A Useful Substructural Logic

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Formal systems seem to come in two general kinds: useful and useless. This is painting things starkly, but the point is important. Formal structures can either be used in interesting and important ways, or they can languish unused and irrelevant. Lewis' modal logics are good examples. The systems **S4** and **S5** are useful in many different ways. They map out structures that are relevant to a number of different applications. **S1**, **S2** and **S3** however, are not so lucky. They are little studied, and used even less. It has become clear that the structures described by **S4** and **S5** are important in different ways, while the structures described by **S1** to **S3** are not so important. In this paper, we will see another formal system with a number of different uses. We will examine a substructural logic which is important in a number of different ways. The logic of *Peirce monoids*, inspired by the logic of relations, is useful in the independent areas of *linguistic types* and *information flow*.

In what follows I will describe the logic of Peirce monoids in its various guises, sketch out its main properties, and indicate why it is important. As proofs of theorems are readily available elsewhere in the literature, I simply sketch the relevant proofs here, and point the interested reader to where complete proofs can be found.

1 Relation algebras

Relation algebras are an interesting generalisation of Boolean algebras. For our purposes we will only need a fragment of the general relation algebras studied by Tarski and others [27]. Take a class of objects and a collection of *binary relations* on this class. There are a number of ways to form new relations from old: the Boolean connectives are some, but there are also others. Pairs of relations can be *composed*: a and b are related by the composition of ρ and σ (written $\rho \cdot \sigma$) just when for some c, $a\rho c$ and $c\sigma b$. Composition is associative, but not commutative or idempotent. The *identity* relation 1' satisfies $1' \cdot \rho = \rho = \rho \cdot 1'$. Any relation ρ has a *converse*, written $\rho^{\smile}$, which satisfies $\rho^{\smile\smile} = \rho$ and a number of other identities.

An abstract *positive relation algebra* is a 6-tuple $\langle R, \cap, \cup, 1, \cdot, 1' \rangle$, such that $\langle R, \cap, \cup, 1 \rangle$ is a distributive lattice with top element 1 (the full relation), $\cdot$ (composition) is an associative binary operation on R with identity 1' (the identity relation). In addition, composition distributes over disjunction:

$$x \cdot (y \cup z) = (x \cdot y) \cup (x \cdot z) \qquad (x \cup y) \cdot z = (x \cdot z) \cup (y \cdot z)$$

Clearly, each of these conditions is true under the intended interpretation of relation algebras. The variety of all algebras satisfying these conditions is called $\mathbf{RA}^+$.

If we add negation and converse ($-$ and $^{\smile}$) we are able to define *residuals* for composition. If we set $x \Rightarrow y = -(x^{\smallsmile} \cdot -y)$ and $y \Leftarrow x = -(-y \cdot x^{\smallsmile})$, it follows that

$$x \cdot y \leq z \text{ if and only if } y \leq x \Rightarrow z \text{ if and only if } x \leq z \Leftarrow y$$

(where $a \leq b$, defined as $a \cup b = b$ is the natural ordering on the algebra). These are called the *residuation conditions*. They show that the behaviour of the 'conditionals' $\Rightarrow$ and $\Leftarrow$ is closely tied to the properties of the composition operation. Although these conditions fall out when we *define* the implications in terms of converse and Boolean negation, they make sense in the absence of these definitions. Since we will not be so interested in Boolean negation and converse, we restrict our attention to positive relation algebras equipped with operators $\Rightarrow$ and $\Leftarrow$ satisfying the residuation conditions. Following Dunn [9], we call these structures *Peirce monoids* and their variety may as well be called '**P**'. These are the structures that have interesting applications.

2 Categorial grammar

Consider a sentence. It is made up of a number of words, which have different *types*. Some are nouns, others are adjectives, others are verbs, and so on. In addition, different parts of sentences have different types. For example, some are noun phrases, others are adverbial phrases, and so on. It is possible to mathematically analyse these parts of speech by means of a formal system. For example, given a basic categorisation of some words:

Kim, Whitney	*NP*
documentary, cartoon	*N*

it is possible to give an account of *other* words in terms of what parts of speech they result in when combined with these. For example: jumps is of type $NP \rightarrow S$, because when combined with a part of speech of type *NP* on the left, it results in something of type S (a sentence). So, Kim jumps, and Whitney jumps are sentences. The word jumps is *not* of type $S \leftarrow NP$, as jumps Kim is not a sentence. So jumps, when combined with something of type *NP* on the right, does not yield a sentence. Instead, jumps Kim is of type $(NP \rightarrow S) \circ NP$. That is, it is something of type $(NP \rightarrow S)$ juxtaposed to the left of something of type *NP*. So, there are three binary operations on types: $\circ$, $\rightarrow$ and $\leftarrow$. The *Lambek Associative Calculus* is given by a Gentzen calculus with antecedents of rules collected as a list [14]. $A_1; A_2; \cdots; A_n \vdash B$ is read as 'the juxtaposition of something of type A_1 with something of type A_2 with ... something of type A_n is also of type B'. This reading motivates a number of rules governing the logic of the connectives. Firstly, we have an axiom, and a structural rule.

$$A \vdash A \qquad X;(Y;Z) \Longleftrightarrow (X;Y);Z$$

The axiom is simple: anything of type A is of type A. The rule simply states that on the left of any sequent, a part of a list of the form $X;(Y;Z)$ may be replaced by $(X;Y);Z$ or *vice versa*. The binary operation ';' truly creates lists, because the order of bracketing is unimportant. Then we have the rules for concatenation.

$$\frac{X \vdash A \quad Y \vdash B}{X;Y \vdash A \circ B} \qquad \frac{X(A;B) \vdash C}{X(A \circ B) \vdash C}$$

These merely ensure that the operation $\circ$ on types mimics the behaviour of the semicolon of concatenation. (In the second rule, $X(A;B)$ is a list in which $A;B$ occurs somewhere. $X(A \circ B)$ is that list with that occurrence of $A;B$ replaced by $A \circ B$.) Finally, the conditional rules.

$$\frac{X \vdash A \quad Y(B) \vdash C}{Y(X; A \rightarrow B) \vdash C} \quad \frac{A; X \vdash B}{X \vdash A \rightarrow B} \qquad \frac{X \vdash A \quad Y(B) \vdash C}{Y(B \leftarrow A; X) \vdash C} \quad \frac{X; A \vdash B}{X \vdash B \leftarrow A}$$

These ensure that the type operations $\rightarrow$ and $\leftarrow$ are right and left residuals for $\circ$ respectively.

The standard cut-elimination proof applies to show that cut is admissible in this system. We will not go into the proof here.

Concatenation in this system acts just like composition in **P**. It is associative, but not commutative or idempotent. The ordering $\leq$ in **P** is a good match for the relation $\vdash$. Similarly, the residuating conditionals have the same properties in each system. We can push the parallels even further.

Consider the full relation 1. It has a parallel as the type *everything* has. Call that type $\top$. Just as $x \leq 1$ in **P**, here we can have

$$X \vdash \top$$

The identity relation 1' is paralleled by the type of the *empty string* (which we may as well write $\Box$). In **P** we have $1' \cdot x = x = x \cdot 1'$. Here we have

$$\Box; X \Longleftrightarrow X \Longleftrightarrow X; \Box$$

These additions to the Gentzen system are trivial. The interesting work comes from interpreting conjunction and disjunction.

2.1 Conjunction and disjunction

One problem in the analysis of sentence structure is the account given of parts of speech such as and, or and but. Assigning types to these words is difficult. In the sentence Jack and Jill walked and talked the two occurrences of and have types $(NP \rightarrow NP) \leftarrow NP$ and $\big((NP \rightarrow S) \rightarrow (NP \rightarrow S)\big) \leftarrow (NP \rightarrow S)$ respectively. In fact, while and always seems to have type $(X \rightarrow X) \leftarrow X$, there seems to be no limit to the number of different types that can be substituted for X. Oehrle gives some examples [16]

Kim gave and Hilary offered Whitney a documentary.	$(S \leftarrow NP) \leftarrow NP$
Kim gave Hilary and Sal offered Whitney a documentary.	$S \leftarrow NP$
Kim gave Whitney a documentary and Hilary a cartoon.	$NP \circ NP$
Kim gave Whitney and offered Hilary a documentary.	$NP \rightarrow S \leftarrow NP$

Operators such as and have caused difficulty for those seeking to give a unified account of syntactic types. One way to proceed is to take and to have the polymorphic type $(X \rightarrow X) \leftarrow X$, where X ranges over the class of all 'conjoinable types'. Another way to proceed is by taking and, or and but to be operators on a par with $\circ$, $\rightarrow$ and $\leftarrow$, and to create new rules for them. This seems to be a category mistake, as the type operators $\rightarrow$, $\leftarrow$ and $\circ$ do not appear in our language. They are operations on types. The variety **P** hints at another way, closely associated to both of these methods, but better motivated. Given that a piece of syntax is of type A and of type B, it is sensible to take it to be of type $A \wedge B$. Similarly, if it is either of type A or of type B, then it is also of type $A \vee B$. Then, we can take and to be of type

$$\bigwedge_{X \in CT} (X \rightarrow X) \leftarrow X$$

where CT is the set of all conjoinable types. (Of course, if CT is infinite (as it may be) we need to introduce countable conjunction which is not difficult to define.)

Makoto Kanazawa gives us one way to model conjunction and disjunction [13]. His proposal is to add the following rules to the Gentzen calculus

$$\frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B} \qquad \frac{X(A) \vdash C}{X(A \wedge B) \vdash C} \qquad \frac{X(A) \vdash C}{X(B \wedge A) \vdash C}$$

$$\frac{X(A) \vdash C \quad X(B) \vdash C}{X(A \vee B) \vdash C} \qquad \frac{X \vdash A}{X \vdash A \vee B} \qquad \frac{X \vdash A}{X \vdash B \vee A}$$

These rules will be familiar to anyone who has seen Girard's linear logic [11]. (These rules have been hinted at elsewhere, dating back at least to Lambek's early work [15]. I have picked out Kazanawa's proposal because it is an easily accessible, contemporary example of the tradition.) Adding these rules to the Gentzen system we have seen produces an interesting formal system. However, it is only an approximation to a correct account of conjunctive and disjunctive types, because the distributive law

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

is not provable in the system as it stands. This is a shortcoming, because under our interpretation of the system the law is valid. If something is both of type A and either of type B or C, then it is either of type A and of B, or of type A and of type C. That is all there is to it.

Distribution is also valid in **P**. We have $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ as a matter of course. To faithfully model the logic of conjunction and disjunction of types we must extend the Gentzen calculus in a way that matches the behaviour of conjunction and disjunction in Peirce monoids.

That way ahead is to follow Dunn and Minc's original work in the Gentzenisation of *relevant* logics [1,8,15a], taken up later by many others [2,6,7,10,19,25]. The guiding idea is to introduce another bunching operator to do for extensional conjunction what the semicolon does for intensional conjunction. So, the intensional/extensional difference gets modelled in the 'punctuation' in our proof theory. (This idea has been generalised by Belnap, who in his 'Display Logic' gives Gentzenisations of logics with connectives governed by more than one set of structural rules [5]. There are also echoes of this idea in Girard's Logique Universelle [12].)

So, in our Gentzenisation we expand the methods of bunching premises together to admit extensional (conjunctive) bunching, designated by the comma, as well as intensional (concatenating) bunching, designated by the semicolon. In this way, we get a Gentzen system which retains the insights we have already seen, and in which distribution is provable. We read X, Y as 'something which is both of type X and of type Y' and $X; Y$ as 'something of type X concatenated with something of type Y'. We can then add extra rules to deal with the new connectives.

$$X, X \Longleftarrow X \qquad X, (Y, Z) \Longleftrightarrow (X, Y), Z \qquad X, Y \Longleftrightarrow Y, X \qquad X, Y \Longrightarrow X$$

These rules mean that any instance of X, X may be replaced with X in a proof; that $X, (Y, Z)$ is replacable by $(X, Y), Z$ anywhere in a proof, and *vice versa*, and so on.

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B} \qquad \frac{X(A, B) \vdash C}{X(A \wedge B) \vdash C}$$

$$\frac{X \vdash A}{X \vdash A \vee B} \qquad \frac{X \vdash B}{X \vdash A \vee B} \qquad \frac{X(A) \vdash C \quad X(B) \vdash C}{X(A \vee B) \vdash C}$$

Given these rules, it is simple to prove distribution. So our system does what we wish of it. However, there might be a cost. It is well known that the systems like linear logic and Kazanawa's extension of the Lambek calculus are cut-free and decidable; and these are desirable properties. Thankfully, the Gentzen system we have before us shares these desirable properties.

FACT 2.1
Cut is admissible.

The proof is too tedious to sketch here, because of its many cases — made more difficult because of the two kinds of bunching connectives present. For details, see [20] in which the proof is offered for not just this system, but many others in its neigbourhood, or Dunn's original work in *Entailment* [1], in which cut elimination is proved for the Gentzenisation of the related relevant logic $\mathbf{R}^{+}$.

FACT 2.2
P is decidable.

There is a simple argument to the effect that in any proof, we need not use too many instances of contraction (deducing $X(Y) \vdash A$ from $X(Y, Y) \vdash A$) and so, proof searches always end in a result. The details are to be found elsewhere [20,25].

The language of the Gentzen system parallels the language of **P** to the last detail. The linguistic $\rightarrow, \leftarrow, \circ, \wedge, \vee, \top$ and $\Box$ mirror $\Rightarrow, \Leftarrow, \cdot, \cap, \cup, 1$ and 1'. We will round this section out by making this claim precise.

Given a Peirce monoid P, an *interpretation* φ of the language of the Gentzen system is a map which takes each atomic type and gives an element of P, and which is then extended to satisfy the following conditions:

$$\varphi(A \rightarrow B) = \varphi(A) \Rightarrow \varphi(B) \quad \varphi(A \leftarrow B) = \varphi(A) \Leftarrow \varphi(B)$$

$$\varphi(A \circ B) = \varphi(A) \cdot \varphi(B) \quad \varphi(A \wedge B) = \varphi(A) \cap \varphi(B)$$

$$\varphi(A \vee B) = \varphi(A) \cup \varphi(B) \quad \varphi(\top) = 1 \quad \varphi(\Box) = 1'$$

$$\varphi(X; Y) = \varphi(X) \cdot \varphi(Y) \quad \varphi(X, Y) = \varphi(X) \cap \varphi(Y)$$

FACT 2.3
The Gentzen calculus is a sound and complete proof system for $\mathbf{P}$. In other words, $X \vdash A$, if and only if for each $P \in \mathbf{P}$, and for each interpretation φ into P, $\varphi(X) \leq \varphi(A)$.

To prove this from left to right, a simple induction on the length of the proof of $X \vdash A$ suffices. Clearly $\varphi(A) \leq \varphi(A)$ for any interpretation φ into a Peirce monoid P. For the induction steps we must show that rules preserve soundness. For example, a structural rule like $X, X \Longleftarrow X$ is fine, because $\varphi(X, X) = \varphi(X)$ for any interpretation, so X, X may be replaced by X anywhere at no charge. Other rules are just as easy.

For right to left, it is simplest to prove the contrapositive: if $X \not\vdash A$ then there is a Peirce monoid P, and an interpretation φ in which $\varphi(X) \not\leq \varphi(A)$. For this, we take a P to be constructed of the equivalence classes of provably equivalent formulae. (A and B are provably equivalent if and only if $A \vdash B$ and $B \vdash A$.) Let the interpretation φ be given by setting $\varphi(A) = [A]$. It is simple to show that if $X \not\vdash A$ then $\varphi(X) \not\leq \varphi(A)$, so our Peirce monoid P of equivalence classes makes all the distinctions we need to prove our result.

This result means that $\mathbf{P}$ is decidable too, because we can use the Gentzen system to provide a decision procedure for the variety. This is a curious result, for Tarski has shown that $\mathbf{RA}$ is undecidable [28]. Removing converse and negation has a significant effect on the algebra.

3 Situations and conditionals

Jon Barwise has recently produced a refinement of the account of conditionals given by situation semantics [3]. It turns out that this has deep connections with the systems we have already seen. For the uninitiated, here is a quick summary of relevant parts of situation semantics and Barwise's recent work on conditionals. A felicitous declarative utterance of a sentence S makes a claim about a *situation*, s. That is, it is about a particular restricted part of the world. It claims that the situation s is of a particular *type*, say T. The situation that the utterance is about is the *demonstrative content* of the utterance. This is typically determined by particular conventions of the language in use, and other relevant facts about the discourse. The type that the utterance describes the situation to be (correctly or incorrectly) is the *descriptive content* of the utterance. This is determined by other conventions of the language in use and other relevant facts.

Though this is couched in terms of utterances and language, it need not be restricted to this application. Situations and types can be used in any context in which we have *limited* information about *limited* parts of a domain. Situation semantics is primarily about partial information, so its calculus is grounded in situations, and not complete possible worlds.

Given this much about types and situations, it is possible to define a few rules about type conjunction and disjunction. Let $s \models T$ be shorthand for 'the situation s *is* of type T' or 's supports T'. Then if we admit conjunction and disjunction of types, we can require the following:

$$s \models T_1 \wedge T_2 \text{ iff } s \models T_1 \text{ and } s \models T_2 \qquad s \models T_1 \vee T_2 \text{ iff } s \models T_1 \text{ or } s \models T_2$$

This much is straightforward. Another condition that is useful is to require that situations be partially ordered. We can *expand* situations to contain more of the domain. It is common to expect situations to be *hereditary* in that as situations are expanded, information is gained, and not lost. This means that for the containment relation $\sqsubseteq$

$$s_1 \sqsubseteq s_2 \Rightarrow (s_1 \models T \Rightarrow s_2 \models T)$$

One important question about information and situations is this: How does information about one situation give us information about another situation? This is related to a question about semantics: What are the descriptive and demonstrative contents of a typical utterence of a sentence *if* S_1 *then* S_2? After all, conditionals tell us how *antecedent* situations are related to *consequent* situations. A partial answer to this question is that the descriptive content is some kind of *constraint* of the kind $T_1 \rightarrow T_2$, where T_1 is the descriptive content of S_1 and T_2 is the descriptive content of S_2. As an example, the utterance

If white exchanges knights on $d5$ she will lose a pawn

has the constraint $T_1 \rightarrow T_2$ as its descriptive content (under the appropriate circumstances), where T_1 is the situation-type in which white exchanges knights on $d5$, and T_2 is the situation-type in which she loses a pawn.

The rest of the answer to the question is less easy. What is the demonstrative content of an utterance *if* S_1 *then* S_2? What bit of the world is the speaker talking about when she makes this claim? What does the speaker classify as being of type $T_1 \rightarrow T_2$? One answer to this is immediate. Whatever it is, it must be a situation, if utterances of conditionals are to be analysed in the same way as other declarative utterances. In our case, the situation is the chess game. We are talking about the white player, and what will happen if she exchanges knights on $d5$. So much is common sense. The demonstrative content of the utterence is a situation. A deeper question is the following: what is it about the situation that makes it of that type? Barwise takes this to be an *information channel*: a connection between situations. These channels might arise because of physical laws, conventions, or logical necessity. However they come about, they are classified by constraints. What it means for a channel $\stackrel{c}{\mapsto}$ to support the constraint $T_1 \rightarrow T_2$ is that if $s_1 \stackrel{c}{\mapsto} s_2$ and s_1 is of type T_1 then s_2 is of type T_2.

In our case it seems that the channel talked about is a situation containing facts about the rules of chess, and the psychology and the chess-playing ability of white's opponent. If the conditional is true, the situation links all exchanging-knight situations to future situations in which white loses a pawn. This is one of many regularities. Generally speaking, we may say that there is a ternary relation $\mapsto$ between situations, where $t \stackrel{s}{\mapsto} u$ means, s is a channel from t to u, or equivalently, relative to s, t, u is an antecedent/consequent pair. Then we have the condition governing the behaviour of $\rightarrow$ for types.

$$s \models A \rightarrow B \text{ iff for each } t, u \text{ where } t \stackrel{s}{\mapsto} u, \text{ if } t \models A \text{ then } u \models B$$

We need to consider the relationship between $\mapsto$ and $\sqsubseteq$. The following monotonicity condition is not too difficult to justify.

$$\text{If } t \stackrel{s}{\mapsto} u \text{ then if } s' \sqsubseteq s,\ t' \sqsubseteq t \text{ and } u \sqsubseteq u' \text{ then } t' \stackrel{s'}{\mapsto} u' \text{ too.}$$

Read $t \overset{s}{\mapsto} u$ as 'applying the information in s to t gets you information contained in u'. This reading is in line with the evaluation clauses for constraints. Given this reading, if s shrinks to s', the result of applying s' to t will still be in u. Similarly if t shrinks, and if u expands the condition will still hold.

Finally, it is helpful to have a situation l which 'records' logical deduction. This means, the situation l simply pairs up situations with themselves (and those contained by them).

$$s \overset{l}{\mapsto} s' \text{ iff } s \sqsubseteq s'$$

A structure satisfying these conditions is called a *bare information frame*.

This does not yet have much to do with the systems discussed earlier. However, this is temporary. We can model the extra connectives simply.

$$s \models \top \text{ always,} \qquad s \models \Box \text{ if and only if } l \sqsubseteq s,$$

So, $\top$ is a piece of trivial information, which we have for free, everywhere. On the other hand, $\Box$ is information which warrants *logic*.

$$s \models A \circ B \text{ iff for some } t, u \text{ where } t \overset{u}{\mapsto} s,\ t \models A \text{ and } u \models B,$$

$$s \models B \leftarrow A \text{ iff for each } t, u \text{ where } s \overset{t}{\mapsto} u, \text{ if } t \models A \text{ then } u \models B.$$

These conditions are straightforward. Consider the statement $s \overset{t}{\mapsto} u$. This means that applying the information in t to s gives information which is already in u. If you like, we can abbreviate this as $s;t \sqsubseteq u$, where we think of $s;t$ as holding the information from s 'filtered through' t. (Barwise calls this operation 'serial composition of channels'.) Then the condition for $\circ$ makes $\circ$ interpret this composition operation. The condition for $\leftarrow$ simply changes the roles in the channeling relation. This makes us read $s \overset{t}{\mapsto} u$ (or $s;t \sqsubseteq u$) as applying s to t instead of t to s. This makes $\leftarrow$ the dual residual of $\circ$ to $\rightarrow$, which is what we would expect.

We are nearly at the point where we have a match between the logic of information frames and the system discussed above. We have the logic of conjunction and disjunction right (they make a distributive lattice), $\top$ is truly a top element and the conditionals $\rightarrow$ and $\leftarrow$ are residuals for the concatenation operation $\circ$. Only two things remain. We do not yet have the associativity of concatenation, and $\Box$ is not yet a *left* identity.

Firstly, associativity. When we add the complex condition below

$$(\exists u)(s \overset{t}{\mapsto} u \text{ and } u \overset{w}{\mapsto} x) \Longleftrightarrow (\exists v)(s \overset{v}{\mapsto} x \text{ and } t \overset{w}{\mapsto} v)$$

we have the associativity of concatenation. (This is a simple exercise in expanding the definition.) However, this condition not only *works*, but it has a justification in terms of the associativity of concatenation. Read $s \overset{t}{\mapsto} u$ as $s;t \sqsubseteq u$, and $u \overset{w}{\mapsto} x$ as $u;w \sqsubseteq x$. Then it certainly looks like we ought to have $(s;t);w \sqsubseteq x$. But then by associativity $s;(t;w) \sqsubseteq x$ and so, for some v, $s;v \sqsubseteq x$ where $t;w \sqsubseteq v$ as desired. So, this condition comes out of our considerations of associativity, and it gives us the desired results. We will call a bare frame which also satisfies this condition an *associative frame*.

Finally, consider $\Box$. We have $s \overset{l}{\mapsto} s'$ if and only if $s \sqsubseteq s'$. This means that if $s \models T$ we must have $s \models T \circ \Box$. And similarly, if $s \models T \circ \Box$ we must have $t \overset{u}{\mapsto} s$ where $t \models T$ and $u \models \Box$. But this means that $l \sqsubseteq u$ and hence $t \overset{l}{\mapsto} s$. But $t \models T$ and this gives $s \models T$. So, $\Box$ certainly is a right identity. To make it a left identity we need to stipulate an extra condition

$$l \overset{s}{\mapsto} s' \Longleftrightarrow s \sqsubseteq s'$$

This is the original condition for l with the channel relation swapped around. This means that l is not only a 'logic' channel when $\mapsto$ is interpreted standardly (as in the clauses for $\rightarrow$) but also a 'logic' channel when $\mapsto$ is interpreted dually (as in the clauses for $\leftarrow$).

We will call an associative frame which satisfies this condition an *associative frame with a two-sided logic*. These frames are a good match for our formal system **P**. Here is how it works. Recall consequence in the Gentzen system. It is modelled by the ordering relation $\leq$ in our Peirce monoids. And it is simple to show that in general, $x \leq y$ if and only if $1' \leq x \Rightarrow y$, or equivalently, $A \vdash B$ if and only if $\Box \vdash A \rightarrow B$. More generally, we have $X \vdash B$ if and only if $\Box \vdash d(X) \rightarrow B$, where $d(X)$ is the 'debunchification' of X. (Set $d(A) = A$, $d(X;Y) = d(X) \circ d(Y)$, $d(X,Y) = d(X) \wedge d(Y)$.) So the consequences of the 'logic' fact $\Box$ record all of the other consequences in our logic.

Fact 3.1
The Gentzen system provides a sound proof procedure for associative information frames with a two-sided logic. That is, if $\Box \vdash A$ then $l \models A$ in all associative information frames.

The proof of this fact is easier if you prove the more general implication $(X \vdash A) \Rightarrow (l \models d(X) \rightarrow A)$. This is a simple induction on the length of the proof of $X \vdash A$.

Fact 3.2
The Gentzen system provides a complete proof procedure for associative information frames with a two-sided logic. That is, if $l \models A$ in all associative information frames, then $\Box \vdash A$.

This proof is quite a bit more complex. (Lengthy presentations are available elsewhere [19,20,23] so we will only sketch the proof.) As usual, you prove the contrapositive. Given an A such that $\Box \nvdash A$, we wish to find an associative information frame with a two-sided logic in which $l \not\models A$. This is constructed from the logic itself. Take the situations in the frame to be the *prime theories* of the logic. That is, the sets of formulae which are (1) closed under consequence (if $A \in \Sigma$ and $A \vdash B$ then $B \in \Sigma$) (2) closed under conjunction (if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$) and (3) prime (if $A \vee B \in \Sigma$ then either $A \in \Sigma$ or $B \in \Sigma$). The logic situation is the set of all consequences of $\Box$ (this can be seen to be a prime theory, with a little work). We take $\Sigma \overset{\Gamma}{\mapsto} \Delta$ if and only if for each $A \rightarrow B \in \Gamma$ if $A \in \Sigma$ then $B \in \Delta$ as you would expect. Finally, we require that $\Sigma \models A$ if and only if $A \in \Sigma$. It remains to be checked that this construction satisfies the conditions for frames. This is not difficult, but it requires a few tedious steps (such as checking that if $A \rightarrow B \notin \Gamma$ then there are appropriate theories Σ and Δ where $A \in \Sigma$, $B \notin \Delta$ where $\Sigma \overset{\Gamma}{\mapsto} \Delta$). The technique for these steps is covered adequately elsewhere [2,19,20,23].

Those who know of the ternary relational semantics for relevant logics will be familiar with these constructions, for information frames simply *are* ternary relational frames. It is interesting to see these frames arising from completely independent considerations.

4 Conclusion

What do these three presentations of the one system have in common? In each, the central work is done by an associative operation, residuated on the left and the right. These residuals are naturally interpreted as conditionals. The fact that the operation is associative makes it important not only for composition of relations (as its original purpose dictated) but also concatenation of strings, or concatenation of channels down which information may flow. In each of these cases, we would not expect the concatenation to be commutative or idempotent, so the logic of the conditionals is much weaker than intuitionistic logic, in which premise combination is associative, commutative *and* idempotent. (We do not have $A \to \big((A \to B) \to B\big)$, $A \to (B \to A)$ or $A \wedge (A \to B) \to B$ as theorems in our system.) This means that the conditionals are *resource sensitive*. The *order* and *number* of antecedents is important. There are many other applications in which such a 'conjunction' appears. For example, actions can be conjoined to make larger actions. This action conjunction is associative, but certainly not commutative. After all, punching someone and then apologising is certainly a different action to apologising and then punching someone. (This opens up another fertile field of comparison, which I was only made aware of after this paper was largely completed. It turns out that **P** is identical to Pratt's logic **ACT** of actions [17,18], without the 'star' operator modelling transitive closure. This places **P** on Pratt's roadmap of two-dimensional logics. Clearly there is much more work to be done here. To name but three possibilities, we ought to examine the relationships to other logics Pratt examines; we ought to interpret this 'star' in the context of linguistic types or situations; and we ought to see whether information frames provide illumination to the logics of action.)

But back to our original train of thought. The second common factor in our field of applications is the behaviour of the extensional connectives $\wedge$ and $\vee$. Linear logic and its cousins (famously 'resource aware' systems) fail to validate distribution. In our applications, distribution is an important and a natural feature. Conjunction and disjunction behave exactly as one expects them to do.

This surprising agreement on composition, the conditional, conjunction and disjunction, is not reflected in negation. Clearly, negation is an interesting issue for each application of **P**. **RA** has its own negation: Boolean negation on relations. There are also more subtle kinds, which incorporate the converse operation. Boolean negation has application in the Lambek calculus, where we can say that x is of type $-A$ just when x is not of type A. Then, nothing is of type $A \wedge -A$, and everything is of type $A \vee -A$, so we have $B \to A \vee -A$ and $A \wedge -A \to B$ as theorems — these are characteristic of Boolean negation. However, this is most certainly *not* the kind of negation important in situation semantics. Barwise and Perry convincingly argue that situations ought to be allowed to be incomplete (neither $s \models A$ nor $s \models -A$) and inconsistent (sometimes $s \models A$ and $s \models -A$). However we model negative information, it ought *not* satisfy these Boolean conditions. For some sketches of how we ought treat negative information, the reader will have to look elsewhere [21,22].

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