# INTRODUCTION 1

## QUESTIONS FOR YOU

1. Recall this reasoning:

> Melbourne is east of Adelaide, and Adelaide is east of Perth, so Melbourne is east of Perth. As Sydney is east of Melbourne, it follows that Sydney is east of Perth.

which we represented by this tree:

$$\dfrac{\textit{Sydney is east of Melbourne} \qquad \dfrac{\textit{Melbourne is east of Adelaide} \quad \textit{Adelaide is east of Perth}}{\textit{Melbourne is east of Perth}}\,(1)}{\textit{Sydney is east of Perth}}\,(2)$$

Construct a *different* proof from the same premises to the same conclusion, using the same principles but combining them in a different way. How many ways are there to do this?

2. Consider this argument.

This is adapted from Howard Posposel and David Marans' text *Arguments: Deductive Logic Exercises* (1978, page 57). It is a useful sourcebook for more arguments to analyse.

> Hunger is caused either by the stomach, by blood acting on the brain, or by all of the body's cells. If the stomach causes hunger, then removing stomach nerves in animals will interfere with normal eating. However, removing those nerves does not interfere with normal eating. So, the stomach doesn't cause hunger. Brain activity always starts with blood entering the brain. It follows that blood acting on the brain doesn't cause hunger. Thus, we can conclude that hunger is caused by all of the body's cells.

Represent this reasoning in the form of a tree, making sure to identify the ultimate conclusion and the intermediate conclusions we draw along the way, and how they are related.

3. Which of these are formulas in our formal language Form, and which are not?

$$p \lor q \qquad p \lor q \to r \qquad \neg\neg p \qquad q\neg p \qquad p \land (q \lor r) \to \bot$$

$$(p \to q) \to ((p \to (q \to r)) \to (p \to r)) \qquad p \land q \land r$$

For those that aren't formulas, are they ambiguous? (Could they be made into correct formulas in different ways by adding parentheses?) If they are, disambiguate them, by listing all of the different ways they can be made formulas, and consider for yourself the different things they could *mean*.

4. For the formulas you identified in the previous question, list all of their *subformulas*.

5. Take some reasoning you've seen in other subjects (philosophy subjects, or anything else) and try to map out that reasoning in the form of a tree. What do you notice? Do you see any logical concepts (conditionals, conjunction, disjunction, negation, or anything else) playing a role in this reasoning?

6. Why can't a formula be *infinitely long*, according to our definition of formulas? What would the problem be with a formula like

$$\cdots\neg\neg\neg\neg p$$

   with an unending series of negations before the $p$, or

$$p_1 \wedge (p_2 \wedge (p_3 \wedge (p_4 \wedge \cdots)))$$

   where the conjunction goes on forever?

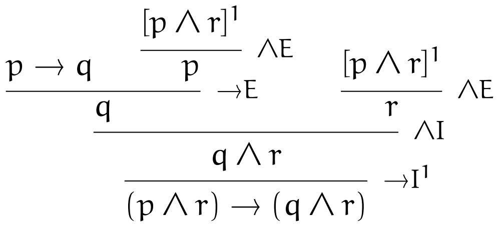
## KEY CONCEPTS AND SKILLS

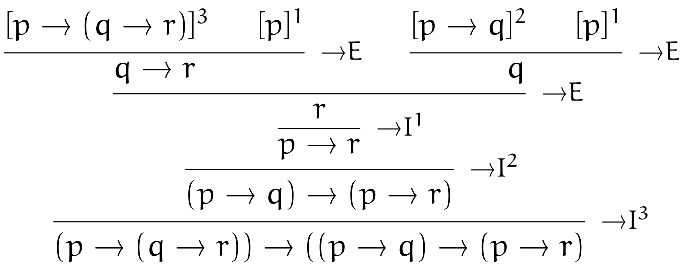
- ☐ You can identify premises and conclusions in a course of reasoning presented in a natural language argument.
- ☐ You understand the definitions of the concepts *partial order* and *tree*. You know how to check if a partial order is also a tree, and you can construct examples of partial orders that aren't trees. You can represent finite trees in tree diagrams.
- ☐ You can represent the structure of reasoning of simple arguments in the form of a tree, distinguishing premises and conclusions, individual inference steps and recognising the ultimate conclusion of a proof.
- ☐ You can construct formulas in the formal propositional language Form. You know how to read formulas, recognising conjunction ($\wedge$), disjunction ($\vee$), the conditional ($\to$) and negation ($\neg$), and you are able to detect whether something is actually a formula or if it is not formed using the formation rules of the formal language Form.
- ☐ You can identify the main connective of a complex formula, and the subformulas of a formula.

# CONNECTIVES: *AND & IF* 2

## QUESTIONS FOR YOU

1. Look at these proofs. Read them from top to bottom, and at every inference step, list which assumptions each formula depends on.

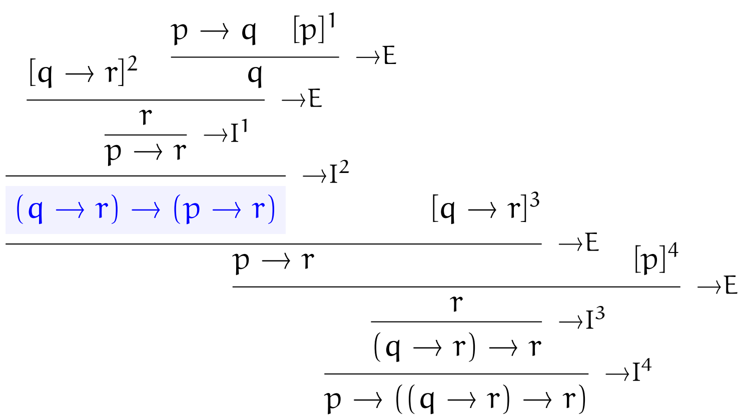




2. Construct proofs for the following arguments:

   - From $p \to q$, $r \to p$ and $r$ to $q$.
   - From $p \to q$ and $r \to p$ to $r \to q$.
   - From the assumption $p \to q$ to $(r \to p) \to (r \to q)$.
   - From $p$ to $q \to (p \wedge q)$.
   - From $p \wedge (q \to r)$ to $q \to (p \wedge r)$.

3. Consider this proof.



It has a detour formula marked in blue. Use the reduction step for detours to eliminate this detour formula. Does the proof you get have another detour formula (a conditional that is introduced and then immediately eliminated)? If so, reduce it, too, and keep reducing detour formulas until the result is a proof with no detours.

4. What do you think of the requirement that a detour (a formula introduced and then eliminated) should always be able to be reduced in a proof? This requirement makes sense for $\wedge$ and for $\rightarrow$ and the rules we have for these concepts. Does it make sense for inference rules for *any* concept? Can you think of proof rules for a concept that might violate this condition?

5. (This is a question to provoke discussion. It does not have a straightforward answer.) What does $\rightarrow$ *mean*? If we can infer $q \rightarrow p$ from $p$ (and the rules we've given seem to have that consequence), does '$\rightarrow$' mean anything like '*if* … *then*'? (Notice that we started with the rule $\rightarrow$E (*modus ponens*), and this seems to be a basic feature of any conditional, and $\rightarrow$I was motivated by arguing that we can infer $A \rightarrow B$ whenever we can infer from $A$ to $B$, which is what we need for *modus ponens*.)

## KEY CONCEPTS AND SKILLS

☐ You should be able to *read* tree proofs using the rules $\wedge$E, $\wedge$I, $\rightarrow$E and $\rightarrow$I. You should be able to check that a proof follows the rules, and you should be able to keep track of which assumptions are active at each stage of the proof.

☐ You should be able to *construct* simple tree proofs using the rules $\wedge$E, $\wedge$I, $\rightarrow$E and $\rightarrow$I.

☐ You can perform *reductions* on tree proofs which involve detours, using the reduction steps.

# CONNECTIVES: *NOT & OR* 3

## QUESTIONS FOR YOU

1. Read the following two proofs, from top to bottom, and at every step, list which assumptions each formula depends on.

$$\dfrac{\dfrac{[p \vee q]^3 \quad \dfrac{\dfrac{\neg p \wedge \neg q}{\neg p}{\scriptstyle \wedge E} \quad [p]^1}{\bot}{\scriptstyle \neg E} \quad \dfrac{\dfrac{\neg p \wedge \neg q}{\neg q}{\scriptstyle \wedge E} \quad [q]^2}{\bot}{\scriptstyle \neg E}}{\bot}{\scriptstyle \vee E^{1,2}}}{\neg(p \vee q)}{\scriptstyle \neg I^3}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p \vee q]^3 \quad \dfrac{\dfrac{(p \to \neg r) \wedge (q \to \neg r)}{p \to \neg r}{\scriptstyle \wedge E} \quad [p]^1}{\neg r}{\scriptstyle \to E} \quad \dfrac{\dfrac{(p \to \neg r) \wedge (q \to \neg r)}{q \to \neg r}{\scriptstyle \wedge E} \quad [q]^2}{\neg r}{\scriptstyle \to E}}{\neg r}{\scriptstyle \vee E^{1,2}} \quad [r]^4}{\bot}{\scriptstyle \neg E}}{\neg(p \vee q)}{\scriptstyle \neg I^3}}{r \to \neg(p \vee q)}{\scriptstyle \to I^4}$$

2. Construct proofs for the following arguments:

   - $p \succ \neg\neg p$
   - $p \to r, q \to s \succ (p \wedge q) \to (r \wedge s)$
   - $p \to r, q \to s \succ (p \vee q) \to (r \vee s)$
   - $\neg p \vee \neg q \succ \neg(p \wedge q)$
   - $\neg\neg\neg p \succ \neg p$

3. Here is a proof for the argument $(p \to r) \wedge (q \to r) \succ (p \wedge q) \to r$. It contains a detour formula, marked in blue. Use the reductions to eliminate the detour.

$$\dfrac{\dfrac{\dfrac{\dfrac{(p \to r) \wedge (q \to r)}{p \to r}{\scriptstyle \wedge E}}{(p \to r) \vee (q \to r)}{\scriptstyle \vee I} \quad \dfrac{[p \to r]^1 \quad \dfrac{[p \wedge q]^3}{p}{\scriptstyle \wedge E}}{r}{\scriptstyle \to E} \quad \dfrac{[q \to r]^2 \quad \dfrac{[p \wedge q]^3}{q}{\scriptstyle \wedge E}}{r}{\scriptstyle \to E}}{r}{\scriptstyle \vee E^{1,2}}}{(p \wedge q) \to r}{\scriptstyle \to I^3}$$

4. Here is a proof, from $(p \to q) \lor r$ to $p \to ((q \lor r) \lor s)$. Does this proof contain any detours?

$$
\dfrac{\dfrac{\dfrac{\dfrac{(p \to q) \lor r \qquad \dfrac{\dfrac{\dfrac{[p \to q]^3 \quad [p]^1}{q}\,{\to}E}{q \lor r}\,{\lor}I}{p \to (q \lor r)}\,{\to}I^1 \qquad \dfrac{\dfrac{[r]^4}{q \lor r}\,{\lor}I}{p \to (q \lor r)}\,{\to}I^2}{p \to (q \lor r)}\,{\lor}E^{3,4} \qquad [p]^5}{q \lor r}\,{\to}E}{(q \lor r) \lor s}\,{\lor}I}{p \to ((q \lor r) \lor s)}\,{\to}I^5
$$

If it does contain detours, what is the detour formula, and where is it introduced and eliminated? Are there any reduction steps to reduce the proof? If there are no detour formulas, how can you explain the presence of the formula $p \to (q \lor r)$ which is not a subformula of the open assumption of the proof ($(p \to q) \lor r$) and is not a subformula of the conclusion ($p \to ((q \lor r) \lor s)$)?

5. Describe in your own words a process for how to construct a proof for an argument. Imagine attempting to program a computer to construct a proof for $X \succ A$. How do you describe the process?

## KEY CONCEPTS AND SKILLS

☐ You should be able to *read* tree proofs using any or all of the rules ($\land E$, $\land I$, $\to E$, $\to I$, $\neg E$, $\neg I$, $\bot E$, $\lor E$, $\lor I$). You should be able to check that a proof follows these rules, and you should be able to keep track of which assumptions are active at each stage of the proof.

☐ You should be able to *construct* simple tree proofs using all the rules.

☐ You should be able to perform *reductions* on tree proofs which involve detours, using the reduction steps.

# FACTS ABOUT PROOFS & PROVABILITY 4

## QUESTIONS FOR YOU

1. Recall that $X$ is said to be inconsistent if and only if there is a proof of $\bot$ from $X$. Which of the following sets are inconsistent? For those that are consistent, prove $\bot$ from those premises. For those sets that aren't, try to explain why they aren't inconsistent.

   (a) $p, q, \neg(p \wedge q)$

   (b) $p \vee q, \neg p \vee \neg q$

   (c) $\neg p, q, p \rightarrow q$

   (d) $p, \neg q, p \rightarrow q$

   (e) $p, q, \neg(p \rightarrow q)$

   (f) $p \rightarrow q, \neg((q \rightarrow r) \rightarrow (p \rightarrow r))$

   (g) $\neg(p \rightarrow q), (q \rightarrow r) \rightarrow (p \rightarrow r)$

2. We say that $A$ and $A'$ are *logically equivalent* if $A \vdash_I A'$ and $A' \vdash_I A$. That is, there is a proof from $A$ to $A'$ and a proof from $A$ to $A'$. Show the following general facts about provability, assuming that $A$ and $A'$ are logically equivalent formulas.

   (a) If $X \vdash_I A$ then $X \vdash_I A'$,

   (b) If $X, A \vdash_I B$ then $X, A' \vdash_I B$,

   (c) $A \wedge B$ is logically equivalent to $A' \wedge B$,

   (d) $A \rightarrow B$ is logically equivalent to $A' \rightarrow B$,

   (e) $B \rightarrow A$ is logically equivalent to $B \rightarrow A'$,

   (f) $\neg A$ is logically equivalent to $\neg A'$,

   (g) $A \vee B$ is logically equivalent to $A' \vee B$.

   (h) Explain why it follows for any complex formula $C(A)$ with $A$ as as subformula, $C(A)$ is logically equivalent to $C(A')$, where $C(A')$ is found by replacing the $A$ in $C$ by $A'$.

3. A set $X$ of formulas is *purely positive* if and only if it does not contain $\neg$ or $\bot$ as a subformula. This question will help you show that *no* set of purely positive formulas is inconsistent. If $X$ is purely positive, we cannot have $X \vdash_I \bot$.

   (a) First explain why if there is a proof for $X \succ \bot$, then there is also proof for $X^p \succ \bot$, where $X^p$ is found by replacing all of the atoms in every formula in $X$ by the one atom $p$.

(b) Then explain why every purely positive formula made up from the atom $p$ is either equivalent to $p$, or equivalent to $p \to p$. (Recall from the previous question that $A$ is equivalent to $B$ if $A \vdash_I B$ and $B \vdash_I A$.)

(c) Then explain why there is no normal proof for $p \succ \bot$, or $p \to p \succ \bot$ or for $p, p \to p \succ \bot$, without appealing to the subformula property in normal proofs.

(d) Then put all of this together to conclude that if $X$ is purely positive, then we never have $X \vdash_I \bot$.

4. Which of these general "facts" about provability are really *facts*. For those that aren't, can you give any reasons why they aren't? For those that are, can you prove them?

   (a) If $A \vdash_I B$ then $\neg B \vdash_I \neg A$.

   (b) If $A \vdash_I B$ then it's not true that $B \vdash_I A$.

   (c) Either $A \vdash_I B$ or $B \vdash_I A$.

   (d) $X, A \vdash_I \neg A$ if and only if $X \vdash_I \neg A$.

   (e) $X \vdash_I A \vee B$ if and only if $X \vdash_I A$ or $X \vdash_I B$.

5. Complete the proof of Theorem 3, by showing that

   - $X \vdash_I A \wedge B$ if and only if $X \vdash_I A$ and $X \vdash_I B$.
   - $X \vdash_I \neg A$ if and only if $X, A \vdash_I \bot$.

   (You can follow the reasoning in the proof of Theorem 3 pretty closely. In particular $\neg A$ is rather like $A \to \bot$.)

6. Consider the following proof, from the premise $(p \to q) \vee r$ to the conclusion $p \to ((q \vee r) \vee s)$. Mark out all of the detour sequences in this proof.

$$
\dfrac{\dfrac{\dfrac{\dfrac{(p \to q) \vee r \quad \dfrac{\dfrac{\dfrac{[p \to q]^3 \quad [p]^1}{q}\,{\to}E}{q \vee r}\,{\vee}I}{p \to (q \vee r)}\,{\to}I^1 \quad \dfrac{\dfrac{[r]^4}{q \vee r}\,{\vee}I}{p \to (q \vee r)}\,{\to}I^2}{p \to (q \vee r)}\,{\vee}E^{3,4} \quad [p]^5}{q \vee r}\,{\to}E}{(q \vee r) \vee s}\,{\vee}I}{p \to ((q \vee r) \vee s)}\,{\to}I^5
$$

   You will notice that there are sequences involving the minor premises of the $\vee$E inference. Permute the $\to$E inference above the $\vee$E inference, so the detour sequences are reduced to length 1.

   Then eliminate those detours in the proof, using reduction steps. Is the resulting proof normal? If so, verify that it has the subformula property. If not, reduce it, and keep reducing it, until you have a normal proof, verifying that this proof indeed has the subformula property.

7. Use DNE to find arguments to show that the following *classical validities* hold.

   (a) $\neg(p \to q) \vdash_C p$,

   (b) $\neg(p \land q) \vdash_C \neg p \lor \neg q$,

   (c) $\vdash_C ((p \to q) \to p) \to p$.

   In the first two cases, first try proving the double negation of the conclusion and then appeal to DNE at the end. For example, first find an argument for $\neg(p \to q) \succ \neg\neg p$ and do *this* by constructing a proof for $\neg(p \to q), \neg p \succ \bot$. This is the same sort of strategy we used for proving $A \lor \neg A$ using DNE.

   Beware, the argument for the last one is rather tricky. Don't be worried if it takes you a while. The hint for this is to not try to prove the double negation of $((p \to q) \to p) \to p$, but prove $\neg\neg p$ from the assumption $(p \to q) \to p$ first.

## KEY CONCEPTS AND SKILLS

- ☐ You should be able to reason about and verify simple general facts about provability ($\vdash_I$), such as the facts expressed in Theorems 1, 2 and 3, and in Questions 2 and 4 in this question set.
- ☐ You should be able to recognise detour formulas and detour sequences in non-normal proofs, make a one step reduction of the detour, including detours involving a formula being introduced in a $\bot$E rule and eliminated in an elimination rule.
- ☐ You should be able to explain the significance of the normalisation theorem, and the subformula property.
- ☐ You can do simple proofs involving DNE.

# MODELS & COUNTEREXAMPLES 5

## QUESTIONS FOR YOU

1. Complete truth tables for these formulas, and decide whether they are tautologies, contradictions or contingencies:

   (a) $\neg(p \land \neg p)$

   (b) $p \land (p \to \bot)$

   (c) $(p \land (p \to \bot)) \to \bot$

   (d) $p \to \neg q$

   (e) $(p \to \neg q) \to (q \to \neg p)$

   (f) $q \to (p \land (p \to q))$

2. Which of these arguments are valid? For those that are, explain why (using valuations), and for those that aren't, provide a counterexample.

   (a) $\neg(p \land q) \succ \neg p \land \neg q$

   (b) $p \to q, q \to r \succ p \to r$

   (c) $p \to q, q \to r \succ \neg p \to \neg r$

   (d) $\neg(p \to q) \succ p$

   (e) $\neg(p \land q) \succ \neg p \lor \neg q$

   (f) $\succ ((p \to q) \to p) \to p$

3. We say that $A$ and $A'$ are *Boolean equivalent* if $A \models_{CL} A'$ and $A' \models_{CL} A$. That is, if $v(A) = v(A')$ for every valuation $v$. Show the following general facts about provability, assuming that $A$ and $A'$ are Boolean equivalent formulas.

   (a) If $X \models_{CL} A$ then $X \models_{CL} A'$,

   (b) If $X, A \models_{CL} B$ then $X, A' \models_{CL} B$,

   (c) $A \land B$ is Boolean equivalent to $A' \land B$,

   (d) $A \to B$ is Boolean equivalent to $A' \to B$,

   (e) $B \to A$ is Boolean equivalent to $B \to A'$,

   (f) $\neg A$ is Boolean equivalent to $\neg A'$,

   (g) $A \lor B$ is Boolean equivalent to $A' \lor B$.

   (h) Explain why it follows for any complex formula $C(A)$ with $A$ as as subformula, $C(A)$ is Boolean equivalent to $C(A')$, where $C(A')$ is found by replacing the $A$ in $C$ by $A'$.

4. Which of these general "facts" about validity are really *facts*. For those that aren't, can you give any reasons why they aren't? For those that are, can you prove them?

   (a) If $A \models_{CL} B$ then $\neg B \models_{CL} \neg A$.

   (b) If $A \models_{CL} B$ then it's not true that $B \models_{CL} A$.

   (c) Either $A \models_{CL} B$ or $B \models_{CL} A$.

   (d) $X, A \models_{CL} \neg A$ if and only if $X \models_{CL} \neg A$.

   (e) $X \models_{CL} A \vee B$ if and only if $X \models_{CL} A$ or $X \models_{CL} B$.

5. A formula is in *disjunctive normal form* (DNF) if it is some number of *disjunctions* of *conjunctions* of *literals*, where a literal is an atom (not including $\bot$) or a negation of an atom (not including $\bot$). We define these notions formally as follows:

   - $A$ is a *literal* iff $A$ is a member of Atom, other than $\bot$, or $A$ is the *negation* of some member of Atom, other than $\bot$.
   - If $A$ is a literal, then we say $A$ is also a *conjunctions of literals* (of conjunction length 1). If $B$ and $C$ are both *conjunctions of literals* (of conjunction lengths $n$ and $m$) respectively, then $(B \wedge C)$ is also a *conjunction of literals* (of conjunction length $n + m$).
   - If $C$ is some conjunction of literals, then we say that $C$ is also a *disjunction of conjunction of literals* (of disjunction length 1). If $D$ and $E$ are disjunctions of conjunctions of literals (with disjunction lengths $k$ and $l$ respectively) then $(D \vee E)$ is a disjunction of conjunctions of literals with disjunction length $k + l$.

   So, for example, $((p \wedge \neg q) \vee \neg r) \vee (\neg p \wedge (r \wedge s))$ is a disjunction of conjunctions of literals, with disjunction length 3. Its disjuncts have conjunction length 2, 1 and 3 respectively.

   In this and the next question, we will show how to find, for any formula $A$, a formula $A'$ Boolean equivalent to $A$, in DNF. In this question, we will show how to find, for some formulas not in disjunctive normal form, another formula equivalent to it, which does not violate the constraints of DNF in that way.

   - $\bot$ is equivalent to $p \wedge \neg p$
   - $A \rightarrow B$ is equivalent to $\neg A \vee B$
   - $\neg\neg A$ is equivalent to $A$
   - $\neg(A \wedge B)$ is equivalent to $\neg A \vee \neg B$
   - $\neg(A \vee B)$ is equivalent to $\neg A \wedge \neg B$
   - $A \wedge (B \vee C)$ is equivalent to $(A \wedge B) \vee (A \wedge C)$
   - $(B \vee C) \wedge A$ is equivalent to $(B \wedge A) \vee (C \wedge A)$

6. For the next part of this question, show that if a formula in Form is *not* a disjunction of conjunction of literals, then it has a *subformula* of one of the following forms:

$$\bot \quad A \to B \quad \neg\neg A \quad \neg(A \land B)$$

$$\neg(A \lor B) \quad A \land (B \lor C) \quad (B \lor C) \land A$$

7. Take the formula $p \to (r \land \neg(p \lor \neg q))$. Choose a subformula of this formula of one of the forms shown in the Question 6, and replace it with an equivalent formula, using the equivalences in Question 5. Is the result in DNF? If not, find another subformula to convert, and continue, until the result is in DNF.

   Do you think this process could work for *any* formula? Could the process of simplification ever go on forever, or will it always terminate in a formula in DNF? Why or why not?

## KEY CONCEPTS AND SKILLS

- ☐ You need to be familiar with the definition of Boolean valuations, given a valuation, you can calculate the value of a complex formula, in terms of the values of its atoms.
- ☐ You should be able to complete truth tables for formulas.
- ☐ You need to know what it means for a formula to be a tautology, a contradiction or a contingency, and you can use truth tables to test for whether a formula is a tautology or a contradiction or contingent.
- ☐ You can test arguments using Boolean valuations.
- ☐ You should be able to reason about and verify simple general facts about validity ($\vdash_I$), such as the facts expressed in Theorems 10 and 12, and in Questions 3 and 4 in these exercises.

# SOUNDNESS & COMPLETENESS 6

## QUESTIONS FOR YOU

1. Complete the proof of the Soundness theorem by completing the cases for $\wedge E$, $\bot E$ and $\vee E$. (For example, for $\wedge E$ we want to show that if $\Pi$, a proof for $X \succ A_1 \wedge A_2$ is truth preserving, then the proof of $A_i$ (whether $i = 1$ or $2$), given by extending $\Pi$ with an $\wedge E$ step, is also truth preserving.)

2. Consider the rules for the biconditional ($\leftrightarrow$).

$$\frac{A \leftrightarrow B \qquad A}{B}\,{\leftrightarrow}E \qquad\qquad \frac{A \leftrightarrow B \qquad B}{A}\,{\leftrightarrow}E \qquad\qquad \frac{\begin{matrix}[A]^1 & [B]^2 \\ \Pi_1 & \Pi_2 \\ B & A\end{matrix}}{A \leftrightarrow B}\,{\leftrightarrow}I^{1,2}$$

   Give Boolean valuation rules for biconditional formulas (explain when $v(A \leftrightarrow B) = 1$, and when it takes the value $0$), such that the Soundness Theorem still holds for the proofs and the models.

3. Let's grant that adding DNE to our proof system is enough to make it complete for Boolean validity. Would adding the following negation rule suffice instead? Why or why not?

$$\frac{\begin{matrix}[\neg A]^i \\ \Pi \\ \bot\end{matrix}}{A}\,\textit{reductio}^i$$

4. Consider the following arguments. Which have counterexamples in this three-valued Heyting algebra?

| $\wedge$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

| $\vee$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{2}$ | $1$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

| $\rightarrow$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ |
| $\frac{1}{2}$ | $0$ | $1$ | $1$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

| $\neg$ | |
|---|---|
| $0$ | $1$ |
| $\frac{1}{2}$ | $0$ |
| $1$ | $0$ |

   (a) $\neg\neg p \succ p$

   (b) $\neg(p \wedge q) \succ \neg p \vee \neg q$

   (c) $\succ (p \rightarrow q) \vee (q \rightarrow p)$

   (d) $\succ \neg p \vee \neg\neg p$

   (e) $(p \rightarrow q) \rightarrow p \succ p$

5. Design a two-place or three-place connective of your own. Start by specifying an introduction rule and an elimination rule for your connective. Like the other introduction and elimination rules, your rules should not use any of the *other* connectives. The rules for $A \sharp B$, for example, should tell us how to deduce something from $A \sharp B$ (in terms of A and B and any other premises or conclusions), and how to infer $A \sharp B$ (in terms of A and B and any other premises or conclusions). Show how to eliminate detours in a proof arising out of your introduction and elimination rules.

Thanks to Dave Ripley for coming up with this exercise.

6. Continuing on from the previous question, find a way to interpret your connective using *valuations*. Show that your introduction and elimination rules are *sound* for the valuation rules you chose.

## KEY CONCEPTS AND SKILLS

☐ You need to understand—and to clearly state for yourself—the definitions of soundness and completeness and the difference between them.

☐ You should understand the proof of soundness of intuitionistic and classical proofs for Boolean validity. In particular, you should understand the shape of the argument (an induction on the construction of the proof in question), and you should be able to prove particular instances for yourself.

☐ You understand what arguments count as counterexamples to the completeness of intuitionistic provability for Boolean validity, and how adding DNE strengthens the system to give completeness.

☐ You can evaluate formulas and arguments in the simple three valued Heyting algebra, when given the truth tables to work from.

☐ You understand the connections between proofs and models, and inferentialism and representationalism.