Contingent Existence and Modal Definedness

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THE UNIVERSITY OF MELBOURNE

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MY TOPIC

I understand $(\forall x)$ & $(\exists x)$.

I understand $(\forall x)$ & $(\exists x)$.

$(\forall L)$ $(\forall R)$ $(\exists L)$ $(\exists R)$

I understand $\Box$ & $\Diamond$.

I understand $\Box$ & $\Diamond$.

$(\Box L)$ $(\Box R)$ $(\Diamond L)$ $(\Diamond R)$

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A cut-free sequent system for two-dimensional modal logic, and why it matters

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ABSTRACT

The two-dimensional modal logic of Davies and Humberstone (1980) [3] is an important aid to our understanding the relationship between *actuality*, *necessity* and *a priori knowability*. I show how a cut-free hypersequent calculus for 2D modal logic not only captures the logic precisely, but may be used to address issues in the epistemology and metaphysics of our modal concepts. I will explain how the use of our concepts motivates the inference rules of the sequent calculus, and then show that the completeness of the calculus for Davies–Humberstone models explains why those concepts have the structure described by those models. The result is yet another application of the completeness theorem.

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Assertion, Denial, Supposition

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y} \qquad \frac{\vdash A \mid X \vdash Y \mid \Delta}{X \vdash \Box A, Y \mid \Delta}$$

Assertion, Denial, Supposition

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y} \qquad \frac{\vdash A \mid X \vdash Y \mid \Delta}{X \vdash \Box A, Y \mid \Delta}$$

How $(\forall x)$, $(\exists x)$, $\Box$ & $\Diamond$ *interact* is another question entirely.

Today I'll show you how.

Today's Plan

Introduction

Free Logic

Modal Logic

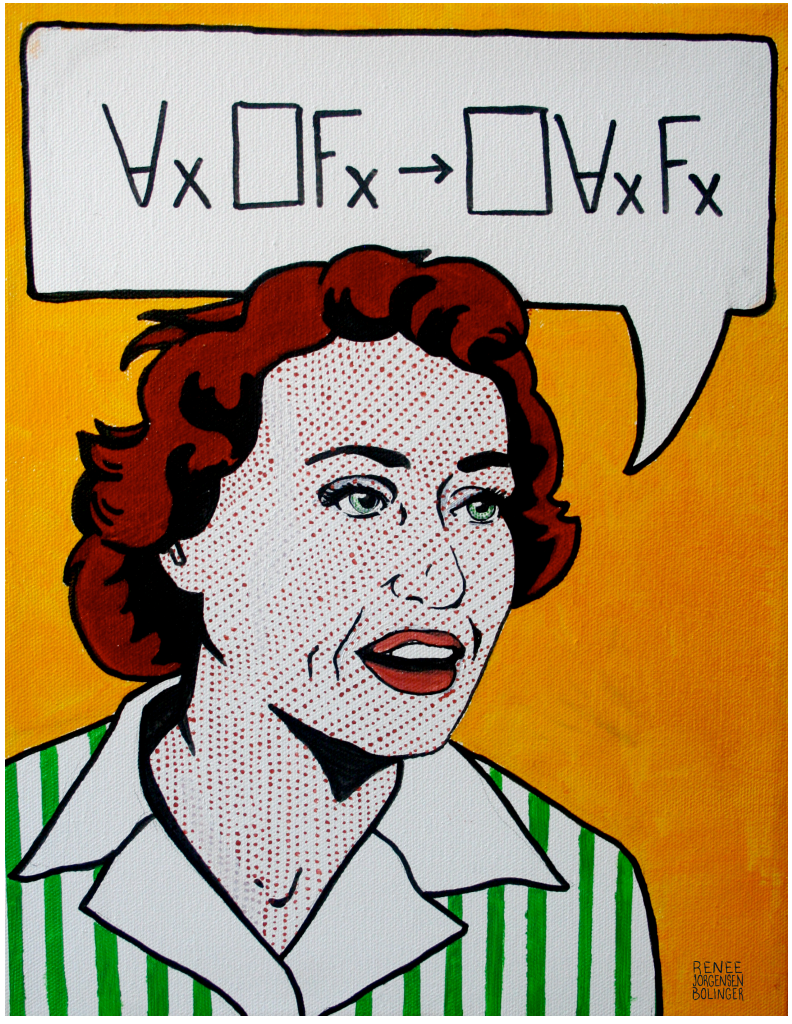
Free Modal Logic

Theorems

The Upshot (?)

INTRODUCTION

Ruth Barcan and the Barcan Formula



Portrait by *Renee Jorgensen Bolinger*

It's *very* easy to prove the Barcan Formula

Just add $(\forall L)$, $(\forall R)$, $(\Box L)$ & $(\Box R)$ and stir.

Barcan Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{F}a \vdash \mathrm{F}a}{\Box \mathrm{F}a \vdash \;\mid\; \vdash \mathrm{F}a}(\Box L)}{\forall x \Box \mathrm{F}x \vdash \;\mid\; \vdash \mathrm{F}a}(\forall L)}{\forall x \Box \mathrm{F}x \vdash \;\mid\; \vdash \forall x \mathrm{F}x}(\forall R)^{*}}{\forall x \Box \mathrm{F}x \vdash \Box \forall x \mathrm{F}x}(\Box R)$$

Barcan Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Fa \vdash Fa}{\Box Fa \vdash \;\mid\; \vdash Fa}(\Box L)}{\forall x \Box Fx \vdash \;\mid\; \vdash Fa}(\forall L)}{\forall x \Box Fx \vdash \;\mid\; \vdash \forall x Fx}(\forall R)^*}{\forall x \Box Fx \vdash \Box \forall x Fx}(\Box R)$$

Barcan Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Fa \vdash Fa}{\Box Fa \vdash \;\mid\; \vdash Fa}(\Box L)}{\forall x \Box Fx \vdash \;\mid\; \vdash Fa}(\forall L)}{\forall x \Box Fx \vdash \;\mid\; \vdash \forall x Fx}(\forall R)^{*}}{\forall x \Box Fx \vdash \Box \forall x Fx}(\Box R)$$

Barcan Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{F}a \vdash \mathrm{F}a}{\Box \mathrm{F}a \vdash \;\mid\; \vdash \mathrm{F}a}(\Box L)}{\forall x \Box \mathrm{F}x \vdash \;\mid\; \vdash \mathrm{F}a}(\forall L)}{\forall x \Box \mathrm{F}x \vdash \;\mid\; \vdash \forall x \mathrm{F}x}(\forall R)^{*}}{\forall x \Box \mathrm{F}x \vdash \Box \forall x \mathrm{F}x}(\Box R)$$

Barcan Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{F}a \vdash \mathrm{F}a}{\Box \mathrm{F}a \vdash \;\mid\; \vdash \mathrm{F}a}(\Box L)}{\forall x \Box \mathrm{F}x \vdash \;\mid\; \vdash \mathrm{F}a}(\forall L)}{\forall x \Box \mathrm{F}x \vdash \;\mid\; \vdash \forall x \mathrm{F}x}(\forall R)^{*}}{\forall x \Box \mathrm{F}x \vdash \Box \forall x \mathrm{F}x}(\Box R)$$

Barcan Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Fa \vdash Fa}{\Box Fa \vdash \;\mid\; \vdash Fa}(\Box L)}{\forall x \Box Fx \vdash \;\mid\; \vdash Fa}(\forall L)}{\forall x \Box Fx \vdash \;\mid\; \vdash \forall x Fx}(\forall R)^*}{\forall x \Box Fx \vdash \Box \forall x Fx}(\Box R)$$

Barcan Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Fa \vdash Fa}{\Box Fa \vdash \;\mid\; \vdash Fa}(\Box L)}{\forall x \Box Fx \vdash \;\mid\; \vdash Fa}(\forall L)}{\forall x \Box Fx \vdash \;\mid\; \vdash \forall x Fx}(\forall R)^*}{\forall x \Box Fx \vdash \Box \forall x Fx}(\Box R)$$

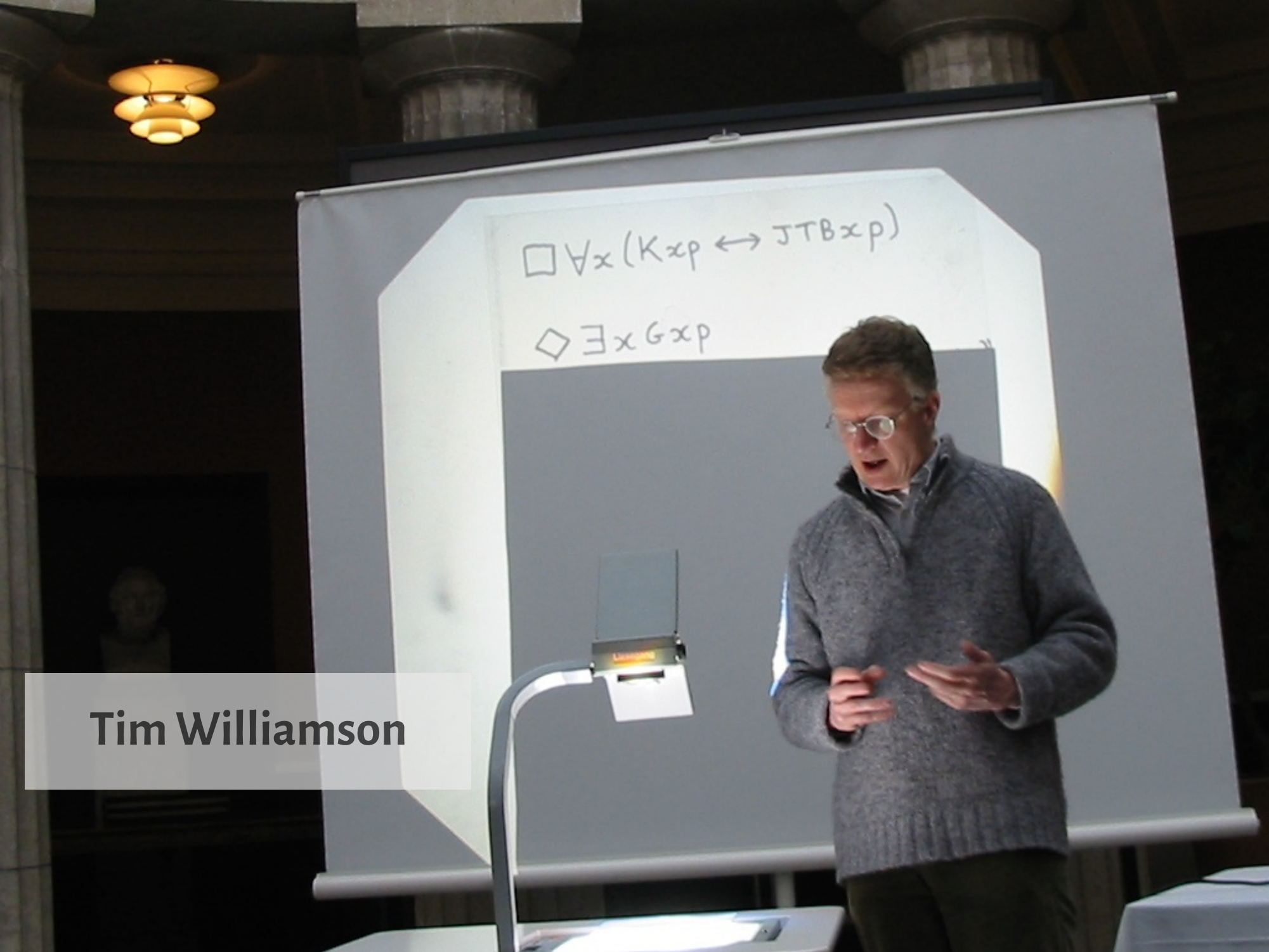
Barcan Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{F}a \vdash \mathrm{F}a}{\Box \mathrm{F}a \vdash \;\mid\; \vdash \mathrm{F}a}(\Box L)}{\forall x \Box \mathrm{F}x \vdash \;\mid\; \vdash \mathrm{F}a}(\forall L)}{\forall x \Box \mathrm{F}x \vdash \;\mid\; \vdash \forall x \mathrm{F}x}(\forall R)^*}{\forall x \Box \mathrm{F}x \vdash \Box \forall x \mathrm{F}x}(\Box R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{F}a \vdash \mathrm{F}a}{\mathrm{F}a \vdash \;\mid\; \vdash \Diamond \mathrm{F}a}(\Diamond R)}{\mathrm{F}a \vdash \;\mid\; \vdash \exists x \Diamond \mathrm{F}x}(\exists R)}{\exists x \mathrm{F}x \vdash \;\mid\; \vdash \exists x \Diamond \mathrm{F}x}(\exists L)^*}{\Diamond \exists x \mathrm{F}x \vdash \exists x \Diamond \mathrm{F}x}(\Diamond L)$$

Some people even *believe* this!

Ed Zalta & Bernard Linsky



Tim Williamson

Max Cresswell

But surely there's *some* space for contingent existence.

Where?

Models for QML with contingent existence invoke free logic for the quantifiers.

Let's start there.

FREE LOGIC

Being DEFINED

$\frac{1}{0}$ is not defined.

$\lim_{x \to \infty} f(x)$ is defined
when f is monotone and bounded.

SOLOMON FEFERMAN*

DEFINEDNESS

ABSTRACT. Questions of definedness are ubiquitous in mathematics. Informally, these involve reasoning about expressions which may or may not have a value. This paper surveys work on logics in which such reasoning can be carried out directly, especially in computational contexts. It begins with a general logic of "partial terms", continues with partial combinatory and lambda calculi, and concludes with an expressively rich theory of partial functions and polymorphic types, where termination of functional programs can be established in a natural way.

Erkenntnis 43: 295–320, 1995.

t is defined

$$t\downarrow$$

A Model $\mathfrak{M}$

- A non-empty *domain* D.
- A predicate F is interpreted as a subset $F^{\mathfrak{M}}$ of D^n (as usual).
- A function symbol f is interpreted as a *partial function* $f^{\mathfrak{M}}$ $D^n \rightharpoonup D$ (*not* as usual).
- (For Feferman, zero-place function symbols—constants—are always defined.)

Interpretations

Given an assignment α of values to variables

- $[\![x]\!]_{\mathfrak{M},\alpha} = \alpha(x)$
- $[\![f t_1 \cdots t_n]\!]_{\mathfrak{M},\alpha} \simeq f^{\mathfrak{M}}([\![t_1]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha})$

Being Defined in a Model

$[\![t]\!]_{\mathfrak{M},\alpha}\downarrow$ iff $[\![t]\!]_{\mathfrak{M},\alpha}$ takes a value.

Satisfaction

- $\mathfrak{M} \vDash_\alpha t\downarrow$ iff $[\![t]\!]_{\mathfrak{M},\alpha}\downarrow$
- $\mathfrak{M} \vDash_\alpha s = t$ iff $[\![s]\!]_{\mathfrak{M},\alpha}\downarrow$, $[\![t]\!]_{\mathfrak{M},\alpha}\downarrow$ and $[\![s]\!]_{\mathfrak{M},\alpha} = [\![t]\!]_{\mathfrak{M},\alpha}$
- $\mathfrak{M} \vDash_\alpha Ft_1 \cdots t_n$ iff for each i, $[\![t_i]\!]_{\mathfrak{M},\alpha}\downarrow$ and $\langle [\![t_n]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha}\rangle \in F^{\mathfrak{M}}$
- ... and compound formulas are treated *as usual*.

The important thing (for me)
is that it has a *natural* proof theory.

Definedness Sequents

$$\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}$$

MODEL THEORETIC READING:

There is no model where
each element of D is defined
each element of X is true
each element of Y is false
and each element of U is undefined.

Definedness Sequents

$$\mathrm{D}; \mathrm{X} \vdash \mathrm{Y}; \mathrm{U}$$

DIRECT READING:

You can't rule D and X *in*
while ruling Y and U *out*.

(Where ruling a *formula* in/out is *asserting*/*denying* it,
and ruling a *term* in/out is taking it to *refer*/*fail to refer*.)

Definedness Sequents

$$\mathrm{D}; \mathrm{X} \vdash \mathrm{Y}; \mathrm{U}$$

DIRECT READING:

You can't rule D and X *in*
while ruling Y and U *out*.

(Where ruling a *formula* in/out is *asserting*/*denying* it,
and ruling a *term* in/out is *admitting* it/*banishing* it.)

Structural Rules for Definednesss

$$\mathsf{t}, \mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{t} \quad \text{t-}Id$$

$$\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{x} \quad \text{x-}Id$$

$$\frac{\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{t} \quad \mathsf{t}, \mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}}{\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}} \ (\text{t-}Cut)$$

$$\frac{\mathsf{t}, \mathsf{t}, \mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}}{\mathsf{t}, \mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}} \ (\text{t-}WL) \qquad \frac{\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{t}, \mathsf{t}}{\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{t}} \ (\text{t-}WR)$$

Structural Rules for Definednesss

$$\mathsf{t}, \mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{t} \quad \text{t-}Id$$

$$\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{x} \quad \text{x-}Id$$

$$\frac{\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{t} \quad \mathsf{t}, \mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}}{\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}} \ (\text{t-}Cut)$$

$$\frac{\mathsf{t}, \mathsf{t}, \mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}}{\mathsf{t}, \mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}} \ (\text{t-}WL) \qquad \frac{\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{t}, \mathsf{t}}{\mathsf{D}; \mathsf{X} \vdash \mathsf{Y}; \mathsf{U}, \mathsf{t}} \ (\text{t-}WR)$$

Variables are always defined

$$D; \Gamma \vdash \Delta; U, x$$

Variables are always defined

$$D; \Gamma \vdash \Delta; U, x$$

Let x be a number ...

Is x *defined*?

Variables are always defined

$$D; \Gamma \vdash \Delta; U, x$$

Let x be a number ...

Is x *defined*?

Of course it is.

Variables are always defined

$$\mathrm{D}; \Gamma \vdash \Delta; \mathrm{U}, x$$

Let x be a number ...

Is x *defined*?

Of course it is.

Rules for Predicates and Functions

$$\frac{t_i, D; \Gamma \vdash \Delta; U}{f t_1 \cdots t_n, D; \Gamma \vdash \Delta; U}\ (fL)$$

$$\frac{t_i, D; \Gamma \vdash \Delta; U}{D; F(t_1, \ldots, t_n), \Gamma \vdash \Delta; U}\ (FL)$$

Rules for $\downarrow$

$$\frac{t, D; \Gamma \vdash \Delta; U}{D; t{\downarrow}, \Gamma \vdash \Delta; U}\ (\downarrow L) \qquad \frac{D; \Gamma \vdash \Delta; U, t}{D; \Gamma \vdash \Delta, t{\downarrow}; U}\ (\downarrow R)$$

Quantifier Rules

$$\frac{D; X \vdash Y; U, t \qquad D; A(t), X \vdash Y; U}{D; \forall x\, A(x), X \vdash Y; U}\ (\forall L)$$

$$\frac{D; X \vdash A(y), Y; U}{D; X \vdash \forall x\, A(x), Y; U}\ (\forall R)^*$$

Sample Derivations

$$\dfrac{\dfrac{(\forall x)(Gx \supset Hx), Fx \vdash Hx \;;\, x \qquad \dfrac{\dfrac{\dfrac{}{Gx \vdash Gx} \quad \dfrac{}{Hx \vdash Hx}\,(\supset L)}{Fx \vdash Fx \quad Gx, Gx \supset Hx \vdash Hx}\,(\supset L)}{Fx \supset Gx, Fx \vdash Hx \;;\, x \qquad Fx \supset Gx, Gx \supset Hx, Fx \vdash Hx}\,(\forall L)}{Fx \supset Gx, (\forall x)(Gx \supset Hx), Fx \vdash Hx}\,(\forall L)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx), Fx \vdash Hx}\,(\supset R)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx) \vdash Fx \supset Hx}\,(\forall R)^*$$

$$(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx) \vdash (\forall x)(Fx \supset Hx)$$

Sample Derivations

$$\dfrac{\dfrac{(\forall x)(Gx \supset Hx), Fx \vdash Hx \; ; \; x \qquad \dfrac{\dfrac{\dfrac{Gx \vdash Gx \quad Hx \vdash Hx}{Fx \supset Gx, Gx \supset Hx \vdash Hx}(\supset L) \quad Fx \vdash Fx}{Fx \supset Gx, Gx \supset Hx, Fx \vdash Hx}(\supset L)}{Fx \supset Gx, Fx \vdash Hx \; ; \; x}(\forall L) \qquad Fx \supset Gx, (\forall x)(Gx \supset Hx), Fx \vdash Hx}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx), Fx \vdash Hx}(\forall L)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx) \vdash Fx \supset Hx}(\supset R)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx) \vdash (\forall x)(Fx \supset Hx)}(\forall R)^*$$

Sample Derivations

$$\dfrac{\dfrac{(\forall x)(Gx \supset Hx), Fx \vdash Hx \;;\, x \qquad \dfrac{\dfrac{\dfrac{}{Gx \vdash Gx} \quad \dfrac{}{Hx \vdash Hx}\,(\supset L)}{Fx \vdash Fx \quad Gx, Gx \supset Hx \vdash Hx}\,(\supset L)}{Fx \supset Gx, Fx \vdash Hx \;;\, x \quad Fx \supset Gx, Gx \supset Hx, Fx \vdash Hx}\,(\forall L)}{Fx \supset Gx, (\forall x)(Gx \supset Hx), Fx \vdash Hx}\,(\forall L)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx), Fx \vdash Hx}\,(\supset R)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx) \vdash Fx \supset Hx}\,(\forall R)^*$$

$$\dfrac{}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx) \vdash (\forall x)(Fx \supset Hx)}$$

Sample Derivations

$$\dfrac{\dfrac{(\forall x)(Gx \supset Hx), Fx \vdash Hx \; ; \, x \qquad \dfrac{\dfrac{\dfrac{Fx \supset Gx, Fx \vdash Hx \; ; \, x \qquad \dfrac{\dfrac{Gx \vdash Gx \quad Hx \vdash Hx}{Gx, Gx \supset Hx \vdash Hx}(\supset L)}{Fx \supset Gx, Gx \supset Hx, Fx \vdash Hx}(\supset L)}{Fx \supset Gx, (\forall x)(Gx \supset Hx), Fx \vdash Hx}(\forall L)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx), Fx \vdash Hx}(\forall L)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx) \vdash Fx \supset Hx}(\supset R)}{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx) \vdash (\forall x)(Fx \supset Hx)}(\forall R)^*$$

Sample Derivations

$$\dfrac{\dfrac{\vdash \ ; \mathbf{x}}{\vdash \mathbf{x}\downarrow}\,(\downarrow R)}{\vdash \forall \mathbf{x}\, \mathbf{x}\downarrow}\,(\forall R)^{*}$$

Sample Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{s;\ \vdash\ ;s}{f(s,t);\ \vdash\ ;s}(fL)}{f(s,t)\downarrow\vdash\ ;s}(\downarrow L)}{f(s,t)\downarrow\vdash s\downarrow}(\downarrow R) \qquad \dfrac{\dfrac{\dfrac{\dfrac{t;\ \vdash\ ;t}{f(s,t);\ \vdash\ ;t}(fL)}{f(s,t)\downarrow\vdash\ ;t}(\downarrow L)}{f(s,t)\downarrow\vdash t\downarrow}(\downarrow R)}{f(s,t)\downarrow\vdash s\downarrow\wedge t\downarrow}(\wedge R)$$

Sample Derivations

$$\dfrac{\dfrac{\mathrm{t};\ \vdash A(\mathrm{t});\mathrm{t} \qquad \mathrm{t};A(\mathrm{t}) \vdash A(\mathrm{t})}{\mathrm{t};\forall \mathrm{x}\, A(\mathrm{x}) \vdash A(\mathrm{t})}\,(\forall L)}{\forall \mathrm{x}\, A(\mathrm{x}), \mathrm{t}{\downarrow} \vdash A(\mathrm{t})}\,({\downarrow}L)$$

MODAL LOGIC

Hypersequents

$$X \vdash Y \mid X' \vdash Y'$$

You can't assert X and deny Y in one context
and assert X' and deny Y' in an an alternative context.

$$\Box p \vdash \quad | \quad \vdash p$$

Hypersequents for $\Box$ and $\Diamond$

$$\frac{X' \vdash Y' \mid X, A \vdash Y \mid \Delta}{X', \Box A \vdash Y' \mid X \vdash Y \mid \Delta}\ (\Box L) \qquad \frac{\vdash A \mid X \vdash Y \mid \Delta}{X \vdash \Box A, Y \mid \Delta}\ (\Box R)$$

Hypersequents for $\Box$ and $\Diamond$

$$\frac{X' \vdash Y' \mid X, A \vdash Y \mid \Delta}{X', \Box A \vdash Y' \mid X \vdash Y \mid \Delta}\ (\Box L) \qquad \frac{\vdash A \mid X \vdash Y \mid \Delta}{X \vdash \Box A, Y \mid \Delta}\ (\Box R)$$

$$\frac{A \vdash \mid X \vdash Y \mid \Delta}{X, \Diamond A \vdash Y \mid \Delta}\ (\Diamond L) \qquad \frac{X' \vdash Y' \mid X \vdash A, Y \mid \Delta}{X' \vdash \Diamond A, Y' \mid X \vdash Y \mid \Delta}\ (\Diamond R)$$

Sample Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p \vdash \ \mid\ \vdash p}(\Box L)}{\Box p \wedge \Box q \vdash \ \mid\ \vdash p}(\wedge L) \quad \dfrac{\dfrac{q \vdash q}{\Box q \vdash \ \mid\ \vdash q}(\Box L)}{\Box p \wedge \Box q \vdash \ \mid\ \vdash q}(\wedge L)}{\Box p \wedge \Box q \vdash \ \mid\ \vdash p \wedge q}(\wedge R)}{\Box p \wedge \Box q \vdash \Box(p \wedge q)}(\Box R)$$

Sample Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p \vdash \ \mid \ \vdash p}(\Box L)}{\Box p \wedge \Box q \vdash \ \mid \ \vdash p}(\wedge L) \quad \dfrac{\dfrac{\dfrac{q \vdash q}{\Box q \vdash \ \mid \ \vdash q}(\Box L)}{\Box p \wedge \Box q \vdash \ \mid \ \vdash q}(\wedge L)}{\Box p \wedge \Box q \vdash \ \mid \ \vdash p \wedge q}(\wedge R)}{\Box p \wedge \Box q \vdash \Box(p \wedge q)}(\Box R)$$

$$\dfrac{\dfrac{\dfrac{\vdash \ \mid \ \vdash \ \mid p \vdash p}{\Box p \vdash \ \mid \ \vdash \ \mid \ \vdash p}(\Box L)}{\Box p \vdash \ \mid \ \vdash \Box p}(\Box R)}{\Box p \vdash \Box\Box p}(\Box R)$$

Sample Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p \vdash \mid \vdash p}(\Box L)}{\Box p \wedge \Box q \vdash \mid \vdash p}(\wedge L) \quad \dfrac{\dfrac{q \vdash q}{\Box q \vdash \mid \vdash q}(\Box L)}{\Box p \wedge \Box q \vdash \mid \vdash q}(\wedge L)}{\Box p \wedge \Box q \vdash \mid \vdash p \wedge q}(\wedge R)}{\Box p \wedge \Box q \vdash \Box(p \wedge q)}(\Box R)$$

$$\dfrac{\dfrac{\dfrac{\vdash \mid \vdash \mid p \vdash p}{\Box p \vdash \mid \vdash \mid \vdash p}(\Box L)}{\Box p \vdash \mid \vdash \Box p}(\Box R)}{\Box p \vdash \Box\Box p}(\Box R)$$

$$\dfrac{\dfrac{p \vdash p \mid \vdash}{p \vdash \mid \vdash \Diamond p}(\Diamond R)}{p \vdash \Box\Diamond p}(\Box R)$$

This can be naturally extended ...

This can be naturally extended …

… to other normal modal logics,

This can be naturally extended …

… to other normal modal logics,

to the logic of actuality,

This can be naturally extended …

… to other normal modal logics,

to the logic of actuality,

to 2D modal logic,

This can be naturally extended …

… to other normal modal logics,

to the logic of actuality,

to 2D modal logic,

and beyond.

This can be naturally extended …

… to other normal modal logics,

to the logic of actuality,

to 2D modal logic,

and beyond.

But here, S5 and simple hypersequents suffice.

FREE MODAL LOGIC

Let's put them *together*!

That seems like a good idea ...

… except for the fact that it is *no help at all*.

… except for the fact that it is *no help at all*.

$$\dfrac{\dfrac{}{\vdash\ ;x\ \mid\ \vdash Fx}\,(x\text{-}Id) \qquad \dfrac{\dfrac{Fx \vdash Fx}{\Box Fx \vdash\ \mid\ \vdash Fx}\,(\Box L)}{\forall x \Box Fx \vdash\ \mid\ \vdash Fx}\,(\forall L)}{\dfrac{\dfrac{\forall x \Box Fx \vdash\ \mid\ \vdash \forall x Fx}{\forall x \Box Fx \vdash \Box \forall x Fx}\,(\Box R)}{}\,(\forall R)^{*}}$$

… except for the fact that it is *no help at all*.

$$\dfrac{\dfrac{}{\vdash\ ;x\ \mid\ \vdash Fx}\,(x\text{-}Id) \qquad \dfrac{\dfrac{Fx \vdash Fx}{\Box Fx \vdash\ \mid\ \vdash Fx}\,(\Box L)}{\forall x \Box Fx \vdash\ \mid\ \vdash Fx}\,(\forall L)}{\dfrac{\dfrac{\forall x \Box Fx \vdash\ \mid\ \vdash \forall x Fx}{\forall x \Box Fx \vdash \Box \forall x Fx}\,(\Box R)}{}\,(\forall R)^{*}}$$

Variables?

Is it incoherent to deny that x exists?

Variables?

Is it incoherent to deny that x exists?

Not at all: it makes perfect sense,

Variables?

Is it incoherent to deny that x exists?

Not at all: it makes perfect sense, *counterfactually*.

Variables?

Is it incoherent to deny that x exists?

Not at all: it makes perfect sense, *counterfactually*.

This is the *heart* of commitment to contingent existence.

Revised Rules

$$\mathrm{D}; \mathrm{X} \vdash \mathrm{Y}; \mathrm{U}, \mathrm{x} \mid \Delta \quad \textit{x-Id}$$

Revised Rules

$$\cancel{\mathrm{D}; \mathrm{X} \vdash \mathrm{Y}; \mathrm{U}, x \mid \Delta \quad x\text{-}Id}$$

Revised Rules

~~$\mathcal{D}; X \vdash Y; \mathcal{U}, x \mid \Delta \quad x\text{-}Id$~~

$$\frac{\mathcal{D}; X \vdash Y; \mathcal{U}, t \mid \Delta \qquad \mathcal{D}; A(t), X \vdash Y; \mathcal{U} \mid \Delta}{\mathcal{D}; \forall x\, A(x), X \vdash Y; \mathcal{U} \mid \Delta}\ (\forall L)$$

Revised Rules

~~$$D; X \vdash Y; U, x \mid \Delta \quad x\text{-}Id$$~~

$$\frac{D; X \vdash Y; U, t \mid \Delta \qquad D; A(t), X \vdash Y; U \mid \Delta}{D; \forall x\, A(x), X \vdash Y; U \mid \Delta}\ (\forall L)$$

$$\frac{y, D; X \vdash A(y), Y; U \mid \Delta}{D; X \vdash \forall x\, A(x), Y; U \mid \Delta}\ (\forall R)^*$$

Revised Rules

~~$$\mathcal{D}; X \vdash Y; \mathcal{U}, x \mid \Delta \quad x\text{-}Id$$~~

$$\frac{\mathcal{D}; X \vdash Y; \mathcal{U}, t \mid \Delta \qquad \mathcal{D}; A(t), X \vdash Y; \mathcal{U} \mid \Delta}{\mathcal{D}; \forall x\, A(x), X \vdash Y; \mathcal{U} \mid \Delta}\ (\forall L)$$

$$\frac{y, \mathcal{D}; X \vdash A(y), Y; \mathcal{U} \mid \Delta}{\mathcal{D}; X \vdash \forall x\, A(x), Y; \mathcal{U} \mid \Delta}\ (\forall R)^*$$

Necessarily, everything exists

$$\dfrac{\dfrac{\dfrac{\vdash \ \mid \mathbf{x};\ \vdash\ ;\mathbf{x}}{\vdash \ \mid \mathbf{x};\ \vdash \mathbf{x}\downarrow}\,(\downarrow R)}{\vdash \ \mid\ \vdash \forall \mathbf{x}\, \mathbf{x}\downarrow}\,(\forall R)^{*}}{\vdash \Box\forall \mathbf{x}\, \mathbf{x}\downarrow}\,(\Box R)$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx \vdash^{?} \Box \forall x\, Fx$$

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$$\forall x \Box Fx \vdash^{?} \Box \forall x\, Fx \quad | \quad \vdash^{?} \forall x\, Fx$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx \vdash^{?} \Box \forall x\, Fx \;\mid\; y;\; \vdash^{?} \forall x\, Fx, Fy$$

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$$\forall x \Box Fx \vdash^{?} \Box \forall x\, Fx \quad | \quad y;\ \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fy \vdash^{?} \Box \forall x\, Fx \;\mid\; y; \;\vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fy \vdash^{?} \Box \forall x\, Fx \quad | \quad y; Fy \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fy \vdash^{!} \Box \forall x\, Fx \quad | \quad y; Fy \vdash^{!} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx \vdash^{?} \Box \forall x\, Fx; y \;\mid\; y; \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fx \vdash^{?} \Box \forall x\, Fx; y \;\mid\; y; \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fx \vdash^{?} \Box \forall x\, Fx; y \quad | \quad y; Fx \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fx \vdash^{?} \Box \forall x\, Fx; y \quad | \quad y; Fx \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fx \vdash^{?} \Box \forall x\, Fx; y \quad | \quad x, y; Fx \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fx \vdash^{?} \Box \forall x\, Fx; y \quad | \quad x, y; Fx \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

$$\forall x \Box Fx, \Box Fx, Fx \vdash^{?} \Box \forall x\, Fx; y \quad | \quad x, y; Fx \vdash^{?} \forall x\, Fx, Fy$$

Now the Barcan Formulas are *Underivable*

. . .

Now the Barcan Formulas are *Underivable*

$$[x\,;\, Fx \;:\; Fy\,;\, y] \;\mid\; [x, y\,;\, Fx \;:\; Fy\,;\,]$$

Now the Barcan Formulas are *Underivable*

$$[x\,;\, Fx \;:\; Fy\,;\, y] \;\mid\; [x, y\,;\, Fx \;:\; Fy\,;\,]$$

The limit of the derivation search is what we can use for a *model*.

THEOREMS

Models

- Worlds, and a (possibly empty!) domain at each world.
- Predicates at each world interpreted as a subset of the domain at that world.
- Function symbols interpreted as partial functions on world domains at worlds.

Given assignments of values to variables

- A term is defined if it takes a value in the domain at the world.
- $\downarrow$ (at a world) expresses being defined (at that world).

And the truth conditions are straightforward:

- $\forall x\, A(x)$ is true at a world iff $A(x)$ is true of every object in the domain of that world.
- $\Box A$ is true at a world if A is true at every world.

Soundness

Each derivable sequent holds in every model.

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(Straightforward proof by induction on derivations.)

Completeness

Every underivable sequent fails in some model.

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Every underivable sequent fails in some model.

(Actually, every sequent underivable *in the system without cut* fails in some model.)

Completeness (*cont.*)

Convert each connective rule to an equivalent one
where the premise(s) contain(s) the conclusion:

$$\frac{\mathcal{D}';\mathcal{X}' \vdash \mathcal{Y}';\mathcal{U}' \mid \mathcal{D};\mathcal{X},A \vdash \mathcal{Y};\mathcal{U} \mid \Delta}{\mathcal{D}';\mathcal{X}',\Box A \vdash \mathcal{Y}';\mathcal{U}' \mid \mathcal{D};\mathcal{X} \vdash \mathcal{Y};\mathcal{U} \mid \Delta}\,(\Box L) \qquad \frac{\vdash A \mid \mathcal{D};\mathcal{X} \vdash \mathcal{Y};\mathcal{U} \mid \Delta}{\mathcal{D};\mathcal{X} \vdash \Box A,\mathcal{Y};\mathcal{U} \mid \Delta}\,(\Box R)$$

become

$$\frac{\mathcal{D}';\mathcal{X}',\Box A \vdash \mathcal{Y}';\mathcal{U}' \mid \mathcal{D};\mathcal{X},A \vdash \mathcal{Y};\mathcal{U} \mid \Delta}{\mathcal{D}';\mathcal{X}',\Box A \vdash \mathcal{Y}';\mathcal{U}' \mid \mathcal{D};\mathcal{X} \vdash \mathcal{Y};\mathcal{U} \mid \Delta}\,(\Box L') \qquad \frac{\vdash A \mid \mathcal{X} \vdash \Box A,\mathcal{Y} \mid \Delta}{\mathcal{X} \vdash \Box A,\mathcal{Y} \mid \Delta}\,(\Box R')$$

Completeness (*cont.*)

$$\frac{D; X \vdash Y; U, t \mid \Delta \quad D; A(t), X \vdash Y; U \mid \Delta}{D; \forall x\, A(x), X \vdash Y; U \mid \Delta}\ (\forall L)$$

becomes

$$\frac{D; \forall x\, A(x), X \vdash Y; U, t \mid \Delta \quad D; A(t), \forall x\, A(x), X \vdash Y; U \mid \Delta}{D; \forall x\, A(x), X \vdash Y; U \mid \Delta}\ (\forall L')$$

Completeness (*cont.*)

$$\frac{\mathcal{D}, y; \mathcal{X} \vdash A(y), \mathcal{Y}; \mathcal{U} \mid \Delta}{\mathcal{D}; \mathcal{X} \vdash \forall x\, A(x), \mathcal{Y}; \mathcal{U} \mid \Delta}\,(\forall R)^*$$

becomes

$$\frac{\mathcal{D}, y; \mathcal{X} \vdash A(y), \forall x\, A(x), \mathcal{Y}; \mathcal{U} \mid \Delta}{\mathcal{D}; \mathcal{X} \vdash \forall x\, A(x), \mathcal{Y}; \mathcal{U} \mid \Delta}\,(\forall R')^*$$

Completeness (*cont.*)

- Any underivable hypersequent can be *filled out* with a backwards derivation search.
- The limit of such a search gives us a model: zones are worlds, LHS terms *exist* at that world and RHS terms *don't exist* there; LHS formulas are *true* at that world and RHS formulas are false there.
- The result is a *genuine model*.

From Soundness and Completeness to Cut Admissibility

- By SOUNDNESS: if Δ is a derivable hypersequent, it is valid in every model—it has no counterexample.
- So, by COMPLETNESS, if Δ is underivable without *cut*, then it would have a counterexample. Since it doesn't have a counterexample, Δ is derivable *without cut*.
- So, *Cut* is at least an admissible rule: if Δ is derivable *with Cut*, then Δ is also derivable *without Cut*.

THE UPSHOT (?)

MEASURED CLAIM

You can *understand* QML
by getting into *this* practice.

MEASURED CLAIM

You can *understand* QML
by getting into *this* practice.

There is *no need* to appeal to
a world-transcending domain
of non-existent objects
to do modal semantics.

The logical structure of
quantified modal discourse
can be accounted for
in terms of the norms
governing that discourse.

MORE EXTREME CLAIM

MORE EXTREME CLAIM

There is *nothing more* to QML
(modal reality, mere *possibilia*)
than what arises out of
our following rules like these.

Modal reality is *grounded* in *our* practice.

WORRYING THOUGHT

Why not also admit *these* quantifiers?

$$\frac{\mathsf{D}; A(t), X \vdash Y; \mathsf{U} \mid \Delta}{\mathsf{D}; \Pi x\, A(x), X \vdash Y; \mathsf{U} \mid \Delta}\ (\Pi L) \qquad \frac{\mathsf{D}; X \vdash A(x), Y; \mathsf{U} \mid \Delta}{\mathsf{D}; X \vdash \Pi x\, A(x), Y; \mathsf{U} \mid \Delta}\ (\Pi R)^*$$

$$\frac{\mathsf{D}; X, A(x) \vdash Y; \mathsf{U} \mid \Delta}{\mathsf{D}; X, \Sigma x\, A(x) \vdash Y; \mathsf{U} \mid \Delta}\ (\Sigma L)^* \qquad \frac{\mathsf{D}; X \vdash A(t), Y; \mathsf{U} \mid \Delta}{\mathsf{D}; X \vdash \Sigma x\, A(x), Y; \mathsf{U} \mid \Delta}\ (\Sigma R)$$

(These don't have the definedness conditions on terms.)

WORRYING THOUGHT

Why not also admit *these* quantifiers?

$$\frac{D; A(t), X \vdash Y; U \mid \Delta}{D; \Pi x\, A(x), X \vdash Y; U \mid \Delta}\,(\Pi L) \qquad \frac{D; X \vdash A(x), Y; U \mid \Delta}{D; X \vdash \Pi x\, A(x), Y; U \mid \Delta}\,(\Pi R)^*$$

$$\frac{D; X, A(x) \vdash Y; U \mid \Delta}{D; X, \Sigma x\, A(x) \vdash Y; U \mid \Delta}\,(\Sigma L)^* \qquad \frac{D; X \vdash A(t), Y; U \mid \Delta}{D; X \vdash \Sigma x\, A(x), Y; U \mid \Delta}\,(\Sigma R)$$

(These don't have the definedness conditions on terms.)

These are well behaved

They're uniquely defining,
and conservative.

(The cut admissibility argument still works.)

The Barcan Derivations are Back

$$\dfrac{\dfrac{\dfrac{\dfrac{Fx \vdash Fx}{\Box Fx \vdash \ \mid\ \vdash Fx}(\Box L)}{\Pi x \Box Fx \vdash \ \mid\ \vdash Fx}(\Pi L)}{\Pi x \Box Fx \vdash \ \mid\ \vdash \Pi x Fx}(\Pi R)}{\Pi x \Box Fx \vdash \Box \Pi x Fx}(\Box R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Fx \vdash Fx}{Fx \vdash \ \mid\ \vdash \Diamond Fx}(\Diamond R)}{Fx \vdash \ \mid\ \vdash \Sigma x \Diamond Fx}(\Sigma R)}{\Sigma x Fx \vdash \ \mid\ \vdash \exists x \Diamond Fx}(\Sigma L)^{*}}{\Diamond \Sigma x Fx \vdash \Sigma x \Diamond Fx}(\Diamond L)$$

(In models, these quantifiers range over the 'outer domain.')

Does this make *any* sense?

$$[\ ;\ \Sigma x\, \phi(x),\ \phi(y)\ :\ \exists x \phi(x)\ ;\ y\ ]$$

Does this make *any* sense?

$$[\ ;\ \Sigma x\, \phi(x),\ \phi(y)\ :\ \exists x \phi(x)\ ;\ y\ ]$$

This depends on what's involved in *ruling* y *out*.

Constraints on defining rules

The rules $(\Pi L)\ (\Pi R)^*\ (\Sigma L)^*\ (\Sigma R)$
satisfy the usual harmony constraints.

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Concerns about these rules
must come on *other* grounds.

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The rules (ΠL) $(\Pi R)^*$ $(\Sigma L)^*$ (ΣR)
satisfy the usual harmony constraints.

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must come on *other* grounds.

What can this tell us about
logicality and *defining rules*?

THANK YOU!

http://consequently.org/presentation/2015/cemd-anu/

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