIONS: P is $\langle\!\langle x : \langle\!\langle y : x \text{ is } x\rangle\!\rangle = \langle\!\langle y:\bot\rangle\!\rangle\rangle\!\rangle$; Q is $\langle\!\langle y : \text{P is P}\rangle\!\rangle$.   AXIOMS: $X, A \Rightarrow A, Y\,[Id]$ $X, \bot \Rightarrow Y\,[\bot L]$

RULES: $\dfrac{X \Rightarrow A, Y \quad X', A \Rightarrow Y'}{X, X' \Rightarrow Y, Y'}\,[Cut]$ $\dfrac{X, \phi(t) \Rightarrow Y}{X, t \text{ is } \langle\!\langle x:\phi(x)\rangle\!\rangle \Rightarrow Y}\,[\text{is }L]$ $\dfrac{X \Rightarrow \phi(t), Y}{X \Rightarrow t \text{ is } \langle\!\langle x:\phi(x)\rangle\!\rangle, Y}\,[\text{is }R]$ $\dfrac{X, t \text{ is } T \Rightarrow Y}{X, t \text{ is } S, S = T \Rightarrow Y}\,[=L]$ $\dfrac{\phi(a) \Rightarrow \psi(a) \quad \psi(a) \Rightarrow \phi(a)}{\Rightarrow \langle\!\langle x:\phi(x)\rangle\!\rangle = \langle\!\langle x:\psi(x)\rangle\!\rangle}\,[=Int]^{*}$

*In each rule, t is an arbitrary term, but in the rule [= Int], the eigenvariable a must not occur in the conclusion.

Figure 1: A derivation of absurdity, and the principles used in it.

and rejecting $\perp$. This is used on the way to derive $\Rightarrow$ P is P, which shows us that there is a clash in rejecting that P is P, on the way to showing that there is invariably a clash, even in the empty sequent ' $\Rightarrow$ '.

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\cfrac{
\cfrac{P \text{ is } P \Rightarrow P \text{ is } P}{P \text{ is } P \Rightarrow a \text{ is } \langle\!\langle y : P \text{ is } P \rangle\!\rangle} [\text{is } R] \quad
\cfrac{
\cfrac{
\cfrac{\perp \Rightarrow \perp}{a \text{ is } \langle\!\langle y : \perp \rangle\!\rangle \Rightarrow \perp} [\text{is } L]
}{a \text{ is } \langle\!\langle y : P \text{ is } P \rangle\!\rangle, \langle\!\langle y : P \text{ is } P \rangle\!\rangle = \langle\!\langle y : \perp \rangle\!\rangle \Rightarrow \perp} [= L]
}{} [Cut]
}{
\cfrac{P \text{ is } P, \langle\!\langle y : P \text{ is } P \rangle\!\rangle = \langle\!\langle y : \perp \rangle\!\rangle \Rightarrow \perp}{P \text{ is } P \Rightarrow \perp} [\text{is } L]
}
$$

Now, in the left branch of this derivation, the first step takes us from P is P to a is $\langle\!\langle y : P \text{ is } P \rangle\!\rangle$, using the criterion for instantiating $\langle\!\langle y : P \text{ is } P \rangle\!\rangle$, [is R]. Since a instantiates $\langle\!\langle y : P \text{ is } P \rangle\!\rangle$ if and only if P is P; if there is a problem with denying that P is P, then that problem remains in denying that a is $\langle\!\langle y : P \text{ is } P \rangle\!\rangle$. For the right, from $\perp \Rightarrow \perp$ we derive a is $\langle\!\langle y : \perp \rangle\!\rangle$ in exactly the same way, but on the left of the arrow. If there is a clash in accepting $\perp$ (as there is), the clash remains if we accept that a is $\langle\!\langle y : \perp \rangle\!\rangle$. Now, any clash in accepting that a is $\langle\!\langle y : \perp \rangle\!\rangle$ remains when we accept that a is $\langle\!\langle y : P \text{ is } P \rangle\!\rangle$ together with the identity $\langle\!\langle y : P \text{ is } P \rangle\!\rangle = \langle\!\langle y : \perp \rangle\!\rangle$. This is the logic of identity, in [= L]. Now, we combine these two conclusions about clashes, with a [Cut]. We must have a clash in accepting that P is P and that $\langle\!\langle y : P \text{ is } P \rangle\!\rangle = \langle\!\langle y : \perp \rangle\!\rangle$ and rejecting $\perp$, since if there was no clash here, then there would either not be a clash in adding the rejection of a is $\langle\!\langle y : P \text{ is } P \rangle\!\rangle$, or in accepting a is $\langle\!\langle y : P \text{ is } P \rangle\!\rangle$ (remember, this clash is a *logical* notion, nothing weaker than that). We have seen that there is a clash on both those sides, so the clash remains when we cut out a is $\langle\!\langle y : P \text{ is } P \rangle\!\rangle$, and there is a clash in accepting that P is P and that $\langle\!\langle y : P \text{ is } P \rangle\!\rangle = \langle\!\langle y : \perp \rangle\!\rangle$ and rejecting $\perp$. But finally, look! What is it to accept that $\langle\!\langle y : P \text{ is } P \rangle\!\rangle = \langle\!\langle y : \perp \rangle\!\rangle$? It is (by [is L]) to accept that P is some x where $\langle\!\langle y : x \text{ is } x \rangle\!\rangle = \langle\!\langle y : \perp \rangle\!\rangle$. It is to accept, in other words, that P is P. So, if there is a clash in accepting that P is P and that $\langle\!\langle y : P \text{ is } P \rangle\!\rangle = \langle\!\langle y : \perp \rangle\!\rangle$ and rejecting $\perp$, that clash remains merely in accepting that P is P and rejecting $\perp$.

The rest of the derivation continues in the same way, and I will not step through the other inferences in this detail. They are displayed in Figure 1. The concluding empty sequent ' $\Rightarrow$,' if derivable, tells us that *every* combination of accepting an rejecting is ruled out on logical grounds. This is, of course, absurd. So, something must have gone wrong *somewhere* in this derivation. The candidates are:

1. [*Id*]. This seems above reproach. To reject [*Id*] is to allow the simultaneous accepting and rejecting of the one statement as acceptable on logical grounds. It would be more than disquieting face rejecting something as fundamental as this on the way to saving truth from paradox.

2. [$\perp L$]. To reject [$\perp L$] is to deny that there is a statement that may be rejected on the grounds of logic alone. While there are some accounts of

logic for which there is no *worst* proposition, none of the systems dis-
cussed by Field have this feature.

3. [*Cut*]. To reject [*Cut*] is to take it that in some contexts which are them-
selves logically coherent, for soem statement A both accepting A and re-
jecting A are ruled out on the basis of logic alone. But what could such
circumstances be? How else do we reason to the rejection of some claim
other than by showing that it is *unacceptable*? At the very least, were
we to reject [*Cut*] we must give a richer and clearer account of the con-
nections between accepting and rejecting than we already have. Field has
gone some way to illuminate these issues in *Saving Truth From Paradox*.
If he is to save naïve property theory from paradox by rejecting [*Cut*], he
must shed more light on these connections.

4. [= *L*]. To reject [= *L*] is to countenance accepting that t is S and that S = T
but rejecting that t is T. What is the point of having an identity predicate
if it allows for *this*? Rejecting [= *L*] will require filling out a theory of
identity, showing not only that plausible rules of identity are respected in
a non-classical theory, but in showing what is going on in contexts like
this where other plausible identity rules are rejected.

5. [= *Int*]. To reject [= *Int*] is to reject a coarse account of properties.[10] It
is to not only accept a finer individuation of properties where logically
equivalent statements pick out distinct properties, but to hold that any at-
tempt at defining coarse properties in terms of fine ones must fail. But at
what point does that construction break down? It seems like a straightfor-
ward construction of equivalence classes or their representatives. To reject
[= *Int*] is also to leave open the vexed question of what the identity con-
ditions for properties *can* be, and *should* be. Just what is a property that
requires that they be more finely individuated than logic requires? Field's
model construction doles out properties in exactly the measure of our lan-
guage. But it would be bizarre to think that this is a adequate picture of
properties. How convenient it would be if properties fit our language like
hand to glove? But which language? My language now? Or yours? How
are we to get to an adequate understanding of the properties picked out by
this naïve theory of properties?

6. [is *L*] & [is *R*]. To reject either of these is to reject the naïve theory of prop-
erties, and to undercut the entire thrust of *Saving Truth From Paradox*.

Which option will you take? Each choice has its costs and its benefits. No choice
is easy. This is not just an issue for Field and those who want to follow the
account in *Saving Truth From Paradox*. It is a paradox for everyone. The point

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[10]But not *very* coarse: recall, we do not mean to identify necessarily coextensive properties,
merely *logically* coextensive ones.

is sharpest, of course, for those like Beall [1], Priest [5], my former self [6] and anyone else who thinks that a non-classical logic could help with the semantic paradoxes. In the paradoxical deduction of Figure 1, there is very little *logic* which we can make non-classical. Connectives and quantifiers do not appear in the deduction: we are left with property abstraction, instantiation, identity and a false proposition. To address the paradox, we must get to the heart of what properties are, how instantiation works, how identity applies to properties, and what we are doing when we are deriving things—how accepting and rejecting are to be managed. Any funny business with the conditional or with negation, with gaps or gluts in truth value, is beside the point. Or at the very least it is beside *this* point.

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