An Inferentialist Account of Identity and Modality

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My Aim

To show how defining rules
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To show how defining rules in a hypersequent setting can give a uniform proof-theoretic semantics of identity
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To show how *defining rules* in a *hypersequent* setting can give a uniform proof-theoretic semantics of *identity* and *modality*,

(1) modal operators for which identity statements are necessary (if true),

(2) modal operators for which identity statements can be contingently true.
My Aim

To show how *defining rules* in a *hypersequent* setting can give a uniform proof-theoretic semantics of *identity* and *modality*, allowing — equally naturally — for (1) modal operators for which identity statements are necessary (if true),
To show how defining rules in a hypersequent setting can give a uniform proof-theoretic semantics of identity and modality, allowing — equally naturally — for (1) modal operators for which identity statements are necessary (if true), and (2) modal operators for which identity statements can be contingently true.
My Plan

My Approach
Defining Identity
Defining Necessity
Combining Them
The Upshot
MY APPROACH
Define concepts by way of rules for their use. (That’s normative pragmatism.)
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Their semantics is not given in the first instance in terms of reference relations to objects. (That’s semantic anti-realism.)
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Their semantics is not given in the first instance in terms of reference relations to objects. (That’s semantic anti-realism.)

The rules are grounded in our communicative and dialogical practice, involving assertion, denial and inference.

Defining Rules for Logical Concepts

\[ \wedge \text{ Conjunction} \]

\[
\frac{X, A, B \rightarrow Y}{X, A \land B \rightarrow Y} \quad ^\wedge Df
\]

Asserting a conjunction has the same force as asserting its conjuncts.
Defining Rules for Logical Concepts

^ Conjunction

→ Conditional

\[
\frac{X \supset A \rightarrow B, Y}{X \supset A \rightarrow B, Y} \rightarrow \text{Df}
\]

Denying a material conditional has the same force as asserting its antecedent and denying its consequent.

Or equivalently, to prove a material conditional suppose its antecedent and prove its consequent.
Defining Rules for Logical Concepts

\[ \begin{align*}
\land & \quad \text{Conjunction} \\
\rightarrow & \quad \text{Conditional} \\
\forall & \quad \text{Universal Quantifier}
\end{align*} \]

\[ X \rightarrow A[x/n], Y \]

\[ X \rightarrow \forall x A, Y \quad \forall \text{Df} \]

To prove that everything has some feature, prove that \( n \) has that feature, making no assumptions about \( n \).

Or equivalently, denying that everything satisfies \( A \) has the same force as denying that \( n \) satisfies \( A \) for some fresh name ‘n’.
How are these *definitions*?

These defining rules show how to add the defined concept to a language that does not possess it . . .
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. . . not by finding a paraphrase in the original language, but by ‘making explicit’ something that was implicit.

In controlled circumstances, we can prove that concepts added by defining rules like these are *conservative* and *uniquely defined*, just like definitions by paraphrase.
In our everyday vocabulary, we make moves *like these* with our informal concepts ‘and’, ‘not’, ‘if’ and ‘all’.
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The logician’s *sharply defined* concepts $\land$, $\rightarrow$, $\forall$, etc. are *simple*, *well-behaved*, *useful* and *can be followed*. 
Why Define?

In our everyday vocabulary, we make moves like these with our informal concepts ‘and’, ‘not’, ‘if’ and ‘all’.

The logician’s sharply defined concepts $\land$, $\rightarrow$, $\forall$, etc. are simple, well-behaved, useful and can be followed.

Defining rules suffice for us to coordinate and to communicate, especially in those contexts where precision is valuable.
Why Define?

In our everyday vocabulary, we make moves *like these* with our informal concepts ‘and’, ‘not’, ‘if’ and ‘all’.

The logician’s *sharply defined* concepts $\land, \rightarrow, \forall$, etc. are *simple, well-behaved, useful* and *can be followed*.

Defining rules suffice for us to *coordinate* and to *communicate*, especially in those contexts where precision is valuable.

These defining rules give rise to Gentzen-style left/right rules in the presence of *Id* and *Cut*, and the resulting rules allow for the usual *cut elimination* argument.
GENERALITY AND EXISTENCE 1:
QUANTIFICATION AND FREE LOGIC

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Abstract. In this paper, I motivate a cut free sequent calculus for classical logic with first order quantification, allowing for singular terms free of existential import. Along the way, I motivate a criterion for rules designed to answer Prior's question about what distinguishes rules for logical concepts, like conjunction from apparently similar rules for putative concepts like Prior's tonk, and I show that the rules for the quantifiers—and the existence predicate—satisfy that condition.

§ 1. Sequents and defining rules. Let's take it for granted for the moment that learning a language involves—at least in part—learning how assertions and denials expressed in that language bear on one another. The basic connection, of course, is that to assert $A$ and to deny $A$ clash. When we learn conjunction, we learn that there is a clash involved in asserting $A$, asserting $B$ and denying $A \wedge B$. Similarly, when we learn disjunction, we learn that there is a clash involved in asserting $A \vee B$, denying $A$ and denying $B$.

One way to systematically take account of the kinds of clashes involved in these acts of assertion and denial is through the language of the sequent calculus. Given collections $\Gamma$ and $\Delta$ of sentences from our language $\mathcal{L}$, a sequent $\Gamma \rightarrow \Delta$ makes the claim that there is a clash involved in asserting each element of $\Gamma$ and denying each element of $\Delta$.

The structural rules of the sequent calculus can be understood in the following way [16].

Identity:

$A \rightarrow A$.

There is a clash involved in asserting $A$ and denying $A$.

Weakening$^1$:

$$
\frac{\Gamma \rightarrow \Delta}{\Gamma, A \rightarrow \Delta} \quad \text{[KL]}
\quad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow A, \Delta} \quad \text{[KR]}.
$$
DEFINING IDENTITY
1. Harmony

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior (1960–61) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if ‘tonk’ has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen (1934) and elaborated by, e.g., Prawitz (1977). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its elimination-rules:

The introductions represent, as it were, the ‘definitions’ of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen 1934: 80)

For example, if the only ground for assertion of ‘\( p \text{ tonk } q \)’ is given by Prior’s rule:

\[
\frac{p}{p \text{ tonk } q} \text{ tonk-I}
\]

then Prior mis-stated the elimination-rule. It should read

\[
\frac{p \text{ tonk } q}{r} \quad \text{tonk-E}
\]
A Defining Rule for Identity

\[
\begin{align*}
X, Fa \rightarrow Fb, Y & \quad X, Fb \rightarrow Fa, Y \\
\hline
X \rightarrow a = b, Y \\
\end{align*}
\]

Denying \( a = b \) has the same significance as taking there to be some feature \( F \) that holds of \( a \) but not \( b \), or vice versa.

Or equivalently, to prove that \( a = b \), prove \( Fb \) from the assumption \( Fa \) (and vice versa), where the predicate \( F \) is arbitrary.

Identity is a kind of indistinguishability.
A Defining Rule for Identity

\[
X, Fa \supset Fb, Y \quad X, Fb \supset Fa, Y \\
\frac{}{X \supset a = b, Y} = Df
\]

Denying \( a = b \) has the same significance as taking there to be some feature \( F \) that holds of \( a \) but not \( b \), or vice versa.
A Defining Rule for Identity

\[ X, Fa \triangleright Fb, Y \quad X, Fb \triangleright Fa, Y \]
\[ \quad =Df \]
\[ X \triangleright a = b, Y \]

Denying \( a = b \) has the same significance as taking there to be some feature \( F \) that holds of \( a \) but not \( b \), or vice versa.

Or equivalently, to prove that \( a = b \), prove \( Fb \) from the assumption \( Fa \) (and vice versa), where the predicate \( F \) is arbitrary.
A Defining Rule for Identity

\[
\frac{X, Fa \rightarrow Fb, Y \quad X, Fb \rightarrow Fa, Y}{X \rightarrow a = b, Y} = \text{Df}
\]

Denying a = b has the same significance as taking there to be *some* feature F that holds of a but not b, or *vice versa*.

Or equivalently, to prove that a = b, prove Fb from the assumption Fa (and *vice versa*), where the predicate F is *arbitrary*.

Identity is a kind of *indistinguishability*. 
An Example Derivation

\[
\begin{align*}
Fa \rightarrow Fa & \quad Fa \rightarrow Fa \quad =Df\downarrow \\
\rightarrow a = a & \quad \lambda Df\downarrow \\
\rightarrow (\lambda x.x = a)a & \\
\end{align*}
\]

\[
\begin{align*}
\rightarrow (\lambda x.x = a)a & \\
\rightarrow (\lambda x.x = a)b & \quad \lambda Df\uparrow \\
a = b \rightarrow (\lambda x.x = a)b & \\
a = b \rightarrow b = a & \quad \lambda Df\uparrow \\
\end{align*}
\]

\[
\begin{align*}
a = b \rightarrow a = b & \quad =Df\uparrow \\
a = b, Fa \rightarrow Fb & \\
\end{align*}
\]

\[
\begin{align*}
a = b, (\lambda x.x = a)a \rightarrow (\lambda x.x = a)b & \\
Spec_{(\lambda x.x = a)}^F & \\
\end{align*}
\]

\[
\begin{align*}
a = b \rightarrow a = b \rightarrow a = b & \\
a = b \rightarrow b = a & \\
\end{align*}
\]
An Example Derivation

\[
\begin{align*}
X, Fa \succ Fb, Y & \quad X, Fb \succ Fa, Y \\
\hline
X \succ a = b, Y & = Df
\end{align*}
\]

\[
\begin{align*}
Fa \succ Fa & \quad Fa \succ Fa \\
\hline
\succ a = a & = Df_{\downarrow}
\end{align*}
\]

\[
\begin{align*}
\succ (\lambda x.x = a)a & \quad a = b \succ a = b \\
\hline
a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b & = Df_{\uparrow}
\end{align*}
\]

\[
\begin{align*}
a = b \succ (\lambda x.x = a)b & \quad Spec_{(\lambda x.x = a)}^F
\hline
a = b \succ b = a & \quad Cut
\end{align*}
\]
An Example Derivation

\[
\begin{align*}
F_a & \vdash F_a & F_a & \vdash F_a \\
\vdash a = a & \quad \lambda Df \downarrow & a = b, F_a & \vdash F_b \\
\vdash (\lambda x.x = a) a & & a = b, (\lambda x.x = a) a & \vdash (\lambda x.x = a) b & \quad \text{Cut} \\
\vdash a = b & \vdash (\lambda x.x = a) b & a = b & \vdash b = a & \quad \lambda Df \uparrow
\end{align*}
\]

\[
\begin{align*}
X \triangleright Y & \quad \text{Spec}^F \quad \Spec^F_{P} \\
X[F/P] & \triangleright Y[F/P]
\end{align*}
\]
An Example Derivation

\[
\begin{align*}
X, Fa \succ Fb, Y & \quad X, Fb \succ Fa, Y \\
\quad & \quad = Df \\
X \succ a = b, Y & \\
\end{align*}
\]

\[
\begin{align*}
Fa \succ Fa & \quad Fa \succ Fa \\
\quad & \quad = Df \downarrow \\
\quad & \quad \lambda Df \downarrow \\
\succ (\lambda x. x = a) a & \\
\end{align*}
\]

\[
\begin{align*}
a = b \succ a = b & \\
\frac{a = b, Fa \succ Fb}{a = b, (\lambda x. x = a) a \succ (\lambda x. x = a) b} & \quad = Df \uparrow \\
\end{align*}
\]

\[
\begin{align*}
a = b \succ (\lambda x. x = a) b & \\
\frac{a = b}{a = b \succ b = a} & \quad \lambda Df \uparrow \\
\end{align*}
\]
An Example Derivation

\[
\begin{align*}
X \triangleright A[x/a], Y & \quad \vdash \lambda Df \\
X \triangleright (\lambda x.A)x, Y
\end{align*}
\]

\[
\begin{align*}
\frac{Fa \triangleright Fa \quad Fa \triangleright Fa}{\triangleright a = a} & = Df \downarrow \\
\frac{a = b \triangleright a = b}{a = b, Fa \triangleright Fb} & = Df \uparrow \\
\frac{a = b, (\lambda x.x = a)a \triangleright (\lambda x.x = a)b}{Spec^F_{(\lambda x.x = a)}} & \\
\frac{a = b \triangleright (\lambda x.x = a)b}{\triangleright a = b} & = Df \uparrow \\
\frac{a = b \triangleright b = a}{\lambda Df \uparrow}
\end{align*}
\]
An Example Derivation

\[
\begin{align*}
F_a \supset F_a & \quad F_a \supset F_a \quad =Df\downarrow \\
\vdash a = a & \quad \lambda Df\downarrow \\
\vdash (\lambda x.x = a)a & \\
\hline \\
\vdash \frac{a = b \supset a = b}{a = b, F_a \supset F_b} \quad =Df\uparrow \\
\frac{a = b, (\lambda x.x = a)a \supset (\lambda x.x = a)b}{Spec_{(\lambda x.x = a)}} \\
\hline \\
\vdash \frac{a = b \supset (\lambda x.x = a)b}{Cut} \\
\frac{a = b \supset b = a}{\lambda Df\uparrow}
\end{align*}
\]
An Example Derivation

\[ X \succ A[x/a], Y \]
\[ \frac{\lambda Df}{X \succ (\lambda x.A)a, Y} \]

\[ Fa \succ Fa \quad Fa \succ Fa \]
\[ \frac{\succ a = a}{\lambda Df \downarrow} \]
\[ \frac{\succ (\lambda x.x = a)a}{\lambda Df \downarrow} \]

\[ a = b \succ a = b \quad a = b, Fa \succ Fb \]
\[ \frac{\succ a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b}{\text{Spec}^F_{(\lambda x.x = a)}} \]
\[ \frac{a = b \succ (\lambda x.x = a)b}{\frac{\lambda Df \uparrow}{\lambda Df \uparrow}} \]
\[ \frac{a = b \succ b = a}{\lambda Df \uparrow} \]
An Example Derivation

\[
\begin{align*}
X, Fa &\rightarrow Fb, Y & X, Fb &\rightarrow Fa, Y \\
\quad &\rightarrow \quad X & \rightarrow a &\equiv b, Y \\
\end{align*}
\]

[Df]

\[
\begin{align*}
X &\rightarrow A[x/a], Y \\
\quad &\rightarrow (\lambda x. A)a, Y \\
\end{align*}
\]

[\lambda Df]

\[
\begin{align*}
X &\rightarrow Y \\
\quad &\rightarrow X[F/P] & Y[F/P] \\
\end{align*}
\]

[Spec^F_p]

\[
\begin{align*}
Fa &\rightarrow Fa & Fa &\rightarrow Fa \\
\quad &\rightarrow a &\equiv a \\
\end{align*}
\]

[\lambda Df]\]

\[
\begin{align*}
\lambda x. x &\equiv a \\
\quad &\rightarrow (\lambda x. x = a)a \\
\end{align*}
\]

[Spec^F]

\[
\begin{align*}
a &\equiv b \\
\quad &\rightarrow a &\equiv b \\
\end{align*}
\]

[\lambda Df]\]

\[
\begin{align*}
a &\equiv b, Fa &\rightarrow Fb \\
\quad &\rightarrow a &\equiv b \\
\end{align*}
\]

[Spec^F_{(\lambda x. x = a)}]

\[
\begin{align*}
\lambda x. x &\equiv a \\
\quad &\rightarrow a &\equiv b, (\lambda x. x = a)a &\rightarrow (\lambda x. x = a)b \\
\end{align*}
\]

[Cut]

\[
\begin{align*}
a &\equiv b &\rightarrow (\lambda x. x = a)b \\
\quad &\rightarrow a &\equiv b \\
\end{align*}
\]

[\lambda Df]\]

\[
\begin{align*}
a &\equiv b &\rightarrow b &\equiv a \\
\end{align*}
\]
From $Df$ to $L/R$: $\equiv R$ is $\equiv Df\Downarrow$

\[
\frac{X, Fa \supset Fb, Y \quad X, Fb \supset Fa, Y}{X \supset a = b, Y} = Df\Downarrow
\]
From $Df$ to $L/R$: These are the $=L$ rules

$$\frac{X \triangleright Pa, Y \quad X', Pb \triangleright Y'}{a = b, X, X' \triangleright Y, Y'} =_{L_1}$$

$$\frac{X \triangleright Pb, Y \quad X', Pa \triangleright Y'}{a = b, X, X' \triangleright Y, Y'} =_{L_2}$$
From $Df$ to $L/R$: These are the $\equiv L$ rules

\[
\begin{align*}
X \triangleright Pa, Y & \quad X', Pb \triangleright Y' \\
\frac{a = b, X, X' \triangleright Y, Y'}{=L_1}
\end{align*}
\]

\[
\begin{align*}
a = b, X \triangleright Pa, Y & \quad a = b, Pb \triangleright Pb \\
\frac{a = b \triangleright a = b}{=Df^\uparrow} \\
a = b, Fa \triangleright Fb & \quad \frac{a = b \triangleright a = b}{Spec^F_p} \\
\end{align*}
\]

\[
\begin{align*}
X \triangleright Pa, Y & \quad a = b, Pa \triangleright Pb \\
\frac{a = b, Pa \triangleright Pb}{Cut} \\
a = b, X \triangleright Pb, Y & \quad X', Pb \triangleright Y' \\
\frac{a = b, X \triangleright Pb, Y \quad X', Pb \triangleright Y'}{Cut}
\end{align*}
\]
From *Df* to *L/R*: These are the $=L$ rules

\[
\frac{X \succ Pb, Y}{a = b, X, X' \succ Y, Y'} = L_2
\]

\[
\frac{a = b \succ a = b}{a = b, Fb \succ Fa} = Df^f
\]

\[
\frac{X \succ Pb, Y}{a = b, Pb \succ Pa} = S_{\text{Spec}}^F
\]

\[
\frac{a = b, X \succ Pa, Y}{a = b, X', Pa \succ Y'} = \text{Cut}
\]

\[
\frac{a = b, X', Pa \succ Y'}{a = b, X, X' \succ Y, Y'} = \text{Cut}
\]
From $Df$ to $L/R$: These are the $\equiv L$ rules

\[
\frac{X \succ Pa, Y \quad X', Pb \succ Y'}{a = b, X, X' \succ Y, Y'} = L_1
\]

\[
\frac{X \succ Pb, Y \quad X', Pa \succ Y'}{a = b, X, X' \succ Y, Y'} = L_2
\]
Spec^F_p: \[
\frac{X \succ Y}{X[F/P] \succ Y[F/P]}
\]

Cut: \[
\frac{X \succ A, Y \quad X', A \succ Y'}{X, X' \succ Y, Y'}
\]
Going Without Some Rules

\[
\begin{align*}
\frac{X \triangleright Y}{X[F/P] \triangleright Y[F/P]} & \quad Spec^P_F \\
\end{align*}
\]

► *Spec is height-preserving admissible* in the the system with *L/R rules + Cut.*
Going Without Some Rules

\[
\frac{X \vdash A, Y}{X, X' \vdash Y, Y'} \quad \frac{X', A \vdash Y'}{\text{Cut}}
\]

- Spec is *height-preserving admissible* in the system with L/R rules + Cut.
- Cut is eliminable in the system with L/R rules.
Eliminating Cut

\[
\begin{align*}
\vdash \delta_1 & : X, Fa \supset Fb, Y \\
\vdash \delta_2 & : X, Fb \supset Fa, Y \\
\vdash X \supset a = b, Y \\
\end{align*}
\]

\[
\frac{\vdash \delta_2}{X, X \supset X \supset a = b, Y} = R
\]

\[
\frac{\vdash \delta_1}{X, X \supset X \supset a = b, Y} = L_1
\]

\[
\begin{align*}
\vdash \delta' & : X' \supset Pa, Y' \\
\vdash \delta'' & : X'', Pb \supset Y'' \\
\vdash a = b, X', X'' \supset Y', Y'' \\
\vdash X' \supset Y, Y', Y'' \\
\end{align*}
\]

\[
\frac{\vdash \delta'}{X', X'' \supset Y, Y', Y''} = L_1
\]

\[
\frac{\vdash \delta''}{X', X'' \supset Y, Y', Y''} = L_1
\]

\[
\frac{\vdash \delta'}{X', X'' \supset Y, Y', Y''} = L_1
\]
Eliminating Cut

\[ \begin{align*}
\delta_1 & \quad \delta_2 \\
X, Fa & \triangleright Fb, Y & X, Fb & \triangleright Fa, Y \\
= & \quad \frac{X \triangleright a = b, Y}{X' \triangleright Pa, Y', X'' \triangleright Pb, Y''} \\
\Rightarrow & \quad X', X'' \triangleright Y, Y', Y''
\end{align*} \]

becomes

\[ \begin{align*}
\delta' & \quad \delta_1[F/P] \\
X' & \triangleright Pa, Y' & X, Pa & \triangleright Pb, Y \\
\text{Cut} & \quad \frac{X, X' \triangleright Pb, Y, Y'}{X'' \triangleright Pb, Y''} \\
\Rightarrow & \quad X, X', X'' \triangleright Y, Y', Y''
\end{align*} \]
But eliminating *Cut* hardly seems worth it!

$$
\frac{X, Fa \rightarrow Fb, Y}{X, Fb \rightarrow Fa, Y} \quad \frac{X, Fb \rightarrow Fa, Y}{X \rightarrow a = b, Y} = R
$$

$$
\frac{X \rightarrow Pa, Y}{a = b, X, X' \rightarrow Y, Y'} = L_1 \quad \frac{X \rightarrow Pb, Y}{a = b, X, X' \rightarrow Y, Y'} = L_2
$$

Violations of the subformula property everywhere.

For analytic derivations, we need different rules.
But eliminating \textit{Cut} hardly seems worth it!

\[
\begin{align*}
X, \text{F}a & \Rightarrow \text{F}b, Y \quad X, \text{F}b & \Rightarrow \text{F}a, Y \\
\frac{}{X \Rightarrow a = b, Y} &= R
\end{align*}
\]

\[
\begin{align*}
X \Rightarrow \text{P}a, Y & \quad X', \text{P}b \Rightarrow Y' \\
\frac{}{a = b, X, X' \Rightarrow Y, Y'} &= L_1
\end{align*}
\]

\[
\begin{align*}
X \Rightarrow \text{P}b, Y & \quad X', \text{P}a \Rightarrow Y' \\
\frac{}{a = b, X, X' \Rightarrow Y, Y'} &= L_2
\end{align*}
\]

Violations of the subformula property \textit{everywhere}.
But eliminating Cut hardly seems worth it!

\[
\frac{X, Fa \vdash Fb, Y \quad X, Fb \vdash Fa, Y}{X \vdash a = b, Y} = R
\]

\[
\frac{X \vdash Pa, Y \quad X', Pb \vdash Y'}{a = b, X, X' \vdash Y, Y'} = L_1 \quad \frac{X \vdash Pb, Y \quad X', Pa \vdash Y'}{a = b, X, X' \vdash Y, Y'} = L_2
\]

Violations of the subformula property everywhere.

For analytic derivations, we need different rules.
From Rules to Axioms: From $\equiv_R$ to $\text{Refl}$

\[
\text{Refl} \\
\Rightarrow a = a
\]
From Rules to Axioms: From \( =R \) to \( \text{Refl} \)

\[
\text{Refl} \\
\therefore \ a = a
\]

\[
\frac{F a \vdash F a \quad F a \vdash F a}{\therefore \ a = a} =R
\]
Replace this:

\[
\begin{aligned}
\delta_1 & \quad \quad \delta_2 \\
X, Fa \succ Fb, Y & \quad X, Fb \succ Fa, Y \\
\hline
X \succ a = b, Y
\end{aligned}
\]

\[=^R\]
From Refl to \( \equiv_R \)

Replace this:

\[ \delta_1 \]
\[ X, Fa \triangleright Fb, Y \]
\[ X, Fb \triangleright Fa, Y \]

\[ \Rightarrow a = b, Y \]

\[ =R \]

With this:

\[ \delta_1 \]
\[ \lambda R \]
\[ \Rightarrow a = a \]
\[ \Rightarrow \lambda x. (a = x) a \]

\[ \delta_1 [F/\lambda x. (a = x)] \]

\[ \lambda R \]
\[ \Rightarrow \lambda x. (a = x) a \triangleright \lambda x. (a = x) b, Y \]

\[ \Rightarrow \lambda x. (a = x) b, Y \]

\[ \Rightarrow a = b, Y \]

\[ \text{Cut} \]
From \( \equiv L \text{ to } \equiv L.a x \text{ and back.} \)

\[
\begin{align*}
\frac{a = b, Pa \supset Pb}{\equiv L.ax_1} \\
\frac{a = b, Pb \supset Pa}{\equiv L.ax_2}
\end{align*}
\]
From \( \equiv L \) to \( \equiv L.\ ax \) and back.

\[
\frac{a = b, Pa \triangleright Pb}{=L.ax_1}
\]

\[
\frac{Pa \triangleright Pa \quad Pb \triangleright Pb}{a = b, Pa \triangleright Pb} =L_1
\]

\[
\frac{Pa \triangleright Pa \quad Pb \triangleright Pb}{a = b, Pb \triangleright Pa} =L_2
\]
From $\equiv L$ to $\equiv L.ax$ and back.

\[
\begin{align*}
\text{From } &\equiv L \text{ to } \equiv L.ax \text{ and back.} \\
\hline
\text{From } &\equiv L \text{ to } \equiv L.ax \text{ and back.} \\
\hline
a = b, Pa \triangleright Pb & \equiv L.ax_1 \\
a = b, Pb \triangleright Pa & \equiv L.ax_2 \\
X \triangleright Pa, Y & \equiv L.ax_1 \\
a = b, Pa \triangleright Pb & \equiv L.ax_1 \\
\text{Cut} & \\
a = b, X \triangleright Pb, Y & \equiv L.ax_1 \\
\text{Cut} & \\
a = b, X, X' \triangleright Y, Y' & \equiv L.ax_1 \\
\hline
X \triangleright Pb, Y & \equiv L.ax_2 \\
a = b, Pb \triangleright Pa & \equiv L.ax_2 \\
\text{Cut} & \\
a = b, X \triangleright Pa, Y & \equiv L.ax_2 \\
\text{Cut} & \\
a = b, X, X' \triangleright Y, Y' & \equiv L.ax_2 \\
\text{Cut} & \\
\end{align*}
\]
We can restrict $=L.ax$ to primitive predicates

\[
\begin{align*}
\frac{a = b, Pa \triangleright Pb}{a = b, Pa \land Qa \triangleright Pb} \quad \text{\(\land L\)}
\end{align*}
\]

\[
\begin{align*}
\frac{a = b, Qa \triangleright Qb}{a = b, Pa \land Qa \triangleright Qb} \quad \text{\(\land L\)}
\end{align*}
\]

\[
\begin{align*}
\frac{a = b, Pa \land Qa \triangleright Pa \land Qb}{a = b, \lambda x.(Px \land Qx)a \triangleright \lambda x.(Px \land Qx)b} \quad \text{\(\land R\)}
\end{align*}
\]
We can restrict $\equiv_{L.ax}$ to *primitive* predicates

\[
\begin{align*}
\text{Let } a &= b, \quad \text{Pb} \succ Pa \\
\text{Then } a &= b, \text{Pb, } \neg Pa \succ \\
\text{And } a &= b, \text{Pb, } \neg Pa \succ \neg Pb \\
\text{So } a &= b, \lambda x. (\neg P x) a \succ \lambda x. (\neg P x) b
\end{align*}
\]
We can restrict $=L.ax$ to *primitive* predicates

\[
\begin{align*}
&=L.ax_2 \\
\frac{a = b, Pac \supset Pbc}{\neg L} \\
\frac{a = b, \forall y Pay \supset Pbc}{\neg R} \\
\frac{a = b, \forall y Pay \supset \forall y Pby}{\lambda} \\
&= a = b, \lambda x. (\forall y Pxy) a \supset \lambda x. (\forall y Pxy) b
\end{align*}
\]
Now eliminate \textit{Cut}

\[
\begin{array}{c}
\frac{a = b, Pa \rightarrow Pb \quad c = b, Pb \rightarrow Pc}{=} \\
\text{---} \quad \text{---}
\end{array}
\]

\[
\frac{a = b, \quad c = b, \quad Pa \rightarrow Pc}{=} \\
\text{---}
\]

\[=L.ax_1 \quad \text{---} \quad =L.ax_2 \quad \text{Cut}\]
Now eliminate \textit{Cut}

\[
\frac{a = b, \ Pa \supset Pb}{a = b, c = b, \ Pa \supset Pc}\quad = \text{L.ax}_1
\]
\[
\frac{c = b, \ Pb \supset Pc}{a = b, c = b, \ Pa \supset Pc}\quad = \text{L.ax}_2
\]

becomes

\[
\frac{a = b, c = b, \ Pa \supset Pc}{a = b, c = b, \ Pa \supset Pc}\quad = \text{L.??}
\]
Now eliminate \textit{Cut}

It suffices to close the axioms under \textit{Cut}.

\[
\begin{align*}
\text{I}^a_b, \text{P}a \rightarrow \text{P}b & \quad \overset{\text{L.ax}^*}{=} \\
\text{I}^a_b & \\
\end{align*}
\]

Where \(\text{I}^a_b\) is any multiset of identities \textit{linking a to b}, and \(\text{P}\) is any primitive predicate.
Now eliminate Cut

It suffices to close the axioms under Cut.

\[
\begin{align*}
&=Lax^* \\
I^a_b, \ Pa \rightarrow Pb
\end{align*}
\]

Where \( I^a_b \) is any multiset of identities linking \( a \) to \( b \), and \( P \) is any primitive predicate.

(a) The empty multiset links \( a \) to \( a \).

(b) \( a = b \) links \( a \) to \( b \) and \( b \) to \( a \).

(c) If \( X \) links \( a \) to \( b \) and \( Y \) links \( b \) to \( c \) then \( X, Y \) links \( a \) to \( c \).
Now eliminate Cut

It suffices to close the axioms under Cut.

\[
\begin{align*}
\text{I}_a^a, \text{P}a &\supset \text{P}b \\
\text{I}_b^a, \text{=L.ax}^* \\
\end{align*}
\]

Where I\(_b^a\) is any multiset of identities linking a to b, and P is any primitive predicate.

(a) The empty multiset links a to a.
(b) a = b links a to b and b to a.
(c) If X links a to b and Y links b to c then X, Y links a to c.

(We can leave ‘P\(\alpha\)’ out if it is \(a = a\).)
Kinds of Identity Rules

\[
\begin{align*}
X, Fa & \triangleright Fb, Y & X, Fb & \triangleright Fa, Y \\
\hline
X \triangleright a = b, Y
\end{align*}
\]

\[=Df\]

\(=Df\) introduces the identity predicate in arbitrary positions, defining the denial of \(a = b\) in terms of distinguishing \(a\) and \(b\).
Kinds of Identity Rules

\[ \Gamma \vdash a = a \quad \text{Refl} \quad \Gamma^a, P \vdash P \quad \text{=}L.ax \]

- \(Df\) introduces the identity predicate in *arbitrary* positions, defining the denial of \(a = b\) in terms of *distinguishing* \(a\) and \(b\).

- \(\text{Refl}\) and \(=L.ax\) are *semantic constraints connecting* primitive predicates.
Kinds of Identity Rules

\[
\begin{align*}
X, \text{Fa} & \Rightarrow \text{Fb}, Y \quad & X, \text{Fb} & \Rightarrow \text{Fa}, Y \quad & =Df \quad \\
X \Rightarrow a = b, Y
\end{align*}
\]

\[
\begin{align*}
\text{Refl} \quad & a = a \\
\text{I}^a_b, \text{Pa} & \Rightarrow \text{Pb} \quad & =L.ax_\ast
\end{align*}
\]

- \(=Df\) introduces the identity predicate in arbitrary positions, defining the denial of \(a = b\) in terms of distinguishing \(a\) and \(b\).

- \(\text{Refl}\) and \(=L.ax_\ast\) are semantic constraints connecting primitive predicates.

- They are equivalent as far as derivability goes.
Identity depends on *predication*

To apply identity rules, we need to agree on what counts as a *singular term* and what counts as a *predicate*.
DEFINING NECESSITY
Modal reasoning trades on supposition and context shift.

Oswald shot Kennedy. But suppose he hadn’t.
Oswald shot Kennedy. But suppose he hadn’t.

I think that Oswald shot Kennedy.
But let’s suppose that you’re right and he didn’t.
Oswald shot Kennedy. But suppose he hadn’t.

I think that Oswald shot Kennedy. But let’s suppose that you’re right and he didn’t.

Oswald shot Kennedy | Oswald didn’t shoot Kennedy
Two notions of necessity

1. **SUBJUNCTIVE/METAPHYSICAL** — how things *could have been* (had things gone differently) — If Oswald *hadn’t* shot Kennedy . . .
Two notions of necessity

1. SUBJUNCTIVE/METAPHYSICAL — how things could have been (had things gone differently) — If Oswald hadn't shot Kennedy . . .

2. INDICATIVE/EPISTEMIC — how things could actually be (given some body of evidence) — If Oswald didn't shoot Kennedy . . .
Two notions of necessity

1. **SUBJUNCTIVE/METAPHYSICAL** — how things *could have been* (had things gone differently) — If Oswald *hadn’t* shot Kennedy . . .

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+ Possible worlds semantics is useful in modelling *both* modal notions.
Two notions of necessity

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+ Possible worlds semantics is useful in modelling *both* modal notions.
+ Let’s consider both these options in the development of a *proof system*. 
An Analogy

A modal operator stands to a kind of context shift as a quantifier stands to a category of terms.
Hypersequents

\[ \Box A \vdash \because \because A \]
A Defining Rule for Necessity

\[ \vdash A \leftarrow X \leftarrow Y \mid S \]

\[ \frac{}{X \leftarrow \Box A, Y \mid S} \Box_{Df} \]

Denying that \( A \) is necessary has the force of denying \( A \) in some context. Or, to prove that \( A \) is necessary, you prove \( A \) in an arbitrary context.

So, necessity expresses a kind of preservation across contexts.

Indicative statements define a kind of 'epistemic' necessity (\( \Box \)).

Subjunctive statements define a kind of 'metaphysical' necessity (\( \Box \)).

The form of the rule is the same in either case, though the content differs.
A Defining Rule for Necessity

\[
\begin{align*}
\neg A \mid X \neg Y \mid S & \quad \Box Df \\
\hline
X \neg \Box A, Y \mid S
\end{align*}
\]

Denying that \( A \) is necessary has the force of denying \( A \) in some context.
A Defining Rule for Necessity

\[
\frac{\nabla A \mid X \nabla Y \mid S}{\nabla X \nabla \square A, Y \mid S} \ \square Df
\]

Denying that \( A \) is necessary has the force of denying \( A \) in some context.

Or, to prove that \( A \) is necessary, you prove \( A \) in an arbitrary context.
A Defining Rule for Necessity

\[ \neg A \mid X \neg Y \mid S \]

\[ \frac{\neg A \mid X \neg Y \mid S}{X \neg \Box A, Y \mid S} \quad \Box Df \]

Denying that \( A \) is necessary has the force of denying \( A \) in some context.

Or, to prove that \( A \) is necessary, you prove \( A \) in an arbitrary context.

So, necessity expresses a kind of \textit{preservation across contexts}. 
A Defining Rule for Necessity

\[ \neg A \mid X \neg Y \mid S \]

\[ \frac{\neg A \mid X \neg Y \mid S}{X \neg \Box A, Y \mid S} \quad \Box Df \]

Denying that \( A \) is necessary has the force of denying \( A \) in some context.

Or, to prove that \( A \) is necessary, you prove \( A \) in an arbitrary context.

So, necessity expresses a kind of \textit{preservation across contexts}.

\textit{Indicative} shifts define a kind of ‘epistemic’ necessity (\( \Box_E \)).
\textit{Subjunctive} shifts define a kind of ‘metaphysical’ necessity (\( \Box_M \)).
A Defining Rule for Necessity

\[ \neg A \mid X \neg Y \mid S \quad \text{□}Df \]
\[ X \neg \Box A, Y \mid S \]

Denying that \( A \) is necessary has the force of denying \( A \) in some context.

Or, to prove that \( A \) is necessary, you prove \( A \) in an arbitrary context.

So, necessity expresses a kind of preservation across contexts.

\textit{Indicative} shifts define a kind of ‘epistemic’ necessity (\( \Box E \)).
\textit{Subjunctive} shifts define a kind of ‘metaphysical’ necessity (\( \Box M \)).

The form of the rule is the same in either case, though the content differs.
An example derivation

\[
\begin{align*}
\Box p & \rightarrow \Box p & \Box Df \uparrow \\
\rightarrow p & \mid \Box p & \Box Df \downarrow \\
\rightarrow \Box p & \mid \Box p & \Box p, \neg \Box p \rightarrow & \neg Df \uparrow \\
\rightarrow \neg \Box p & \mid \Box p & \Box Df \downarrow \\
\rightarrow \neg \Box p & \mid \rightarrow \neg \Box p & \neg Df \downarrow \\
\rightarrow \neg \Box p & \rightarrow \Box \neg \Box p & \Box Df \downarrow
\end{align*}
\]

Cut
Both kinds of shift are important to creatures like us

- We disagree. We have reason to come to shared positions.
Both kinds of shift are important to creatures like us

- **We disagree.** We have reason to come to shared positions.
- **We plan.** We have reason to consider options (prospectively) or to replay scenarios (retrospectively).
Both kinds of shift are important to creatures like us.

- We disagree. We have reason to come to shared positions.

- We plan. We have reason to consider options (prospectively) or to replay scenarios (retrospectively).

- Our everyday practice is messy and informal. Formalising an aspect of it helps us isolate distinctive features of that practice.
Both kinds of shift are important to creatures like us

- We disagree. We have reason to come to shared positions.

- We plan. We have reason to consider options (prospectively) or to replay scenarios (retrospectively).

- Our everyday practice is messy and informal. Formalising an aspect of it helps us isolate distinctive features of that practice.

- None of this requires an antecedent commitment to an ontology of worlds.
Combining Indicative and Subjunctive Shifts

You can combine the two kinds of context shift into one hypersequent system easily enough (see my “A Cut-Free Sequent System for Two-Dimensional Modal Logic”) but we don’t need to worry about those details here.
From $\Box Df$ to $\Box L/\Box R$

\[
\begin{align*}
\vdash A \mid X \vdash Y \mid S \\
\hline
X \vdash \Box A, Y \mid S
\end{align*}
\]
From □Df to □L/□R

\[
\begin{align*}
\therefore A &| X \triangleright Y | S \\
\frac{X \triangleright \Box A, Y | S}{\Box R}
\end{align*}
\]
From $\Box Df$ to $\Box L/\Box R$

$$\vdash A \mid X \triangleright Y \mid S$$
$$\frac{X \triangleright \Box A, Y \mid S}{\Box R}$$

$$\Box A \triangleright \Box A$$
$$\vdash A \mid \Box A \triangleright X, A \triangleright Y \mid S$$
$$\frac{\Box A \triangleright | X \triangleright Y \mid S}{\Box A \triangleright | X \triangleright Y \mid S} \text{ Cut}$$
From $\Box Df$ to $\Box L/\Box R$

\[
\begin{align*}
\begin{array}{c}
\vdash A \mid X \triangleright Y \mid S \\
\hline
X \triangleright \Box A, Y \mid S
\end{array} & \Box R \\
\begin{array}{c}
\vdash X, A \triangleright Y \mid S \\
\hline
\Box A \triangleright \mid X \triangleright Y \mid S
\end{array} & \Box L
\end{align*}
\]
COMBINING THEM
Doesn’t identity become necessary, *automatically*?

\[
\begin{align*}
\text{Refl} & \quad \Rightarrow a = a \\ 
\Rightarrow \Box(a = a) & \quad \Box I \\ 
\Rightarrow \lambda x. \Box(a = x)a & \quad \lambda Df_\downarrow \\ 
\Rightarrow a = b, \lambda x. \Box(a = x)a \succ \lambda x. \Box(a = x)b & \quad =L.ax_1 \\
\Rightarrow a = b \succ \Box(a = x)b & \quad \lambda Df_\uparrow \\
\Rightarrow a = b \succ \Box(a = b) & \quad \text{Cut}
\end{align*}
\]
Williamson *defends* the necessity of identity
to mess with the modal or temporal logic of identity in order to avoid ontological inflation would be a lapse of methodological good taste, or good sense, for it means giving more weight to ontology than to the vastly better developed and more successful discipline of logic.

More specifically, the classical modal or temporal logic is a strong, simple, and elegant theory. To weaken, complicate, and uglify it without overwhelming reason to do so merely in order to block the derivation of the necessity or permanence of identity would be as retrograde and wrong-headed a step in logic and metaphysics as natural scientists would consider a comparable sacrifice of those virtues in a physical theory.

Giovanna Corsi “Counterpart Semantics” 2002

Maria Aloni “Individual Concepts in Modal Predicate Logic” 2005
Hesperus

Phosphorous
Hesperus = Phosphorous
\( h = p \mid h \neq p \)
Identity and Context Shift

\[ a = b \implies \Box a \implies \Box b \]

Is it consistent to grant that \( a = b \) in one context, while distinguishing \( a \) from \( b \) in another?
Identity and Context Shift

\[ a = b \rightarrow \neg (Fa \leftrightarrow Fb) \]

Is it consistent to grant that \( a = b \) in one context, while distinguishing \( a \) from \( b \) in another?

Indicative shifts (*disagreement*): This seems consistent.
(We *learned* that \( h = p \). The possibility that \( h \neq p \) was open to us.)
Identity and Context Shift

\[ a = b \supset \neg \neg Fa \supset \neg Fb \]

Is it consistent to grant that \( a = b \) in one context, while distinguishing \( a \) from \( b \) in another?

Indicative shifts (disagreement): This seems consistent. (We learned that \( h = p \). The possibility that \( h \neq p \) was open to us.)

Subjunctive shifts (planning): This is much less plausible. (\( h = p \). Were I to travel to \( h \), then of necessity, I am going to \( p \).)
More from Williamson

... we are not interested in epistemic readings of ‘it is possible that’.

Subjunctive Context Shifts: *Necessary* Identity Rules

\[
\begin{align*}
X &\succ Y | X', F_a \succ F_b, Y' | S \\
\hline
X &\succ Y | X', F_b \succ F_a, Y' | S \\
\hline
X &\succ a = b, Y | X' \succ Y' | S
\end{align*}
\]

=\text{Df}
Subjunctive Context Shifts: \textit{Necessary} Identity Rules

\[
\frac{X \triangleright Y \mid X', \; Fa \triangleright Fb, \; Y' \mid S \quad X \triangleright Y \mid X', \; Fb \triangleright Fa, \; Y' \mid S}{X \triangleright a = b, \; Y \mid X' \triangleright Y' \mid S} = \text{Def}
\]

\[
\frac{\triangleright a = a}{\text{Refl}} \quad \frac{I_b^a \mid Pa \triangleright Pb}{= \text{L.ax}_*}
\]

$I_b^a$ is a multiset of identities, linking $a$ and $b$, distributed across the zones of the hypersequent, in the LHS of each sequent.

(And again, we can leave $Pa$ out if it is $a = a$.)
Subjunctive Context Shifts: *Necessary* Identity Rules

\[
\frac{X \triangleright Y \mid X', Fa \triangleright Fb, Y' \mid S}{X \triangleright a = b, Y \mid X' \triangleright Y' \mid S} \quad \text{ref}\]

\[
\frac{\triangleright \ a = a \quad \ I_b^a \mid Pa \triangleright Pb}{=} \quad \text{L.ax}_*\]

$I_b^a$ is a multiset of identities, linking $a$ and $b$, distributed across the zones of the hypersequent, in the LHS of each sequent.

(And again, we can leave $Pa$ out if it is $a = a$.)

To deny $a = b$ is to take $a$ and $b$ to be distinguishable.
A Cut-Free Derivation

\[
\begin{array}{c}
a = b \quad \Rightarrow \\
\Rightarrow a = b \quad \Rightarrow \\
a = b \quad \Rightarrow \Box a = b
\end{array}
\]

\[\text{=} \text{L.ax}_*\]

\[\Box R\]
Epistemic Shifts: Contingent Identity

- BLOCK THIS: $a = b \rightarrow | F_a \rightarrow F_b$
Epistemic Shifts: Contingent Identity

- BLOCK THIS: \( a = b \succ \mid Fa \succ Fb \)
- DERIVE THIS: \( a = b, Fa \succ Fb \)
Epistemic Shifts: Contingent Identity

- **BLOCK THIS:** \( a = b \supset F_a \supset F_b \)
- **DERIVE THIS:** \( a = b, F_a \supset F_b \)

This requires distinguishing kinds of predicates, because we also want to reject \( a = b, \lambda x.\Box_E (a = x) a \supset \lambda x.\Box_E (a = x) b \).
Epistemic Shifts: Contingent Identity

- **BLOCK THIS:** $a = b \supset F a \supset F b$
- **DERIVE THIS:** $a = b, F a \supset F b$

This requires distinguishing kinds of predicates, because we *also* want to reject $a = b, \lambda x. \Box_E (a = x) a \supset \lambda x. \Box_E (a = x) b$.

- To distinguish $a$ from $b$ we want to find some *feature* $a$ has that $b$ lacks (or *vice versa*).
Epistemic Shifts: Contingent Identity

- **BLOCK THIS:** $a = b \rightarrow | Fa \rightarrow Fb$

- **DERIVE THIS:** $a = b, Fa \rightarrow Fb$

This requires distinguishing kinds of predicates, because we *also* want to reject $a = b, \lambda x. \Box_E (a = x) a \rightarrow \lambda x. \Box_E (a = x) b$.

- To distinguish $a$ from $b$ we want to find some feature $a$ has that $b$ lacks (or *vice versa*). However, epistemic modalities do not always generate features.
Epistemic Shifts: Contingent Identity

- **BLOCK THIS**: \( a = b \not\supset F a \supset F b \)
- **DERIVE THIS**: \( a = b, F a \supset F b \)

This requires distinguishing kinds of predicates, because we also want to reject \( a = b, \lambda x. \Box_E (a = x) a \not\supset \lambda x. \Box_E (a = x)b \).

- To distinguish \( a \) from \( b \) we want to find some feature \( a \) has that \( b \) lacks (or vice versa). However, epistemic modalities do not always generate features.
- After all, it’s no argument against \( h = p \) that it’s epistemically necessary that \( h = h \) and not epistemically necessary that \( h = p \).
Two Kinds of Predicates

F: feature predicates  G: general predicates
Two Kinds of Predicates

F: feature predicates      G: general predicates

Feature predicates are closed under the classical connectives and quantifiers and $\lambda$. 
Two Kinds of Predicates

F: feature predicates      G: general predicates

- Feature predicates are closed under the classical connectives and quantifiers and $\lambda$.
- Applying $\lambda$ into $\Box_E$ contexts creates a general predicate.
Two Kinds of Predicates

- **F**: feature predicates
- **G**: general predicates

- Feature predicates are closed under the classical connectives and quantifiers and $\lambda$.

- Applying $\lambda$ into $\Box_E$ contexts creates a *general* predicate.

- There are two kinds of *Spec* rule: one for feature predicates, and one for general predicates.
Contingent Identity Rules

\[
\begin{align*}
X, F_a & \not\rightarrow F_b, Y \mid S & X, F_b & \not\rightarrow F_a, Y \mid S \\
\hline
a = b, X & \not\rightarrow Y \mid S
\end{align*}
\]
Contingent Identity Rules

\[
\begin{align*}
X, Fa & \not\rightarrow Fb, Y | S & X, Fb & \not\rightarrow Fa, Y | S \\
\hline
a = b, X \not\rightarrow Y | S
\end{align*}
\]

I \_b \uparrow P_a \not\rightarrow P_b

\[
\begin{align*}
\not\rightarrow a & \equiv a & \equiv L.ax_
\end{align*}
\]

I \_b is a multiset of identities, linking a and b.

P is a feature predicate.

(And again, we can leave P\_a out if it is a = a.)
Contingent Identity Rules

\[
X, Fa \nRightarrow Fb, Y \mid S \quad X, Fb \nRightarrow Fa, Y \mid S
\]

\[
\Rightarrow a = b, X \nRightarrow Y \mid S
\]

\[
\Rightarrow a = a \quad \text{Refi}
\]

\[
I_{b}^{a}, Pa \nRightarrow Pb
\]

\[
= L.ax_{*}
\]

\(I_{b}^{a}\) is a multiset of identities, linking \(a\) and \(b\).

\(P\) is a feature predicate.

(And again, we can leave \(Pa\) out if it is \(a = a\).)

To deny \(a = b\) is to take there to be some way to distinguish \(a\) and \(b\).
Two Dimensions — with actuality

\[
\begin{array}{c|c|c|c}
X_1^1 \succ @ Y_1^1 & X_2^1 \succ Y_2^1 & \cdots & X_{m_1}^1 \succ Y_{m_1}^1 \\
X_1^2 \succ @ Y_1^2 & X_2^2 \succ Y_2^2 & \cdots & X_{m_2}^2 \succ Y_{m_2}^2 \\
\vdots & \vdots & \ddots & \vdots \\
X_1^n \succ @ Y_1^n & X_2^n \succ Y_2^n & \cdots & X_{m_n}^n \succ Y_{m_n}^n \\
\end{array}
\]
Features of these rules

- The modal rules are conservative and uniquely defining, relative to the choice of context shift.
The modal rules are conservative and uniquely defining, relative to the choice of context shift. (If □ is introduced twice, given the same hypersequent separator, those operators are equivalent.)
Features of these rules

- The modal rules are conservative and uniquely defining, relative to the choice of context shift. (If □ is introduced twice, given the same hypersequent separator, those operators are equivalent.)

- The necessary identity rules are conservative and uniquely defining, relative to the choice of predicates in the language.
Features of these rules

- The modal rules are conservative and uniquely defining, relative to the choice of context shift. (If □ is introduced twice, given the same hypersequent separator, those operators are equivalent.)

- The necessary identity rules are conservative and uniquely defining, relative to the choice of predicates in the language. (If two identity predicates are added using the same form of rules, they are equivalent, provided that both identity predicates count as predicates for the purposes of both sets of rules.)
Features of these rules

- The modal rules are conservative and uniquely defining, relative to the choice of context shift. (If □ is introduced twice, given the same hypersequent separator, those operators are equivalent.)

- The necessary identity rules are conservative and uniquely defining, relative to the choice of predicates in the language. (If two identity predicates are added using the same form of rules, they are equivalent, provided that both identity predicates count as predicates for the purposes of both sets of rules.)

- The contingent identity rules are conservative and uniquely defining, relative to the choice of feature predicates in the language.
Features of these rules

- The modal rules are conservative and uniquely defining, relative to the choice of context shift. (If □ is introduced twice, given the same hypersequent separator, those operators are equivalent.)

- The necessary identity rules are conservative and uniquely defining, relative to the choice of predicates in the language. (If two identity predicates are added using the same form of rules, they are equivalent, provided that both identity predicates count as predicates for the purposes of both sets of rules.)

- The contingent identity rules are conservative and uniquely defining, relative to the choice of feature predicates in the language. (If two identity predicates are added using the same form of rules, they are equivalent, provided that both identity predicates count as feature predicates for the purposes of both sets of rules.)
SLOGAN: The *limit* of the process of filling out an underivable (hyper)sequent describes a *model*. 
SLOGAN: The *limit* of the process of filling out an underivable (hyper)sequent describes a *model*.

\[
\frac{A \rightarrow Y \mid S}{\rightarrow Y \mid S} \quad \frac{A \rightarrow B \mid S \quad B \rightarrow C \mid S}{A \rightarrow C \mid S} \quad \frac{A \rightarrow B \mid S \quad B \rightarrow C \mid S}{A \rightarrow C \mid S}
\]
SLOGAN: The limit of the process of filling out an underivable (hyper)sequent describes a model.

\[
\begin{align*}
X \triangleright A, Y | S & \quad X, A \triangleright Y | S & \quad \triangleright A | X \triangleright \Box A, Y | S \\
X \triangleright Y | S & \quad \Box R & \quad X \triangleright \Box A, Y | S
\end{align*}
\]

The zones in correspond to worlds, the formulas in the LHS are true and those in the RHS are false.
Necessary Identity Models

- ‘=’ determines an equivalence relation on terms.
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Necessary Identity Models

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- ‘=’ determines a congruence relation for each predicate at each world.
- Take the quotient of the terms by identity to form your ‘domain’.
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- These are regular models for constant domain S5 with necessary identity.
- (The usual rules for quantifiers work, too.)
Contingent Identity Models

\[ a = b, \; c = d \rightarrow b = c \mid b = c \rightarrow a = d \mid a = b, \; b = c \rightarrow a = d \]
Contingent Identity Models

\[ a = b, c = d \implies b = c \mid b = c \implies a = d \mid a = b, b = c \implies a = d \]
Contingent Identity Models

\[ a = b, c = d \Rightarrow b = c \mid b = c \Rightarrow a = d \mid a = b, b = c \Rightarrow a = d \]
Contingent Identity Models

\[ F_a, F_b, F_c, F_d, F_a, F_d, F_b, F_c, F_a, F_b, F_c, F_d \]

\[ a = b, c = d \Rightarrow b = c \mid b = c \Rightarrow a = d \mid a = b, b = c \Rightarrow a = d \]
Contingent Identity Models

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\[ F_a, F_b \quad F_c, F_d \quad F_a, F_d \quad F_b, F_c \quad F_a, F_b, F_c \quad F_d \]

\[ G_a, G_b, G_c, G_d \]

\[ a, b, c, d \]

F: feature predicate

G: general predicate

Greg Restall

An Inferentialist Account of Identity and Modality
Contingent Identity Models

\[ a = b, c = d \rightarrow b = c \]
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\( F_a, F_b \)
\( F_c, F_d \)
\( F_a, F_d \)
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\( F_a, F_b, F_c \)
\( F_d \)

\( G_a \)
\( G_b, G_c, G_d \)
\( G_a \)
\( G_b, G_c, G_d \)
\( G_a \)
\( G_b, G_c, G_d \)

\( F: \) feature predicate
\( G: \) general predicate

\( G_x \equiv \Box F_x \)
Identities: metaphysically necessary, epistemically contingent

We would expect to have to resolve disagreements about identities. Identities are the kinds of things we can learn by engagement with the world around us, and they are open to us until we learn them.
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On the other hand, once we have learned an identity, that should constrain our view of how things *could go*, or how things *could have gone*. 

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So, a ‘property’ of the form $\lambda x. \Box_E (a = x)$ is not the kind of thing that is preserved by an identity. (Even if it holds of $a$, and if $a = b$, it need not hold of $b$.)
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These distinctions can be explained and motivated in terms of our dialogical practice, and its aims. The result is a uniform treatment of epistemic and metaphysical modality and their distinctive interactions with identity.
There is More to Do

Quantifiers
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- Existence
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- Sortals, Conceptual Covers
- Existence
- Comparisons with other systems
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THANK YOU!