Structural Rules in Natural Deduction with Alternatives

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Abstract

Natural deduction with alternatives extends Gentzen–Prawitz-style natural deduction with a single structural addition: negatively signed assumptions, called *alternatives*. It is a mildly bilateralist, single-conclusion natural deduction proof system in which the connective rules are unmodified from the usual Prawitz introduction and elimination rules — the extension is purely structural. This framework is general: it can be used for (1) classical logic, (2) relevant logic without distribution, (3) affine logic, and (4) linear logic, keeping the connective rules fixed, and varying purely structural rules.

The key result of this paper is that the two principles that introduce kinds of *irrelevance* to natural deduction proofs: (a) the rule of explosion (from a contradiction, anything follows); and (b) the structural rule of vacuous discharge; are shown to be two sides of a single coin, in the same way that they correspond to the structural rule of weakening in the sequent calculus. The paper also includes a discussion of assumption classes, and how they can play a role in treating additive connectives in substructural natural deduction.

1 Proofs and Sequents

Gentzen–Prawitz-style natural deduction is an elegant way to present proofs. In this proof calculus, each connective is governed by introduction and elimination

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rules, and the structural features of proofs — the conditions governing propositions as such, unlike connective rules, which govern propositions of particular forms — are given by the proof's tree structure, together with rules governing discharge of assumptions [12]. To illustrate, consider the natural deduction system for intuitionistic linear logic. The simplest proof in this system is a formula standing on its own:

$$A$$

This is the *identity proof*, in which the conclusion is identical to the undischarged assumption, A. In this limiting case, the first thing that follows from the assumption of A is A itself, in zero inference steps. To keep matters simple, let's consider two connectives, the conditional ($\to$) and negation ($\neg$).

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\,{\to}I^i \qquad \dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\,{\to}E \qquad \dfrac{\begin{matrix}[A]^i \\ \Pi \\ \sharp\end{matrix}}{\neg A}\,\neg I^i \qquad \dfrac{\begin{matrix}\Pi \\ \neg A\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{\sharp}\,\neg E$$

Each inference rule builds a larger proof from smaller proofs (here marked with a Π or a Π'). (To be precise, in these statements of rules, a Π with a formula below it represents a proof *with that formula as its conclusion*. If, in addition, it has a formula above it (perhaps surrounded with brackets), it represents the proof with that formula among its assumptions.) In the elimination rules $\to E$ and $\neg E$, we form a proof by combining two proofs. For $\to E$ we combine one proof (Π) of $A \to B$ with another (Π') of A to form a proof of B. The resulting proof has, as its assumptions, all those assumptions used in Π, together with those used in Π'. For $\neg E$, we combine a proof of $\neg A$ with a proof of A. This introduces a new kind of conclusion, the symbol '$\sharp$'. This is not a formula, but is a punctuation mark,[1] indicating that the proof has reached a contradiction, because we have proved (from the assumptions granted in Π and Π') a contradictory pair of conclusions.

The mark '$\sharp$' is exploited in the rule $\neg I$ which allows us to backtrack when we have reached such a contradiction, by 'blaming' it on one of the undischarged assumptions — by discharging it and concluding its negation. A similar sort of move is made in the conditional introduction rule $\to I$. Here we prove a condition $A \to B$ by first proving B on the basis of the assumption A. We *discharge* the assumption A to conclude $A \to B$ on the basis of the remaining assumptions.

Here is an example proof, illustrating the use of all four rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \to \neg q \qquad [p]^1}{\neg q}\,{\to}E \qquad [q]^2}{\sharp}\,\neg E}{\neg p}\,\neg I^1}{q \to \neg p}\,{\to}I^2$$

[1]This treatment of falsity follows Neil Tennant [21]

This proof represents the process of reasoning from the premise $p \to \neg q$ as follows: we assume p to derive $\neg q$. We assume q and get a contradiction. We 'blame' that contradiction on the assumption of p, discharging it, to conclude $\neg p$, and so, we have proved $\neg p$ having assumed q, so we discharge that assumption, to conclude $q \to \neg p$.

The undischarged assumption of the proof, $p \to \neg q$, stands unbracketed as a leaf of the tree, and the conclusion, $q \to \neg p$ is at the root. Each transition in the proof is governed by an introduction or elimination rule. The two introduction steps discharge one assumption: the negation introduction discharges the assumption p (tagged with a '1') while the conditional introduction discharges the q (tagged with a '2').

The system with these rules models the implication/negation fragment of intuitionistic linear logic. (See Girard's fundamental paper [3] for an introduction to linear logic, and Troelstra's *Lectures on Linear Logic* [22] for a presentation of natural deduction for intuitionistic linear logic in a *sequent* format.) It is *intuitionistic* linear logic because (as is familiar) the proof system provides no way to prove p from $\neg\neg p$. It is *linear* because each introduction step is restricted to discharge one and only one occurrence of an assumption. As a result, we cannot (for example) prove $p \to q$ from the assumption of $p \to (p \to q)$ (that would require discharging *two* copies of p) and neither can we prove $p \to q$ from the assumption p (that would require discharging *zero* copies of p).

* * *

To extend this system to stronger logics, including intuitionistic logic, we can keep these connective rules largely unchanged, by adding purely *structural* rules to the calculus, managing the assumption classes used in the discharging rules $\neg I$ and $\to I$. To extend the system first to the system of *relevant* implication, we allow for more than one assumption instance to be discharged at once, we allow for proofs like this:

$$\dfrac{\dfrac{\dfrac{p \to (p \to q) \qquad [p]^1}{p \to q}{\scriptstyle \to E} \qquad [p]^1}{q}{\scriptstyle \to E}}{p \to q}{\scriptstyle \to I}$$

Proofs such as this allow for *duplicate* discharge, in which the set of discharged formula instances has size at least two.

To extend the system to *minimal* logic, we modify the discharge policy further, by allowing for for *any* number of instances of the indicated assumption to be discharged, including *zero*. With this in place, we have a very short proof of $p \to (q \to p)$.

$$\dfrac{\dfrac{[p]^1}{q \to p}{\scriptstyle \to I}}{p \to (q \to p)}{\scriptstyle \to I^1}$$

Here, *zero* instances of the assumption q are discharged at the first $\to E$ step. Let's say that a policy for discharging assumption classes allows for *vacuous* discharging if and only if it allows for proofs like this, where the set of discharged assumptions is empty.

Given that we can either allow or ban vacuous discharge, and allow or ban duplicate discharge, we have four different proof systems in one, given this simple set of rules. This is a natural deduction proof system for intuitionistic logic if we allow both vacuous and duplicate discharge. If we ban duplicate discharge while allowing vacuous discharge, we get *affine logic*. If we allow duplicate discharge while banning vacuous discharge, we get relevant logic, and if we ban both vacuous and duplicate discharge, we get *linear* logic.

Well, *almost*. There is one small wrinkle in this simple story. There is no vacuous discharge in the following proof, from $\neg p$ and p to q. This proof is not allowed in linear logic or in relevant logic, but it is intuitionistically acceptable:

$$\dfrac{\dfrac{\neg p \quad p}{\sharp}\,{\neg E}}{q}\,{\sharp E}$$

This proof does not use vacuous discharge (there is no discharge at all in the $\neg E$ inference). Instead, it uses the new primitve inference rule, an elimination principle for $\sharp$:

$$\dfrac{\begin{array}{c}\Pi \\ \sharp\end{array}}{A}\,{\sharp E}$$

The $\sharp E$ rule is another properly structural proof principle, governing the logical power of reaching an inconisistent state, and not governing any connective in particular. To extend our proof system all the way up to intuitionistic logic, we need to add $\sharp E$ as well as allowing duplicate and vacuous discharge. Conversely, to convert intuitionistic logic into a properly *relevant* logic, we must not only ban vacuous discharge—you must also ban $\sharp E$.

* * *

It is straightforward to verify that the natural deduction system with no vacuous discharge and no duplicate discharge gives us proofs for the implication/negation fragment of intuitionistic linear logic. This logic is given by the following single-conclusion sequent system, in which sequents consist of a *multiset* of formulas on the left and either a single formula on the right, or an *empty* right hand side. (The empty right hand side plays the same sort of role in a sequent as the contradiction marker $\sharp$ does in the conclusion of a proof.) We use 'C' to range over possible inhabitants of the conclusion position, so here, 'C' is either a formula or the empty RHS, while 'A' and 'B' always stand for formulas, and 'X' and 'X′' range over arbitrary multisets of formulas. The structural rules are *Id* and *Cut*:

$$A \succ A \; \mathit{Id} \qquad \dfrac{X \succ A \quad X', A \succ C}{X, X' \succ C}\,\mathit{Cut}$$

The connectives are governed by the expected left and right rules.

$$\frac{X \succ A \qquad X', B \succ C}{X, X', A \to B \succ C}{\to}L \qquad \frac{X, A \succ B}{X \succ A \to B}{\to}R \qquad \frac{X \succ A}{X, \neg A \succ}\neg L \qquad \frac{X, A \succ}{X \succ \neg A}\neg R$$

FACT 1 *There is a linear natural deduction proof from the premises* X *to the conclusion* C *if and only if there is a derivation of the sequent* $X \succ C$ *in the linear sequent calculus. (Following our convention concerning 'C', this means that there is a sequent derivation of* $X \succ$ *if and only if there is a natural deduction proof from* X *to* $\sharp$*. We allow 'C' to take the appropriate form whether occuring as a conclusion of a proof or the* RHS *of a sequent.)*

Proof: From left to right, this can be verified by a simple induction on the construction of the proof. The base case proof is the identity proof A, which corresponds exactly to the identity sequent $A \succ A$. Now, for the induction steps, consider the ways to generate new proofs from old. For ${\to}I$, suppose we have a proof from assumptions X together with one given occurrence of the assumption A to conclusion B and we discharge that occurrence of A in a ${\to}I$ step to deduce $A \to B$. The induction hypothesis delivers us a derivation of $X, A \succ B$, which can be extended to a derivation of $X \succ A \to B$, by ${\to}R$, as desired.

For ${\to}E$, suppose we have a proof from X to $A \to B$ and another from X' to A, and we combine these into a proof from X, X' to B. The induction hypothesis delivers us derivations of $X \succ A \to B$ and $X' \succ A$. Using *Cut* and ${\to}R$ we can construct the desired derivation of $X, X' \succ B$ like this:

$$\dfrac{X \succ A \to B \qquad \dfrac{X' \succ A \qquad \dfrac{}{B \succ B}Id}{X', A \to B \succ B}{\to}L}{X, X' \succ B}Cut$$

The cases for the negation rules parallel the conditional rules precisely, so leaving these as an exercise, I will declare this part of the proof done.

For the right-to-left direction of the equivalence, we show how we can construct a proof from X to C, given a derivation of $X \succ C$ (whether C is a formula or $\sharp$). If our derivation is a simple appeal to *Id* ($A \succ A$) we have the atomic proof featuring the assumption A standing alone as both assumption and conclusion. For *Cut*, we paste together a proof from X to A to a proof from X', A to C to construct the combined proof from X and X' to C, going through A as an intermediate step.[2]

$$\begin{array}{ccc} & & X \\ & & \Pi_1 \\ X' & & A \\ \hdashline & \Pi_2 & \\ & C & \end{array}$$

[2] Here, the dashed line above Π_2 indicates that the subproof Π_2 has the formulas listed in X' and A together as its undischarged leaves.

The connective rules on the left and right correspond neatly to the corresponding applications of the elimination and introduction rules. For $\to L$, suppose we already have a proof Π_1 from X to A and a proof Π_2 from X', B to C we construct a proof from $X, X', A \to B$ to C like this:

$$\dfrac{X' \quad \dfrac{A \to B \qquad \begin{matrix} X \\ \Pi_1 \\ A \end{matrix}}{B}{\to}E}{\begin{matrix} \Pi_2 \\ C \end{matrix}}$$

Similarly, given a proof from X, A to B, we can discharge that instance of A in the assumptions in one $\to I$ step to construct a proof from X to $A \to B$. The reasoning for the negation rules has the same shape, so again, we can declare the proof complete. ∎

So, we can see that the sequent calculus and the natural deduction system for linear implication and negation mirror each other.

To extend the sequent calculus to model relevant logic, affine logic and intuitionistic logic, we can add the structural rules of contraction (on the left) and weakening (both on the left and on the right), like so:[3]

$$\dfrac{X, A, A \succ C}{X, A \succ C}\,W \qquad \dfrac{X \succ C}{X, A \succ C}\,KL \qquad \dfrac{X \succ}{X \succ B}\,KR$$

Using *contraction* (W), we can implement in the sequent calculus the behaviour of *duplicate discharge* in natural deduction. If we wish to discharge more than one instance of the assumption formula A in a $\to I$ step, then in the derivation, you may contract those copies of A in the left of the sequent down to one, with W, and then you are in a position to apply $\to R$. Using *weakening* on the left (KL), we can do the work of *vacuous* discharge in natural deduction. Wherever we would vacuously discharge an assumption formula in some inference, in the sequent calculus we insert that formula using KL to be in a position to apply the right rule, introducing a conditional or a negation.

However, once we add these structural rules, the parallel between the sequent calculus and natural deduction is less direct and straightforward than it is in the linear case. Consider the following derivation of the sequent $p \to (p \to q), p \succ q$, using contraction:

$$\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ p \qquad q \succ q}{p \to q, p \succ q}{\to}L}{p \to (p \to q), p, p \succ q}{\to}L}{p \to (p \to q), p \succ q}\,W$$

[3] We use 'W' for contraction and 'K' for weakening, following the names from Combinatory Logic. Haskell Curry named the contraction combinator 'W' (for the combinator satisfying $(Wxy) = (xyy)$), since 'W' is reminiscent of repetition [1]; while Schönfinkel's 'K' (for the combinator satisfying $(Kxy) = x$) stands for '*Konstanzfunktion*' [18].

This sequent derivation in some sense 'says' that there is a proof of q from $p \to (p \to q)$ and p—from one copy of each. There indeed *is* a natural deduction proof from $p \to (p \to q)$ and p to q, but there is no such proof that simply uses two steps of $\to E$, in the way that this derivation uses two steps of $\to L$. In our natural deduction system, the job of contraction is accomplished at the points where we *discharge* assumptions, in $\to I$ and $\neg I$ inferences. Our proof which uses only one copy of p among the assumptions goes like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \to (p \to q) \qquad [p]^1}{p \to q}{\scriptstyle \to E} \qquad [p]^1}{q}{\scriptstyle \to E}}{p \to q}{\scriptstyle \to I^1} \qquad p}{q}{\scriptstyle \to E}$$

This proof manages to get to the conclusion q from one copy each of the premises $p \to (p \to q)$ and p, but it does so at the cost of making an initial detour, constructing $p \to q$ and immediately breaking it down again. It does more work than seems appropriate in deriving q from those premises. This is our first hint that we may not yet have the clearest understanding of the behaviour of structural rules, like weakening and contraction, in Prawitz-style natural deduction.

* * *

However, there is a more pressing issue concerning the behaviour of structural rules in natural deduction, and that is the extension of our simple natural deduction system to extend to *classical* logic, and to the classical variants of the implication/negation fragments of linear logic, relevant logic and affine logic. If we extend the sequent calculus to allow for more than one formula on the right, like this —

$$A \succ A \; {\scriptstyle Id} \qquad \dfrac{X \succ A, Y \qquad X', A \succ Y'}{X, X' \succ Y, Y'}{\scriptstyle Cut}$$

$$\dfrac{X \succ A, Y \qquad X', B \succ Y'}{X, Y, A \to B \succ Y, Y'}{\scriptstyle \to L} \qquad \dfrac{X, A \succ B, Y}{X \succ A \to B, Y}{\scriptstyle \to R}$$

$$\dfrac{X \succ A, Y}{X, \neg A \succ Y}{\scriptstyle \neg L} \qquad \dfrac{X, A \succ Y}{X \succ \neg A, Y}{\scriptstyle \neg R}$$

— it is well known that we get fully dualising behaviour from these rules. For example, we can derive *double negation elimination* as well as *introduction*. The fully left–right symmetric sequent calculus allows for this symmetric pair of derivations:

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}{\scriptstyle \neg L}}{p \succ \neg\neg p}{\scriptstyle \neg R} \qquad \dfrac{\dfrac{p \succ p}{\succ \neg p, p}{\scriptstyle \neg R}}{\neg\neg p \succ p}{\scriptstyle \neg L}$$

Can we extend Prawitz-style natural deduction with purely structural rules, so as to do justice to derivations like these, which make use of more than one formula on the right hand side? This is one motivation for a *bilateralist* proof system, in which there is a full symmetry between premise and conclusion, between assertion and denial, and between left and right. The most direct attempts to expand natural deduction in this fully symmetric direction is to propose proof systems with *multiple conclusions* [14–16, 19], in addition to the multiple premises available in a Gentzen–Prawitz-style proof. This extension of natural deduction in a fully bilateral format is well-motivated, but to get the details correct, one must move beyond tree-like structures to graphs [19, see Parts I and II], and the correspondence with natural deduction becomes less direct.[4]

Another way to extend natural deduction in a bilateral direction is to allow for *negatively* decorated formulas (for *rejection* or *denial*), as well as *positive* formulas [20]. In modern renderings of this kind of bilateralist natural deduction, we assign every formula in a proof a *sign*, either '$+$' or '$-$', for assertion and denial respectively [5, 17]. This provides a neat way to pair full symmetry between positive and negative position, in a structure with many *premises* and a single *conclusion*. A proof from $+A$, $-B$, $+C$ to $+D$ can do duty for the sequent $A, C \succ B, D$, since it reassures us that there is no way for A and C to be *true* while B and D are *false*, or equally, it is inconsistent to *accept* A and C and *reject* B and D, or to put things in terms of speech acts, to *assert* A and C and *deny* B and D. This sequent corresponds to other proofs, too, such as a proof from $-B$, $-D$, $+C$ to $-A$. By decorating formulas with '$+$' or '$-$', we can move them between premise to conclusion position in a proof as desired. Since formulas can appear both positively and negatively signed, instead of each connective being defined by *two* rules, they have *four*: introduction and elimination rules for both positively and negatively signed occurrences. Such *fully bilateralist* natural deduction systems are interesting and powerful, but as we will see, they add to the natural deduction framework more than is strictly necessary to ford the chasm between intuitionist and classical natural deduction, and the substructural variants thereof. It is possible to be bilateralist in a much less drastic manner, and to still get all the power of classical reasoning. In the rest of this paper, we will see how.

* * *

Before introducing the structural addition to proofs that suffices for mild bilateralism, there is one more modification to natural deduction that is worth mentioning, the *Restart* rule of Michael and Murdoch Gabbay [2]. The restart rule:

$$\frac{A}{B}\,\textit{Restart}$$

[4] A simple case that shows the problem is this. If we would like a downward branching disjunction rule (from $A \vee B$ you branch to two conclusions, one A and the other B) in parallel to the upward branching conjunction rule (infer $A \wedge B$ from two premises, one A and one B), then there seems to be no way to construct a proof from $p \vee p$ to $p \wedge p$ without in some way breaking the tree shape of proofs.

is an addition to natural deduction for intuitionistic logic that is indeed sufficient to capture classical logic. Of course the rule does not apply without restrictions. A proof using the *Restart* rule is complete only when *below* every application of a *Restart* inference from A to B there is at least one further occurrence of A. Surprising as it is, natural deduction extended with this rule is indeed sound and complete for classical logic. Before explaining why, let's see a complete proof of the classical tautology $((p \to q) \to p) \to p$ using *Restart*:

$$\dfrac{\dfrac{[(p \to q) \to p]^2 \qquad \dfrac{\dfrac{[p]^1}{q}\,\textit{Restart}}{p \to q}\,{\to}I^1}{p}\,{\to}E}{((p \to q) \to p) \to p}\,{\to}I^2$$

In this proof, the *Restart* in the first inference is paid off when we return to p in the second last inference. Here is why the restart rule is sound and complete for classical logic. Suppose have a proof from premises X to a conclusion A. So, we have $X \succ A$. Then, if we *restart* to introduce B, the 'score' is now $X \succ B, A$. The A does not *go away*, as it were. We just set it aside (as an alternative conclusion) to insert another conclusion in its place. The *Restart* rule at the point of application is a kind of weakening on the right (*KR*). To make explicit the idea that the proof still has a single formula in the conclusion, let's represent the sequent in the form $X \succ C; Y$ where C is the formula (or $\sharp$, perhaps) in conclusion position, and Y collects together the other conclusions we have discarded along the way whenever we have applied *Restart*.

What, then, is the point of the side condition to the effect that we must return to the discarded formula A? When we return to a previously discarded conclusion, A, the score is $X \succ A; A, Y$. We declare the restart step *complete* and the formula is removed from the discard pile: so the score is then $X \succ A; Y$. This side condition, therefore, is an application of *contraction* on the right hand side of the sequent (*WR*). If we complete *every* restart step in a proof, the discard pile is empty, the score has the shape $X \succ A;$ —and the proof is indeed a justificiation of the conclusion on the basis of the undischarged assumptions.

The *Restart* rule is an ingenious addition to natural deduction that happens to be tailor-made for classical logic. However, the rule encodes both contraction and weakening, so it is ill-suited to substructural variants of classical logic. Furthermore, it is difficult to see how it can be motivated on explicitly *bilateralist* lines. Nonetheless, it contains the kernel of the idea of how we can make a small structural modification of natural deduction that suffices for this range of logical systems, and as we will see, this modification can be motivated by bilateralist considerations.

2 Natural Deduction with Alternatives

In any natural deduction proof, we have some collection X (possibly empty) of undischarged assumptions, and a concluding formula B, or a contradiction marker $\sharp$. If we wish the 'score' of our proof to encompass the whole range of sequents of the form $X \succ Y$ (as seems to be desirable, in order to match our classical systems), then if the conclusion formula is selected from the collection Y of formulas on the right hand side, we need some way to take care of the *remaining* formulas on the right, if there are any.

Let's use the notation that seemed natural when considering the restart rule, and think of the score in our proof as taking the shape $X \succ C; Y$ where C is the conclusion of the proof (whether a formula or $\sharp$), X collects together the undischarged assumptions, and Y is yet to be accounted for. The distinguished position in the right hand side of the sequent is the *focus*. At any stage of a proof, there is either a formula in the focus position (the conculding formula of the proof), or the focus is empty, in which case the proof concludes in $\sharp$. The restart rule manipulated the score by allowing us to remove a formula from the focus, and to place something else in its place (in this case, any other formula we please). If we wish to model any of our substructural logics, this is altogether too generous, since this corresponds to weakening our sequent by adding a new formula to the RHS. If we wish to move a formula out of focus, there is only one thing, in general, we can put in its place, if we wish to refrain from weakening. That is $\sharp$, or in sequent vocabulary, *nothing*.

The appropriate sequent rule to remove a formula from conclusion position has the following shape:

$$\frac{X \succ A; Y}{X \succ \ ; A, Y}\uparrow$$

Here, there is no contraction or weakening. We simply remove a formula from the focus position, and leaving nothing in its place. Formula occurrences are neither deleted (as happens in *contraction*) nor added (in *weakening*). A natural mate for the $\uparrow$ rule is its converse:

$$\frac{X \succ \ ; A, Y}{X \succ A; Y}\downarrow$$

This rule takes a formula out of the discard pile to return it to focus. Again, there is no implicit contraction or weakening involved.

Let's now consider how we can achieve the effect of these moves in a natural deduction framework. First, for the $\uparrow$ step, we move from a proof in which a given formula A is the conclusion, to a proof in which the conclusion is now $\sharp$, a contradiction. In this new proof, the formula A is now added to the discard pile, or the collection of alternative conclusions. In natural deduction proofs, one option to represent this formula A is among the leaves of the proof (the context against which the conclusion is derived), but we must find some way to distinguish this former conclusion — now set aside — from the other undischarged assumptions, also in the leaves of the proof tree. We do this with a *sign*, as with other bilateralist natural deduction systems. To emphasise the *negative* role played by these formulas, we will use a slash for the sign. (The slash through the entire formula should also bring to mind that it is not another connective, able to be composed with other connectives.) The corresponding proof step then takes this form:

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad \cancel{A}}{\sharp}\uparrow$$

This looks rather like the $\neg E$ rule in that a contradiction is derived from A and a negative version of A. However, there are two differences. The first is obvious: negation is an embeddable, composable content of a judgement — the negation of a formula can occur inside other formulas — while the slash here is a structural feature of proofs, and cannot be so embedded. The second is more subtle, but no less important: the negation elimination rule composes *two* proofs, one for A and the other for $\neg A$, into a single refutation, a proof ending in $\sharp$. The $\uparrow$ rule, on the other hand, does not compose two proofs. There is no proof ending in $\cancel{A}$. In this proof calculus, slashed formulas will appear only in leaves, and never as the conclusion of a proof. These formulas represent the conclusions we have temporarily set aside, and are stored among the leaves. Furthermore, unlike $\neg E$ which dictates the behaviour of a specific kind of formula, the $\uparrow$ rule is purely structural, allowing for the rearrangement of information around the proof structure, independently of the particular content or shape of the formula involved.

Why is this rule labelled with '$\uparrow$'? When we apply it, the formula A — which was the conclusion of the proof — is lifted up from the conclusion and stored among the leaves of the proof, where it takes its place as part of the context against which the conclusion is proved. For this reason, we also call it the *store* rule, and the conclusion formulas, temporarily stored up in the leaves are also called *alternatives*, since they are alternative candidates for conclusion, temporarily set aside for the sake of the argument. The converse of the *store* rule must do the reverse. It must *retrieve* an item kept in storage, to return it to the focus of the proof, its conclusion. Here is the appropriate shape in natural deduction:

$$\dfrac{\begin{matrix}[\cancel{A}]^i \\ \Pi \\ \sharp\end{matrix}}{A}\downarrow^i$$

Once we have proved a contradiction, we are in a position to select a stored formula (one instance only, in linear natural deduction) and discharging it, we return it to the conclusion. Before the retrieve step, the score was $X \succ\ ; A, Y$, and after, it is $X \succ A; Y$, when the A is retrieved from the storehouse of alternatives, to return to its place as a conclusion.

With these rules, we can mimic multiple-conclusion sequent derivations, despite the asymmetric shape of tree proofs. Here are proofs of double negation elimination, and Peirce's Law, the latter now making explicit how weakening ($\sharp E$) and contraction (duplicate discharge) play a role:

$$\dfrac{\dfrac{\neg\neg p \qquad \dfrac{\dfrac{[p]^1 \qquad [\cancel{p}]^2}{\sharp}\,\uparrow}{\neg p}\,\neg I^1}{\sharp}\,\neg E}{p}\,\downarrow^2 \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\cancel{p}]^2}{\sharp}\,\uparrow}{q}\,\sharp E}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [\cancel{p}]^2}{\sharp}\,\uparrow}{p}\,\downarrow^2}{((p \to q) \to p) \to p}\,{\to}I^3$$

This proof system is a purely structural extension of Prawitz-style natural deduction, changing it only with the addition of two structural rules, *store* and *retrieve*. This calculus is *bilateralist* because modifying the rules in this way allows for the *context* in which a formula is proved to have a twofold structure. A proof of A from the assumption formulas X and the alternatives Y is a proof corresponding to the sequent $X \succ A; Y$, and the intuitive interpretation is that A follows, provided that we have the means to rule X *in* and rule Y *out*.

Although this natural deduction calculus is bilateralist, it is bilateralist in a much milder manner than other bilateralist generalisations of natural deduction. We do not tag *every* formula in the proof, or add to the connective rules, and neither have we had to change the topology of proofs from the familiar tree structure. The *context* against which formulas are proved has been enlarged, but the remaining rules of the familiar natural deduction calculus are unchanged.

Although I have presented this natural deduction system as a more flexible sibling of Gabbay and Gabbay's natural deduction with restart, its origins go back further than their work. The proof system here is derived from Michel Parigot's $\lambda\mu$-calculus for classical logic [9–11]. The original contribution of this paper is twofold: first, rewriting the rules to make the connection with natural deduction and the sequent calculus more explicit, and second, formulating the store and retrieve rules so that the formulation applies equally to substructural systems of natural deduction. It is to the consideration of structural rules that we will now return, before finishing this paper with an indication of how rules for other connectives can be formulated, and a proof that the rules are indeed sound and complete for the substuctural multiple-conclusion sequent logics in question.

3 Weakening and Explosion

We have already seen that adding *irrelevance* to linear natural deduction comes in two distinct ways. Vacuous discharge, and $\sharp E$.

$$
\begin{prooftree}
\AxiomC{$[p]^1$}
\RightLabel{$\to I$}
\UnaryInfC{$q \to p$}
\RightLabel{$\to I^1$}
\UnaryInfC{$p \to (q \to p)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\neg p$}
\AxiomC{p}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\sharp E$}
\UnaryInfC{q}
\end{prooftree}
$$

These are distinct features of the natural deduction calculus. They are so distinct that we can have a proof system (for *minimal logic*) in which we have vacuous discharge without $\sharp E$. This is no longer so in the classical setting, in the presence of the store and retrieve rules. Given the retrieve rule, $\sharp E$ is no longer a separate distinct rule — it is simply the *vacuous* case of the retrieve inference. We can step from $\sharp$ to any given formula A by retrieving *zero* copies of the stored formula A. The proof from $\neg p$ and p to q now takes this form:

$$
\begin{prooftree}
\AxiomC{$\neg p$}
\AxiomC{p}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow$}
\UnaryInfC{q}
\end{prooftree}
$$

The natural deduction system with alternative rules *unifies* these two distinct kinds of irrelevance, by showing that they both count as forms of vacuous discharge.

The connection between $\sharp E$ and vacuous retrieval is a tight one, since if we have the store and retrieve rules with vacuous discharge of assumptions then we get the effect of $\sharp E$ whether we add vacuous discharge of *alternatives* as a primitive rule or not. Vacuous discharge comes as a package deal, in the presence of the store and retrieve rules. It is well known from minimal logic that from a contradiction we can infer an arbitary *negation*, including $\neg\neg q$ by vacuous discharge of the assumption $\neg q$, and so, using a store and retrieve two-step, we can infer the arbitrary q anyway:

$$
\begin{prooftree}
\AxiomC{$\neg p$}
\AxiomC{p}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I$}
\UnaryInfC{$\neg\neg q$}
\AxiomC{$[q]^1$}
\AxiomC{$[\cancel{q}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg q$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{q}
\end{prooftree}
$$

So, the store and retrieve rules of natural deduction with alternatives gives us a vantage point from which we can see the phenomena of *irrelevance* arising from one single source, the vacuous appeal to context, whether positive or negative.

4 Varieties of Conjunction

Let's add conjunction to our natural deduction system. It is well known that if we use the familiar Prawitz rules $\& I$ and $\& E$, we see that we can get the effect of vacuous discharge, by laundering our unused assumption (here q through an $\& I/\& E$

two step).

$$\dfrac{\dfrac{\dfrac{p \qquad [q]^1}{p \,\&\, q}\,\&I}{p}\,\&E}{q \to p}\to I^1$$

So, if we wish to do without weakening, we should not use $\&I$ together with $\&E$. One option is to start with the rule $\&I$ and to scout around for a rule that fits neatly with it, whether contraction or weakening are present, or not. The resulting connective is a *multiplicative* conjunction, and we will write '$\otimes$' to set multiplicative conjunction apart from other conjunctions. Given the familiar introduction rule $\otimes I$, the matching elimination rule is natural:

$$\dfrac{\begin{matrix}\Pi_1 \\ A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ B\end{matrix}}{A \otimes B}\otimes I \qquad\qquad \dfrac{\begin{matrix}\Pi_1 \\ A \otimes B\end{matrix} \qquad \begin{matrix}[A]^i, [B]^j \\ \Pi_2 \\ C\end{matrix}}{C}\otimes E^{i,j}$$

To eliminate a conjuction $A \otimes B$ we can derive anything we can derive from the conjuncts individually. In a linear context, we discharge one copy each of each conjunct. In the presence of contraction, we may discharge more copies. In the presence of weakening, we may discharge *zero* copies. The result is the expected behaviour of multiplicative conjunction in our systems, and we need not spend any time considering its distinctive behaviour, because in a sense, it brings nothing new to the table. Multiplicative conjunction is *definable* in terms of negation and the conditional in the way you expect: $A \otimes B$ is equivalent to $\neg(A \to \neg B)$, and the inference rules are derivable from the rules at hand. First, we can reconstruct the $\otimes$ introduction rule by combining two elimination steps with one introduction:

$$\dfrac{\begin{matrix}\Pi_1 \\ A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ B\end{matrix}}{A \otimes B}\otimes I \qquad\qquad \dfrac{\dfrac{\dfrac{[A \to \neg B]^1 \qquad \begin{matrix}\Pi_1 \\ A\end{matrix}}{\neg B}\to E \qquad \begin{matrix}\Pi_2 \\ B\end{matrix}}{\sharp}\neg E}{\neg(A \to \neg B)}\neg I^1$$

Dually, the job of the $\otimes$ elimination rule can be performed by two introduction steps with one elimination, combined with one storage and one retrieval:

$$\dfrac{\begin{matrix}\Pi_1\\ A\otimes B\end{matrix}\qquad \begin{matrix}[A]^i,[B]^j\\ \Pi_2\\ C\end{matrix}}{C}\,{\otimes}E^{i,j}
\qquad\qquad
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \neg(A\to\neg B)\end{matrix}\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[A]^i,[B]^j\\ \Pi_2\\ C\end{matrix}\qquad [\cancel{C}]^k}{\sharp}\,\uparrow}{\neg B}\,\neg I^j}{A\to\neg B}\,{\to}I^i}{\sharp}\,\neg E}{C}\,\downarrow^k$$

So, adding muliplicative conjunction gives us no increase in expressive power, over and above the rules already at hand.[5]

* * *

So, what of the other kind of conjunction, the *additive* conjunction, which is found when we start with Prawitz's elimination rule? Here the elimination rules are trivial, but the corresponding introduction rule is harder to find. At the level of *sequents* the target rules are straightforward:

$$\dfrac{X, A \succ C, Y}{X, A\wedge B \succ C, Y}\,\wedge L \qquad \dfrac{X, B \succ C, Y}{X, A\wedge B \succ C, Y}\,\wedge L \qquad \dfrac{X \succ A, Y \qquad X \succ B, Y}{X \succ A\wedge B, Y}\,\wedge R$$

The left rules correspond to the expected elimination rules for conjunction: if we can prove something from A we could have proved it from $A \wedge B$ instead — and the same goes if we could have proved it from B. The *right* rule, on the other hand, is hard to model in natural deduction. The intended behaviour is that if we can prove A and prove B *from the same context of assumptions* (whether positive or negative), then we can prove the conjunction $A \wedge B$ from that same context of assumptions. This is hard to model in the usual tree structure of natural deduction proofs. Consider the usual introduction rule:

$$\dfrac{\begin{matrix}\Pi_1\\ A\end{matrix}\qquad\begin{matrix}\Pi_2\\ B\end{matrix}}{A\wedge B}\,\wedge I$$

[5] We do not have space to consider normalisation of proofs here, but indeed, the expected normalisation behaviour for $\otimes I/E$ detours follows from the normalisation rules for the other connectives, together with $\uparrow$ and $\downarrow$. Cut elimination for the linear sequent calculus is very easy to show (in the absence of contraction, each cut reduction shrinks a derivation), and cut elimination for the extensions with contraction or weakening follows from standard techniques [8, 13]. Parigot shows strong normalisation for his classical natural deduction calculus (which differs slightly in structural rules from the calculus presented here, in ways that make no difference in the presence of contraction and weakening), and a close analysis of the reduction steps in Parigot's argument can apply in the four natural deduction systems presented here [9, 11]. However, a detailed consideration of normalisation must wait for another occasion.

Here, the rule *combines* the assumptions from from Π_1 and Π_2 into the larger collection of assumptions for the new proof. This does not have the desired effect, in the linear context.

One option, explored by Ernst Zimmermann [23], is to constrain $\wedge I$ in such a way as to require that the contexts in Π_1 and Π_2 are identical, but to then choose one side to *discharge* all assumptions in the context at the application of $\wedge I$:

$$\dfrac{\begin{matrix} X \\ \Pi_1 \\ A \end{matrix} \qquad \begin{matrix} [X]^i \\ \Pi_2 \\ B \end{matrix}}{A \wedge B}\,\wedge I^i$$

A rule of this form certainly has the desired shape: if we can prove A and prove B from the same context, then the result will be a proof of $A \wedge B$ from the very same context. However, the rule has one structural shortcoming, and this is that proofs no longer *compose*. That is, the following two proofs are acceptable:

$$\dfrac{p \wedge q}{p}\,\wedge E \qquad\qquad \dfrac{p \qquad [p]^1}{p \wedge p}\,\wedge I^1$$

However, we cannot compose these two proofs to form a proof from $p \wedge q$ to $p \wedge p$.

$$\dfrac{\dfrac{p \wedge q}{p}\,\wedge E \qquad [p]^1}{p \wedge p}\,\cancel{\wedge I^1}$$

This is not a proof, since the conjunction introduction rule is no longer a correct application in context, since the proofs of p no longer come from the same context.[6] So, while Zimmermann's discharging rule for additive conjunction is ingenious, I will set it aside for another option.

* * *

It will help to return to the discussion of structural rules from the first section, and to pay closer attention to the behaviour of assumptions in natural deduction proofs. An assumption class is a collection of occurrences of assumptions (of the same formula) in a proof, which are discharged together in one inference steps [6]. In our linear natural deduction system for $\rightarrow$ and $\neg$, assumption classes are always single formula occurrences. In the presence of multiple discharge, we allow

[6]Notice that the corresponding sequent derivation with a *Cut*, composing the derivation of $p \wedge q \succ p$ with that of $p \succ p \wedge p$ is unproblematic.

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,\wedge L \qquad \dfrac{p \succ p \qquad p \succ p}{p \succ p \wedge p}\,\wedge R}{p \wedge q \succ p \wedge p}\,Cut$$

for larger assumption classes, and in the presence of vacuous discharge, we allow for assumption classes to be empty. In proofs, we indicate assumption classes, where necessary, by superscript numerals. To treat additive conjunction — and to give a more detailed analysis of the behaviour of the structural rules — we will more closely examine this behaviour, by splitting the treatment of multiple discharge into two distinct phases. The first is the merging of two assumption classes into one, and the second is the discharge of that single assumption class. In this way, we will have the intermediate phase of the single assumption class occurring undischarged at two places in the proof. Since we indicate discharge with a notation with two components (the brackets and the superscript), we will use one component (the superscript) to indicate the assumption class, and the other (the brackets) to indicate discharge. With this notation in mind, consider the following two proofs, which differ only in one respect:

$$\dfrac{\dfrac{p \to (p \to q) \quad p}{p \to q}{\scriptstyle \to\!E} \qquad p}{q}{\scriptstyle \to\!E} \qquad\qquad \dfrac{\dfrac{p \to (p \to q) \quad p^1}{p \to q}{\scriptstyle \to\!E} \qquad p^1}{q}{\scriptstyle \to\!E}$$

In the first, the two occurrences of p occur in different assumption classes. In the second, the two occurrences are members of the same assumption class. In this second proof, the one act of assumption (an assumption that p) has been used *twice* in two separate $\to\!E$ inferences. In the first proof, the two assumptions p may have been assumed in two separate acts, or they may be justified by two separate processes.[7] The first proof is for the sequent $p \to (p \to q), p, p \succ q$, while the second is for $p \to (p \to q), p \succ q$, since there is only a single assumption class for p in use. The lacuna mentioned in Section 1 concerning contraction and natural deduction is now dealt with in a new way, using assumption classes.

One key feature of this treatment of assumption classes is their interaction with proof composition. If I compose my proof from $p \to (p \to q), p$ to q with another proof, say, from $p \wedge r$ to p, the composition should be a proof from $p \to (p \to q), p \wedge r$ to q, since we replace the assumption of p by the proof of p from the new assumption $p \wedge r$. Writing out the whole proof, we get this:

$$\dfrac{\dfrac{p \to (p \to q) \quad \dfrac{p \wedge r^1}{p}{\scriptstyle \wedge E}}{p \to q}{\scriptstyle \to\!E} \qquad \dfrac{p \wedge r^1}{p}{\scriptstyle \wedge E}}{q}{\scriptstyle \to\!E}$$

Here, the tree format requires that we insert the new proof at two places, and now the two occurrences of the new assumption $p \wedge r$ come from a single assumption class. In this way, we can compose proofs naturally, and without restriction.

[7] In a *type* theory, in which all formulas are types of terms, the difference is recorded by the identity or difference of variables used in assumptions. In the first proof, the formula p types two distinct variables, while in the latter, it types one variable, occurring twice in the proof.

With the addition of our explicit treatment of assumption classes, we need to revisit the formulation of each of our rules. The introduction rules $\to I$, $\neg I$ and the retrieve rule $\downarrow$ make use of assumption classes directly. In each case, each formula occurrence (unslashed assumptions in the case of the introduction rules, and slashed alternatives in the case of the retrieve rule) in a single assumption class is discharged, while the remaining assumption classes in the proof are undisturbed.

$$\dfrac{\begin{matrix}[A]^i\\ \Pi\\ B\end{matrix}}{A \to B}{\to}I^i \qquad \dfrac{\begin{matrix}[A]^j\\ \Pi\\ \sharp\end{matrix}}{\neg A}{\neg}I^j \qquad \dfrac{\begin{matrix}[\cancel{A}]^k\\ \Pi\\ \sharp\end{matrix}}{A}{\downarrow}^k$$

In addition, in the $\to E$ rule or a $\neg E$, in which two proofs are combined into one, if *contraction* is not in use, the assumption classes in the context of both proofs are kept separate. For example, a proof from X and Y to $A \to B$, combined with a proof from X' and Y' to A, using a $\to E$ step, gives us a proof from X, X' and Y, Y' to B. The assumption classes are not combined. If we are allowing contraction, in our proof we allow for some merging of assumption classes, as we have seen in the proof from $p \to (p \to q)$, p to q, in which two assumption occurrences of p are merged into the one assumption class. To represent this operation on assumption classes, let us use $\mathcal{C}$ and $\mathcal{C}'$ as natural deduction contexts (of assumptions and alternatives, grouped into classes). These inference steps take the form:

$$\dfrac{\begin{matrix}\mathcal{C}\\ \Pi\\ A \to B\end{matrix} \quad \begin{matrix}\mathcal{C}'\\ \Pi'\\ A\end{matrix}}{B}{\to}E \qquad \dfrac{\begin{matrix}\mathcal{C}\\ \Pi\\ \neg A\end{matrix} \quad \begin{matrix}\mathcal{C}'\\ \Pi'\\ A\end{matrix}}{\sharp}{\neg}E$$

Here, the context of the whole proof has the form $\mathcal{C} + \mathcal{C}'$ where this is the *disjoint* union of context classes in the case of linear natural deduction. If *contraction* is allowed, the requirement that this be a *disjoint* union is dropped: an *arbitrary* union is allowed.

With this treatment of assumption classes in hand, we can return to the additive conjunction rules. The rules take this format:

$$\dfrac{\begin{matrix}\mathcal{C}\\ \Pi_1\\ A\end{matrix} \quad \begin{matrix}\mathcal{C}\\ \Pi_2\\ B\end{matrix}}{A \wedge B}{\wedge}I \qquad \dfrac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{A}{\wedge}E \qquad \dfrac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{B}{\wedge}E$$

where the condition in the introduction rule is that the assumption classes in Π_1 and Π_2 are *identical*, and after the $\wedge I$ step, the assumption classes are *combined*, so that the assumption class for the whole proof remains $\mathcal{C}$. (Rules for additive connectives in substructural natural deduction of this form are given by Sara Negri [7], but the discussion of the behaviour in terms of assumption classes and distinguishing the phases of *identification* and *discharge* is original to this paper.)

Here is an example proof, from the premise $(p \to q) \land (p \to r)$ to the conclusion $p \to (q \land r)$.

$$\dfrac{\dfrac{\dfrac{(p \to q) \land (p \to r)^1}{p \to q}{\scriptstyle \land E} \quad [p]^2}{q}{\scriptstyle \to E} \qquad \dfrac{\dfrac{(p \to q) \land (p \to r)^1}{p \to r}{\scriptstyle \land E} \quad [p]^2}{r}{\scriptstyle \to E}}{\dfrac{q \land r}{p \to (q \land r)}{\scriptstyle \to I^2}}{\scriptstyle \land I}$$

In this proof, at the $\land I$ step, we have two subproofs, each from $(p \to q) \land (p \to r)$ and p, and the assumption classes of both of these subproofs are combined, using the labels 1 and 2. So the rule is appropriately applied, and in addition, we discharge the single assumption class for p to derive $p \to (q \land r)$ in the last inference step.

We have considered how assumption classes can be combined in the presence of contraction. It remains to consider the role of weakening. In the simple natural deduction proof from p to $q \to p$, with one $\to I$ inference, *zero* instances of q are discharged. This means that in proofs with weakening, we must allow assumption classes to be *empty*. Once assumption classes can be empty, there will be many more ways for different proofs to come from the same context. Consider the following sequent derivation, using *weakening*, to derive $p, q \succ p \land q$.

$$\dfrac{\dfrac{p \succ p}{p, q \succ p}{\scriptstyle KL} \qquad \dfrac{q \succ q}{p, q \succ q}{\scriptstyle KL}}{p, q \succ p \land q}{\scriptstyle \land R}$$

What proof might correspond to this sequent derivation? The proof we might expect should have the shape

$$\dfrac{p \quad q}{p \land q}{\scriptstyle \land I}$$

but for this to be a correct proof, we must understand the sense in which the two subproofs (the atomic proofs of p and of q) have the same context. In the presence of weakening, the atomic proof of p is indeed a proof of p from that occurrence of p, but it is also a proof of p from p, q, where the assumption class for q is *empty*, while the assumption class for p has one inhabitant. In the presence of weakening, a proof is not only a proof from a single context $\mathcal{C}$, but also any *extension* of $\mathcal{C}$ by any finite number of empty assumption classes. In this way, an atomic proof p corresponds not only to the sequent $p \succ p$, but $p, q \succ p$ (adding the empty positive assumption class q), $p \succ p, r$ (adding the empty class of occurrences of $\cancel{r}$), and any other sequent of the form $X, p \succ p, Y$ for finite X and Y. The effect of this condition in natural deduction proofs is twofold: first, in the discharging inferences $\to I$, $\neg I$ and $\downarrow$, in which an empty class of occurrencs may be discharged, as expected. Second, as we have seen in the above example, it may also play a role in the $\land I$ rule, which can apply even when the *non-empty* assumption classes occurring in

$$
\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\,{\to}I^i
\qquad
\dfrac{\begin{array}{cc}\mathcal{C} & \mathcal{C}'\\ \Pi & \Pi'\\ A \to B & A\end{array}}{B}\,{\to}E
\qquad
\dfrac{\begin{array}{c}[A]^j\\ \Pi\\ \sharp\end{array}}{\neg A}\,\neg I^j
\qquad
\dfrac{\begin{array}{cc}\mathcal{C} & \mathcal{C}'\\ \Pi & \Pi'\\ \neg A & A\end{array}}{\sharp}\,\neg E
$$

$$
\dfrac{\begin{array}{cc}\mathcal{C} & \\ \Pi & \\ A & \cancel{A}\end{array}}{\sharp}\,\uparrow
\qquad
\dfrac{\begin{array}{c}[\cancel{A}]^k\\ \Pi\\ \sharp\end{array}}{A}\,\downarrow^k
\qquad
\dfrac{\begin{array}{cc}\mathcal{C} & \mathcal{C}\\ \Pi_1 & \Pi_2\\ A & B\end{array}}{A \wedge B}\,\wedge I
\qquad
\dfrac{\begin{array}{c}\mathcal{C}\\ \Pi\\ A \wedge B\end{array}}{A}\,\wedge E
\qquad
\dfrac{\begin{array}{c}\mathcal{C}\\ \Pi\\ A \wedge B\end{array}}{B}\,\wedge E
$$

Figure 1: The Natural Deduction Rules

the proofs of A and of B are not identical, since we can add extra *empty* assumption classes to either proof, until the contexts match.

Figure 1 compiles the rules for our natural deduction system. These rules can be read in four different ways, depending on the presence or absence of contraction and weakening.

- If contraction is *absent*, the contexts $\mathcal{C}$ and $\mathcal{C}'$ in the ${\to}E$ and $\neg E$ rules are required to be *disjoint*. If contraction is *present*, at each ${\to}E$ and $\neg E$ step, some assumption classes are permitted to be *merged*.
- If weakening is *absent*, the assumption classes in discharge rules are non-empty, and each proof has a *unique* context, of non-empty classes appearing in the leaves of the proof. If weakening is *present*, each proof has not only a minimal context $\mathcal{C}$ of formula occurrences present in the leaves, but is also a proof from any *wider* context $\mathcal{C}'$ with empty assumption classes added. As a result, any two proofs can be combined in a $\wedge I$ inference, by the addition of empty assumption classes to each side, to ensure that the assumption classes match.

5 Soundness and Completeness

It remains to show that this system of natural deduction with alternatives corresponds tightly with the traditional sequent calculi, and it is to this result that we turn. For clarity, we will split this result into two cases. First, for the *linear* calculus, and then we will end with the result for calculi with structural rules.

FACT 2 *There is a linear natural deduction proof with alternatives, from the premises* X *and alternatives* Y *to the conclusion* C *if and only if there is a derivation of the sequent* $X \succ C, Y$ *in the classical linear sequent calculus.*

FACT 3 *There is a natural deduction proof with alternatives (a)* using duplicate discharge, *or (b)* using vacuous discharge *from the premises* X *and alternatives* Y *to the conclusion* C *if and only if there is a derivation of the sequent* $X \succ C, Y$ *in the classical linear sequent calculus with the addition of (a)* contraction, *or (b)* weakening.

To verify both of these facts, it is useful to draw out a simple lemma, which has the effect that we treat natural deduction proofs as representing sequents of the form $X \succ Y$, in which we disregard which formula is in focus, since focus can be moved freely.

LEMMA 4 [*FOCUS SHIFT*] *There is a proof from assumptions* X *and alternatives* Y *to conclusion* A *iff there is a proof from assumptions* X *and alternatives* A, Y *to conclusion* $\sharp$. (*The second proof uses vacuous discharge or duplicate discharge if and only if the first proof does.*) *Similarly, there is a proof from* X *and* A, Y *to* B *iff there is a proof from* X *and* B, Y *to* A.

Proof: This is an immediate application of the store and retrieve rules. Any proof from X and Y to A, extended with one $\uparrow$ step is a proof from X and A, Y to $\sharp$. Conversely, any proof from X and A, Y to $\sharp$, extended with one $\downarrow$ step, is a proof from X and Y to A. If we have a proof from X and A, Y to B, on one $\uparrow$ step this is a proof from X and A, B, Y to $\sharp$, which in one $\downarrow$ steop is a proof from X and B, Y to A. ■

With the Focus Shift Lemma at hand, we can complete the proof of Fact 2. This proof follows the structure of the proof of Fact 1 (see page 5) directly, except we allow for the presence of alternatives (on the proof side) and sequents with more than one formula on the right (on the derivation side) and we add cases for the new rules in each system.

Proof: The left-to-right direction is an induction on the construction of the proof from X and Y to C. The base case is unchanged from our earlier reasoning: a proof of A corresponds to the identity derivation $A \succ A$. For the induction steps, we suppose we are generating a new proof, by some inference step, from proofs for which the induction hypothesis holds. For the connective rules for the conditional and negation, the argument is exactly the same as in our earlier reasoning, except we have to verify that the derivation steps corresponding to natural deduction inferences are correct in the presence of proofs with alternatives. Consider the case for $\rightarrow E$. This step is applied in a natural deduction proof when we have a proof from X and Y to $A \rightarrow B$ and a proof from X' and Y' to A, which we combine, to produce a proof from X, X' and Y, Y' to B. The induction hypothesis ensures we have derivations of $X \succ A \rightarrow B, Y$ and $X' \succ A, Y'$. Using *Cut* and $\rightarrow L$ we can construct the desired derivation of $X, X' \succ B, Y, Y'$ like this:

$$\dfrac{X \succ A \rightarrow B, Y \qquad \dfrac{X' \succ A, Y' \qquad \dfrac{}{B \succ B}\,Id}{X', A \rightarrow B \succ B, Y'}\,{\rightarrow}L}{X, X' \succ B, Y, Y'}\,Cut$$

The cases for the other rules for the conditional and negation follow in just the same manner as this, making the obvious changes to allow for sequents with a more general RHS.

Next, consider the rules for additive conjunction. If we extend our proof with a $\wedge E$ step, we extend a proof from X, Y to $A \wedge B$ to a proof from the same context to the conclusion A (or B). The induction hypothesis ensures that we have a derivation of $X \succ A \wedge B, Y$. This can be extended to derivations of $X \succ A, Y$ and $X \succ B, Y$ straightforwardly:

$$\dfrac{X \succ A \wedge B, Y \qquad \dfrac{\dfrac{}{A \succ A}\,\mathit{Id}}{A \wedge B \succ A}\,{\wedge L}}{X \succ A, Y}\,\mathit{Cut} \qquad \dfrac{X \succ A \wedge B, Y \qquad \dfrac{\dfrac{}{B \succ B}\,\mathit{Id}}{A \wedge B \succ B}\,{\wedge L}}{X \succ B, Y}\,\mathit{Cut}$$

If our proof ends in a $\wedge I$ inference, with conclusion $A \wedge B$, from context X, Y, then we have two proofs, one to A and the other, to B, from the same context X, Y. This means we have two derivations, one of $X \succ A, Y$, and the other, of $X \succ B, Y$. They can be extended like this

$$\dfrac{X \succ A, Y \qquad X \succ B, Y}{X \succ A \wedge B, Y}\,{\wedge R}$$

to give us the derivation we need. So, we have completed the cases for the connective rules for the left-to-right part of our fact. It remains to consider the structural *store* and *retrieve* rules. If our proof ends in a *store* ($\uparrow$) step, we convert a proof from X and Y to A to a proof from X and A, Y to $\sharp$. The induction hypothesis delivers us a derivation of $X \succ A, Y$, and we want a derivation corresponding to our new proof from X and A, Y to $\sharp$, which is *also* a derivation from $X \succ A, Y$, so the *store* rule is inert at the level of sequent derivations without focus. (This is one lesson of the *focus shift* lemma.) So, too, is the *retrieve* ($\downarrow$) rule, which simply reverses the effect of a *store* step. So, with this noted, we complete the proof of the left-to-right direction of our fact.

For the right-to-left direction of the equivalence, we show how we can construct a proof from context X, Y to C, given a derivation of $X \succ C, Y$ (whether C is a formula or $\sharp$). As before, if our derivation is a simple appeal to *Id* ($A \succ A$) we have the atomic proof featuring the assumption A standing alone as both assumption and conclusion. Or, given that $A \succ A$ is a derivation corresponding to a proof of $\sharp$ from the context A and A, this derivation *also* corresponds to the proof

$$\dfrac{A \qquad \cancel{A}}{\sharp}\,\uparrow$$

consisting of a single *store* inference. Notice that this proof is found by a simple modification of the original identity proof of A. We could, here, appeal to the *focus shift* lemma instead, rather than explicitly constructing every focus variant of our first proof.

For the other structural rule, *Cut*, we have derivations of $X \succ A, Y$ and $X', A \succ Y'$. By the induction hypothesis, we have a proof from context X and Y to A, and a proof from context X', A and Y' to $\sharp$.[8] We paste these proofs together to construct the combined proof from X, X' and Y, Y' to $\sharp$, going through A as an intermediate step, just as we did in the proof of Fact 1.

$$\begin{array}{c} \begin{array}{cc} & \begin{array}{c} X\ \cancel{Y} \\ \hdashline \Pi_1 \end{array} \\ X'\ \cancel{Y'} & A \end{array} \\ \hdashline \Pi_2 \\ \sharp \end{array}$$

For the proofs corresponding to the remaining focusings of the sequent $X, X' \succ Y, Y'$, we appeal to the focus shift lemma.

As before, the connective rules on the left and right correspond neatly to the corresponding applications of the elimination and introduction rules. For $\to L$, suppose we already have a proof Π_1 from X and Y to A and a proof Π_2 from X', B and Y' to C. We construct a proof from $X, X', A \to B$ and Y, Y' to C like this:

$$\begin{array}{c} \begin{array}{cc} & \begin{array}{r} X\ \cancel{Y} \\ \hdashline \Pi_1 \end{array} \\ X'\ \cancel{Y'} & \dfrac{A \to B \qquad A}{B}\,{\to}E \end{array} \\ \hdashline \Pi_2 \\ C \end{array}$$

(Again, if we wished to construct a proof of a different conclusion, shifting the focus, we appeal to the focus shift lemma.) Similarly, given a proof from X, A and Y to B, we can discharge that assumption class of instances A in one $\to I$ step to construct a proof from X to $A \to B$. The reasoning for the negation rules has the same shape, so it remains only to consider the additive conjunction rules. For $\wedge L$, we have a derivation of $X, A \succ Y$ (and so, a proof of $\sharp$ from X, A and Y), and we extend this to a derivation of $X, A \wedge B \succ Y$. So, we want a proof of $\sharp$ from $X, A \wedge B$ to Y. This is trivial, since we can extend our proof by replacing every instance A in the indicated assumption class by $\wedge E$ inference from $A \wedge B$ to A, being careful to merge each assumption $A \wedge B$ into one assumption class. The result is a proof from $X, A \wedge B$ and Y to $\sharp$, as desired:

$$\begin{array}{c} \begin{array}{cc} X\ \cancel{Y} & \dfrac{A \wedge B}{A}\,\wedge E \end{array} \\ \hdashline \Pi \\ \sharp \end{array}$$

The same goes for a derivation from $X, B \succ Y$ to $X, A \wedge B \succ Y$, using the $\wedge E$ step from $A \wedge B$ to B. (The focus shift lemma deals with the proofs corresponding to different selections of the conclusion from the context.)

[8]We could pick out a given formula from the family Y' of alternatives, if Y' is non-empty, but allowing the focus to remain on $\sharp$ is the general case, so we use this case here.

Our final case is the conjunction *right* rule, for which we have derivations of $X \succ A, Y$ and of $X \succ B, Y$, which we extend into a derivation of $X \succ A \wedge B, Y$. By hypothesis, we have proofs of A from X and Y and of B from the same context, X and Y. So, we can extend these in one $\wedge I$ step, in which we *identify* the assumption classes, pairing each assumption class from the context of the proof of A with exactly one assumption class from the context of the proof of B. The result is a proof of $A \wedge B$ from exactly the same context X and Y as desired, and we can declare our proof complete, modulo another appeal to the focus shift lemma. ■

The only remaining item is to prove Fact 3, which requires attention to the conditions for *contraction* and *weakening* in proofs and in sequent derivations.

Proof: We extend the reasoning of the previous proof, first by considering what additions we need to make to account for *contraction*, and then, for *weakening*. First, let's consider *contraction*. For the left-to-right direction, we wish to constuct a sequent derivation (perhaps using the contraction rule) of $X \succ A, Y$ from a natural deduction proof of A from the context X and Y in which we allow for the merging of assumption classes in the inferences $\to E$ and $\neg E$. The reasoning for atomic proofs is the same as before, since no contraction can take place with only one formula in the context. Take a proof ending in a $\to E$ step in which some classes are merged. We have a proof from X and Y to $A \to B$ and another, from X' and Y' to A, and by induction hypothesis, we have a derivation of $X \succ A \to B, Y$ and of $X' \succ A, Y'$. As in the proof of the previous fact, we have a derivation of $X, X' \succ B, Y, Y'$, by way of a $\to L$ inference and a *Cut*. The context X, X' and Y, Y' is too large, because this is the disjoint combination of the two contexts. The application of some contraction steps is enough to pare down the context so there is a member of the multiset on the LHS and that on the RHS for each assumption class in the proof. This is the only change required to produce a sequent derivation using contraction, and we can declare the left-to-right direction of this part of our proof complete.

For the right-to-left case, we show that from any derivation of $X \succ Y$ we can construct a proof of $\sharp$ from the context X and Y, as well as any focus shift of that proof. Notice that contraction steps can occur at *any* point of a derivation, not only at the steps immediately before $\to E$ and $\neg E$ inferences. To take account of that, we prove a more general fact, that from any derivation of $X \succ Y$ we can construct a proof of $\sharp$ from the context X and Y as well as any *contraction* of that context (in which assumption classes are merged), as well as any focus shift of such a proof. The base case, corresponding to the sequent $A \succ A$ corresponds to the atomic proof of A and the proof of $\sharp$ from A, $\cancel{A}$, neither of which may be contracted.

For the other structural rule, *Cut*, we have derivations of $X \succ A, Y$ and $X', A \succ Y'$. By the induction hypothesis, we have a proof from context X and Y to A (and of any contraction X^* and Y^* of that context), and a proof from context X', A and Y' to $\sharp$ (and from any contraction $X^{*\prime}, A$ and $Y^{*\prime}$ of that context). We paste these proofs together to construct the combined proof from $X^*, X^{*\prime}$ and $Y^*, Y^{*\prime}$ to $\sharp$, going through A as an intermediate step, just as we did in the proof of Fact 2.

$$
\begin{array}{cc}
 & X^* \; \cancel{Y^*} \\
 & \vdots \\
 & \Pi_1 \\
X^{*\prime} \; \cancel{Y^{*\prime}} & A \\
\multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c}{\Pi_2} \\
\multicolumn{2}{c}{\sharp}
\end{array}
$$

Perhaps the new context $X^*, X^{*\prime}$ and $Y^*, Y^{*\prime}$ may be contracted further. If so, there is a point in the proof (either in an inference in Π_1 or an inference in Π_2) where the two distinct assumption classes to be contracted first enter the proof. This must be in either a $\to E$ step or a $\neg E$ step, because in the other inference steps, we do not join proofs with different assumption classes. At this inference, then, we can contract the desired assumption classes, to ensure that in the whole proof we have contracted the context to the desired extent. The reasoning in the *Cut* rule can apply to the other steps in a derivation where different contexts are combined. These rules are $\to L$ and $\neg L$, and the corresponding proofs have $\to E$ and $\neg E$ steps, at which we can contract the corresponding assumption classes, as desired. With this modification, contractions in our derivations can be dealt with directly. If our derivation moves to $X, A \succ Y$ from $X, A, A \succ Y$, the induction hypothesis ensures that we have a proof of $\sharp$ from X, A, A and Y and any contraction of this context. This means it is immediate that we have a proof of $\sharp$ from X, A and Y and any contraction of *this* context, too. The same reasoning applies to contraction on the left, and we can declare the right-to-left case for contraction complete.

Now consider proofs with the *weakening* conditions in force. To confirm the left-to-right direction of our fact, we wish to construct, for any C from context X and Y, a derivation of $X \succ C, Y$. The atomic case of a proof consisting of the lone assumption A now counts as a proof of A from the context X, A and Y for any finite X and Y. We have a derivation of $X, A \succ A, Y$ in our sequent calculus by applying weakening on the left and the right the appropriate number of times from the identity sequent $A \succ A$. With the atomic case dealt with, the remaining proof steps are straightforward. The only modifications needed for our earlier argument (whether the linear calclulus, or the calculus with contraction) is to note that we allow for discharging of empty assumption classes in the $\to I$, $\neg I$ and $\downarrow$ inferences. So for the connective rules, at the corresponding $\to R$, and $\neg R$ steps in the sequent calculus we must weaken in the vacuously discharged formula before applying the rule. For the structural rule $\downarrow$, a vacuous application corresponds in the sequent calculus to an explicit step of weakening on the right. Finally, consider the $\wedge I$ inference. Suppose we have a proof of A from the context $\mathcal{C}$ and a proof of B from the same context, with the weakening conditions in play, and we extend this proof to conclude $A \wedge B$ from the same context. This means we have some derivation of a sequent $X \succ A, Y$ and another derivation of a sequent $X' \succ B, Y'$ where the contexts X, Y and X', Y' are the assumption classes explicitly appearing in the proof. However, we add new empty assumption classes to both contexts, sufficient to allow the contexts to match. That is, we have the wider context $\mathcal{C} = X'', Y''$ where

X'' subsumes both X and X', and similarly, Y'' subsumes Y and Y'. By hypothesis, we have derivations for $X'' \succ A, Y''$ and $X'' \succ B, Y''$, and so, by $\wedge R$ this may be extended to a derivation for $X'' \succ A \wedge B, Y''$ as desired. The reasoning for the other rules works in the same way.

For the right-to-left reasoning, we wish to show that for any derivation of a sequent $X \succ Y$ (using the structural rule of weakening) we have a natural deduction proof (using the weakening conditions) of $\sharp$ from the context X and Y, as well as any focus shift of that proof. Here, the proof is quick because we have defined natural deduction proofs with weakening in such a way that if we have a derivation of some conclusion from the context X, Y it counts as a proof from any weakened context, too. So, any appeal to the structural rule of weakening in the derivation is *inert* at the level of the natural deduction proof. (The atomic proof A counts as a proof from A to A as well as a proof from A, B to A in which the B is unused.) It is straightforward to check that the process for defining a natural deduction proof from a sequent derivation will — if we simply do not attempt to translate the appeals to weakening into the application of any particular rule — generate a natural deduction proof in which the weakening conditions are applied, and with that, we can declare this result proved. ∎

So, with this result established, we can see that with the shift from a *unilateral* context X (of things *positively* granted) to a *bilateral* context X and Y (where some things have been ruled *in* and others ruled *out*) we have a simple extension of Gentzen–Prawitz-style natural deduction, sufficient to give an account not only of classical proof, but of proof in classical flavours of linear, relevant and affine logic, too.

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