Collection Frames for Substructural Logics

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LANCOG WORKSHOP ON SUBSTRUCTURAL LOGIC

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Joint work with Shawn Standefer

Our Aims

To *better understand*,
to *simplify* and to *generalise*
the ternary relational semantics
for substructural logics.

Our Plan

Ternary Relational Frames

Multiset Relations

Multiset Frames

Soundness

Completeness

Beyond Multisets

TERNARY RELATIONAL FRAMES

$$\langle P, N, \sqsubseteq, R \rangle$$

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- P: a non-empty set
- $N \subseteq P$
- $\sqsubseteq \subseteq P \times P$
- $R \subseteq P \times P \times P$

Ternary Relational Frames for Positive Substructural Logics

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4. $y \sqsubseteq y'$ iff $(\exists x)(Nx \wedge Rxyy')$.

Modal Frames

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No conditions!

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No conditions!

Binary relations are *everywhere*.

Intuitionist Frames

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Partial orders are *everywhere*.

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Where can you find a structure like *that*?

One, Two, Three,...

$$\langle P, N, \sqsubseteq, R \rangle$$

One, Two, Three,...

$$\langle P, N, \sqsubseteq, R\rangle$$

$$N \subseteq P \qquad \sqsubseteq \subseteq P \times P \qquad R \subseteq P \times P \times P$$

$$R^2(xy)zw =_{df} (\exists v)(Rxyv \wedge Rvzw)$$
$$R'^2x(yz)w =_{df} (\exists v)(Ryzv \wedge Rxvw)$$

… and more

$$R^2(xy)zw \;=_{df}\; (\exists v)(Rxyv \wedge Rvzw)$$
$$R'^2x(yz)w \;=_{df}\; (\exists v)(Ryzv \wedge Rxvw)$$

$$R^2, R'^2 \subseteq P \times P \times P \times P$$

In RW^+

$$Rxyz \iff Ryxz$$

$$R^2(xy)zw \iff R'^2x(yz)w$$

In RW^+ and in R^+

$$Rxyz \iff Ryxz$$

$$R^2(xy)zw \iff R'^2x(yz)w$$

$$Rxxx$$

The Behaviour of N, $\sqsubseteq$ and R

$$N\ \overline{z} \qquad \underline{x} \sqsubseteq \overline{z} \qquad R\ \underline{x}\underline{y}\overline{z}$$

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$$N\,\overline{z} \qquad \underline{x} \sqsubseteq \overline{z} \qquad R\,\underline{x}\,\underline{y}\,\overline{z}$$

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- The position of an *underlined* variable is closed *downwards* along $\sqsubseteq$.
- The position of an *overlined* variable is closed *upwards* along $\sqsubseteq$.

$$R\,\overline{z} \qquad \underline{x}\,R\,\overline{z} \qquad \underline{x}\,\underline{y}\,R\,\overline{z}$$

- The position of an *underlined* variable is closed *downwards* along ⊑.
- The position of an *overlined* variable is closed *upwards* along ⊑.

Collection Relations

$$R\,z \qquad x\,R\,z \qquad xy\,R\,z$$

$$X \; R \; z$$

X is a finite *collection* of elements of P; z is in P.

What kind of finite collection?

Leaf-Labelled Trees *Lists* *Multisets* *Sets* *more* …

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MULTISET RELATIONS

(Finite) Multisets

$$[1, 2] \qquad [1, 1, 2] \qquad [1, 2, 1] \qquad [1] \qquad [\;\;]$$

$$R \subseteq \mathcal{M}(P) \times P$$

Finding our Target

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R generalises $\sqsubseteq$.

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So, it should satisfy analogues of *reflexivity* and *transitivity*.

Reflxivity

$$[x]\ R\ x$$

Generalised Transitivity

$X\ R\ x$

Generalised Transitivity

$$X \; R \; x \qquad [x] \cup Y \; R \; y$$

Generalised Transitivity

$$X \, R \, x \qquad [x] \cup Y \, R \, y \qquad\qquad X \cup Y \, R \, y$$

Generalised Transitivity

$$(X \mathrel{R} x \wedge [x] \cup Y \mathrel{R} y) \Rightarrow X \cup Y \mathrel{R} y$$

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$X \cup Y\ R\ y$

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$X \cup Y \mathrel{R} y$ $\qquad$ $X \mathrel{R} x$

Generalised Transitivity

$$(X \; R \; x \wedge [x] \cup Y \; R \; y) \Rightarrow X \cup Y \; R \; y$$

$$X \cup Y \; R \; y \qquad\qquad X \; R \; x \qquad [x] \cup Y \; R \; y$$

Generalised Transitivity

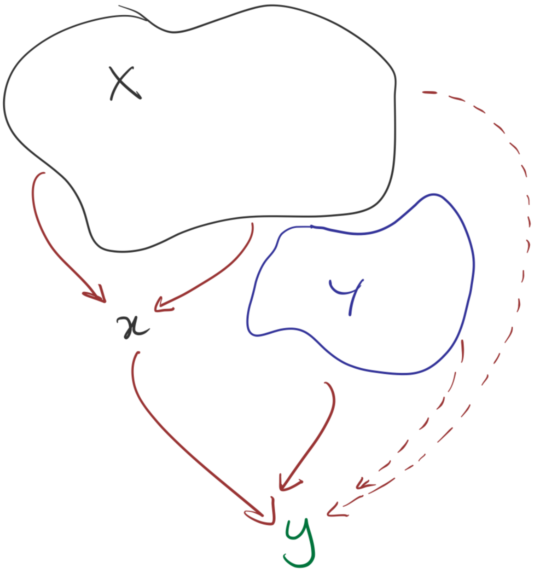
$$(X\ R\ x \wedge [x] \cup Y\ R\ y) \Rightarrow X \cup Y\ R\ y$$

$$X \cup Y\ R\ y \;\Rightarrow\; (\exists x)(X\ R\ x \;\wedge\; [x] \cup Y\ R\ y)$$

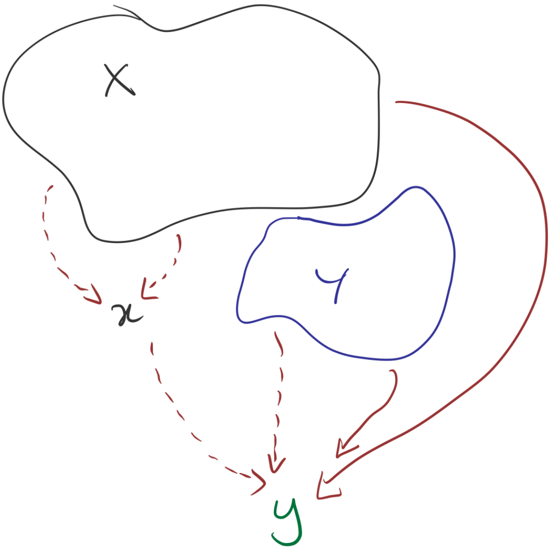
Generalised Transitivity

$$(\exists x)(X\ R\ x \wedge [x] \cup Y\ R\ y) \Leftrightarrow X \cup Y\ R\ y$$

Left to Right



Right to Left



Compositional Multiset Relations

$R \subseteq \mathcal{M}(P) \times P$ is *compositional* iff for each $X, Y \in \mathcal{M}(P)$ and $y \in P$

- $[y]\ R\ y$
- $(\exists x)(X\ R\ x \wedge [x] \cup Y\ R\ y) \iff X \cup Y\ R\ y$

Examples on $\mathcal{M}(\omega) \times \omega$

$X\ R\ y$ iff…

SUM $y = \Sigma X$ (where $\Sigma[] = 0$)

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SOME PROD.	for some $X' \leq X, y = \Pi X'$

Examples on $\mathcal{M}(\omega) \times \omega$

$X \; R \; y$ iff…

SUM $y = \Sigma X$ (where $\Sigma[\,] = 0$)

PRODUCT $y = \Pi X$ (where $\Pi[\,] = 1$)

SOME SUM for some $X' \leq X, y = \Sigma X'$

SOME PROD. for some $X' \leq X, y = \Pi X'$

MAXIMUM $y = \max(X)$ (where $\max[\,] = 0$)

Sum

$$X \; R \; y \text{ iff } y = \Sigma X$$

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REFL. $n = \Sigma[n]$

$$X \; R \; y \text{ iff } y = \Sigma X$$

REFL. $n = \Sigma[n]$

TRANS. $y = \Sigma(X \cup Y) = \Sigma X + \Sigma Y = \Sigma([\Sigma X] \cup Y)$.

Some Product

$$X \; R \; y \text{ iff for some } X' \leq X, y = \Pi X'$$

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$$X \mathrel{R} y \text{ iff for some } X' \leq X, y = \Pi X'$$

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TRANS. $Z \leq X \cup Y$ iff for some $X' \leq X$ and $Y' \leq Y$, $Z = X' \cup Y'$,
so $X \cup Y \mathrel{R} y$ iff for some $X' \leq X$ and $Y' \leq Y$, $y = \Pi(X' \cup Y')$.
But $\Pi(X' \cup Y') = \Pi X' \times \Pi Y' = \Pi([\Pi X'] \cup Y')$, and $X \mathrel{R} \Pi X'$.

Membership?

$$X \; R \; y \text{ iff } y \in X$$

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Right to left: Suppose $y \in X \cup Y$. Is there some $x \in X$ where $y \in [x] \cup Y$?

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But this fails when $X = [\ ]$.

Membership is a compositional relation on $\mathcal{M}'(\omega) \times \omega$, on *non-empty* multisets.

Between?

$$\min(X) \leq y \leq \max(X)$$

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$$\min(X) \leq y \leq \max(X)$$

This is also compositional on $\mathcal{M}'(\omega) \times \omega$.

MULTISET FRAMES

Order

Consider the binary relation $\sqsubseteq$ on P given by setting $x \sqsubseteq y$ iff $[x]\ R\ y$.

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If $[x]\ R\ y$ and $[y]\ R\ z$,
then since $[x]\ R\ y$ and $[y] \cup [\,]\ R\ z$,
we have $[x]\ R\ z$, as desired.

R respects order

$$\underline{X}\ R\ \overline{y}$$

Propositions

If $x \Vdash p$ and $[x]\ R\ y$ then $y \Vdash p$

Truth Conditions

- $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$.

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- $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$.
- $x \Vdash A \to B$ iff for each y, z where $[x, y]Rz$, if $y \Vdash A$ then $z \Vdash B$.

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This models the logic RW^+.

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Our frames *automatically* satisfy
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Our frames *automatically* satisfy the RW^+ conditions:

$$[x, y]Rz \Leftrightarrow [y, x]Rz$$

$$(\exists v)([x, y]Rv \wedge [v, z]Rw) \Leftrightarrow (\exists u)([y, z]Ru \wedge [x, u]Rw)$$

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1. N is non-empty.
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5. $Rxyz \Leftrightarrow Rxyz$
6. $(\exists v)(Rxyv \wedge Rvzw) \Leftrightarrow (\exists u)(Ryzu \wedge Rxuw)$

$$\langle P, R \rangle$$

- P: a non-empty set
- $R \subseteq \mathcal{M}(P) \times P$

1. R is compositional. That is, $[x]\ R\ x$ and $(\exists x)(X\ R\ x \wedge [x] \cup Y\ R\ y) \Leftrightarrow X \cup Y\ R\ y$

SOUNDNESS

Soundness Proof

Standard argument, by induction on the length of a proof.

It is straightforward in a natural deduction sequent system for RW^+.

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Show that if $\Gamma \succ A$ is derivable, then for any model, if $x \Vdash \Gamma$ then $x \Vdash A$.

Soundness Proof

Standard argument, by induction on the length of a proof.

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Show that if $\Gamma \succ A$ is derivable, then for any model, if $x \Vdash \Gamma$ then $x \Vdash A$.

Extend $\Vdash$ to structures by setting

$$x \Vdash \epsilon \text{ iff } [\,]\ R\ x$$

$$x \Vdash \Gamma, \Gamma' \text{ iff } x \Vdash \Gamma \text{ and } x \Vdash \Gamma'$$

$$x \Vdash \Gamma; \Gamma' \text{ iff for some } y, z \text{ where } [y, z]\ R\ x,\ y \Vdash \Gamma \text{ and } y \Vdash \Gamma'$$

COMPLETENESS

The canonical RW^+ frame is a multiset frame.

BEYOND MULTISETS

Membership, Betweenness, . . .

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$$(\exists x)(X\ R\ x \wedge [x] \cup Y\ R\ y) \Leftrightarrow X \cup Y\ R\ y$$

Membership, Betweenness, ...

$$(\exists x)(X \; R \; x \wedge [x] \cup [\,] \; R \; y) \Leftrightarrow X \cup [\,] \; R \; y$$

Membership, *Betweenness*, . . .

$$(\exists x)(X\ R\ x \land Y(x)\ R\ y) \Leftrightarrow Y(X)\ R\ y$$

Non-Empty Multisets

Membership, *Betweenness*, ...

$$(\exists x)(X \; R \; x \wedge Y(x) \; R \; y) \Leftrightarrow Y(X) \; R \; y$$

If $Y(x)$ is a multiset containing x and X is a multiset, $Y(X)$ is the multiset found by *replacing* x in $Y(x)$ by X, in the natural way.

e.g., if $Y(x)$ is $[1, 2, 3, x]$ then $Y([3, 4])$ is $[1, 2, 3, 3, 4]$.

Frames on non-empty multisets model RW^+ without t.

There are *no* normal points.

Frames on non-empty multisets model RW^+ without t.

There are *no* normal points.

They model *entailment* but not *logical truth*.

(Sequents $\Gamma \succ A$ with a non-empty right hand side.)

$$R \subseteq \mathcal{P}^{\text{fin}}(P) \times P$$

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$$\{x\}\ R\ x$$

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$$\{x\}\ R\ x$$

$$(\exists x)(X\ R\ x \land Y(x)\ R\ y) \Leftrightarrow Y(X)\ R\ y$$

Contraction

Since $\{x\}$ R x, we have $\{x, x\}$ R x.

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Set frames are models of R^+.

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Set frames are models of R^+.

OPEN QUESTION: Is the logic of set frames *exactly* R^+?

We can take collections to be *lists* (order matters) or *leaf-labelled binary trees* (associativity matters), and the generalisation works well.

We can model the Lambek Calculus (lists),

or the basic substructural logic B^+ (trees).

Lists, Trees

We can take collections to be *lists* (order matters) or *leaf-labelled binary trees* (associativity matters), and the generalisation works well.

We can model the Lambek Calculus (lists), or the basic substructural logic B^+ (trees).

The *empty list* is straightforward and natural.

The *empty tree* is less straightforward.

(To get the logic B^+ take the empty tree to be a *left* but not a *right* identity.)

Finite Structures

There is a general mathematical theory of finite structures.
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What *other* finite structures give rise
to natural logics like these?

The Upshot

- The collection of conditions on N, $\sqsubseteq$, R in ternary frames are not *ad hoc*, but arise out of a single underlying phenomenon, the *compositional relation*.

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The Upshot

- The collection of conditions on N, $\sqsubseteq$, R in ternary frames are not *ad hoc*, but arise out of a single underlying phenomenon, the *compositional relation*.
- Identifying compositional relations on structures is a way to look for *natural* models of substructural logics.
- Different logics are found by varying the *collections* being related, whether sets, multisets, lists, leaf-labelled binary trees or something else.

THANK YOU!