Collection Frames for Substructural Logics

Greg Restall

THE UNIVERSITY OF
MELBOURNE

LANCOG WORKSHOP ON SUBSTRUCTURAL LOGIC

LISBON ◊ 26 SEPTEMBER 2019

Joint work with Shawn Standefer
Our Aims

To *better understand,* to *simplify* and to *generalise* the ternary relational semantics for substructural logics.
Our Plan

Ternary Relational Frames
Multiset Relations
Multiset Frames
Soundness
Completeness
Beyond Multisets
TERNARY
RELATIONAL
FRAMES
Ternary Relational Frames for Positive Substructural Logics

\[ \langle P, N, \sqsubseteq, R \rangle \]
Ternary Relational Frames for Positive Substructural Logics

\[ \langle P, N, \sqsubseteq, R \rangle \]

- **P**: a non-empty set
- **N \subseteq P**
- **\sqsubseteq \subseteq P \times P**
- **R \subseteq P \times P \times P**
Ternary Relational Frames for Positive Substructural Logics

\[ \langle P, N, \sqsubseteq, R \rangle \]

- \( P \): a non-empty set
- \( N \subseteq P \)
- \( \sqsubseteq \subseteq P \times P \)
- \( R \subseteq P \times P \times P \)

1. \( N \) is non-empty.
\( \langle P, N, \sqsubseteq, R \rangle \)

- **P**: a non-empty set
- **N \subseteq P**
- **\( \sqsubseteq \subseteq P \times P \)**
- **R \subseteq P \times P \times P**

1. N is non-empty.
2. \( \sqsubseteq \) is a partial order (or preorder).
Ternary Relational Frames for Positive Substructural Logics

\[ \langle P, N, \sqsubseteq, R \rangle \]

- **P**: a non-empty set
- **N \subseteq P**
- **\( \sqsubseteq \subseteq P \times P \)**
- **R \subseteq P \times P \times P**

1. N is non-empty.
2. \( \sqsubseteq \) is a partial order (or preorder).
3. R is downward preserved in the its two positions and upward preserved in the third, i.e. if \( R \times' y' z \) and \( x \sqsubseteq x', y \sqsubseteq y', z \sqsubseteq z' \), then \( R \times y z' \).
Ternary Relational Frames for Positive Substructural Logics

\[ \langle P, N, \sqsubseteq, R \rangle \]

- **P**: a non-empty set
- **N \subseteq P**
- **\sqsubseteq \subseteq P \times P**
- **R \subseteq P \times P \times P**

1. **N** is non-empty.
2. **\sqsubseteq** is a partial order (or preorder).
3. **R** is downward preserved in the its two positions and upward preserved in the third, i.e. if \( Rx' y' z \) and \( x \sqsubseteq x' \), \( y \sqsubseteq y' \), \( z \sqsubseteq z' \), then \( Rxyz' \).
4. \( y \sqsubseteq y' \iff (\exists x)(Nx \land Rxyy') \).
Modal Frames

$\langle P, R \rangle$
Modal Frames

\[ \langle P, R \rangle \]

- \( P \): a non-empty set
- \( R \subseteq P \times P \)
Modal Frames

\[ \langle P, R \rangle \]

- $P$: a non-empty set
- $R \subseteq P \times P \quad \text{No conditions!}$
Modal Frames

\[ \langle P, R \rangle \]

- \( P \): a non-empty set
- \( R \subseteq P \times P \)

No conditions!

Binary relations are everywhere.
\langle P, \sqsubseteq \rangle
Intuitionist Frames

\[ \langle P, \sqsubseteq \rangle \]

- P: a non-empty set
- \( \sqsubseteq \subseteq P \times P \)
Intuitionist Frames

\[ \langle P, \sqsubseteq \rangle \]

- P: a non-empty set
- \( \sqsubseteq \subseteq P \times P \)

1. \( \sqsubseteq \) is a partial order (or preorder).
Intuitionist Frames

\[ \langle P, \sqsubseteq \rangle \]

- **P**: a non-empty set
- \( \sqsubseteq \subseteq P \times P \)

1. \( \sqsubseteq \) is a partial order (or preorder).

Partial orders are *everywhere*. 
Ternary Relational Frames for Positive Substructural Logics

\[ \langle P, N, \sqsubseteq, R \rangle \]

- **P**: a non-empty set
- **N \subseteq P**
- **\sqsubseteq \subseteq P \times P**
- **R \subseteq P \times P \times P**

1. **N** is non-empty.
2. **\sqsubseteq** is a partial order (or preorder).
3. **R** is downward preserved in the its two positions and upward preserved in the third, i.e. if \(Rx'y'z\) and \(x \sqsubseteq x', y \sqsubseteq y', z \sqsubseteq z'\), then \(Rxyz'\).
4. \(y \sqsubseteq y'\) iff \((\exists x)(N x \land Rx y y')\).
Where can you find a structure like *that*?
One, Two, Three,…

\[ \langle P, N, \sqsubseteq, R \rangle \]
One, Two, Three,…

\[\langle P, N, \subseteq, R \rangle\]

\[N \subseteq P \quad \subseteq \subseteq P \times P \quad R \subseteq P \times P \times P\]
... and more

\[
R^2(xy)zw \equiv \exists v (Rxvy \land Rvw) \\
R'^2x(yz)w \equiv \exists v (Ryzv \land Rxvw)
\]
\[ R^2(x y)z w =_{df} (\exists v)(R x y v \land R v z w) \]
\[ R'^2x(y z)w =_{df} (\exists v)(R y z v \land R x v w) \]

\[ R^2, R'^2 \subseteq P \times P \times P \times P \]
In $\text{RW}^+$

\[
\begin{align*}
R_{xyz} & \iff R_{yxz} \\
R^2(xy)zw & \iff R'^2x(yz)w
\end{align*}
\]
In $\text{RW}^+$ and in $\mathbb{R}^+$

\[
\begin{align*}
R_{xyz} & \iff R_{yxz} \\
R^2(xy)zw & \iff R'^2x(yz)w \\
R_{xxx} &
\end{align*}
\]
The Behaviour of $\mathbb{N}$, $\sqsubseteq$ and $\mathbb{R}$
The Behaviour of $\mathbb{N}$, $\sqsubseteq$ and $\mathbb{R}$

$\mathbb{N} \sqsubseteq \bar{z}$ \quad $x \sqsubseteq \bar{z}$ \quad $\mathbb{R} \underline{xy\bar{z}}$

- The position of an \textit{underlined} variable is closed \textit{downwards} along $\sqsubseteq$. 
The Behaviour of $\mathbb{N}$, $\sqsubseteq$ and $\mathbb{R}$

$\mathbb{N} \overline{z}$ \quad $x \sqsubseteq \overline{z}$ \quad $\mathbb{R} \underline{xy\overline{z}}$

- The position of an *underlined* variable is closed *downwards* along $\sqsubseteq$.
- The position of an *overlined* variable is closed *upwards* along $\sqsubseteq$. 
The Behaviour of $\mathbb{N}$, $\sqsubseteq$ and $\mathbb{R}$

- The position of an underlined variable is closed *downwards* along $\sqsubseteq$.
- The position of an overlined variable is closed *upwards* along $\sqsubseteq$. 
The Behaviour of \( \mathbb{N}, \sqsubseteq \) and \( \mathbb{R} \)

\[ R \bar{z} \quad x \ R \bar{z} \quad xy \ R \bar{z} \]

- The position of an \underline{underlined} variable is closed \underline{downwards} along \( \sqsubseteq \).
- The position of an \underline{overlined} variable is closed \underline{upwards} along \( \sqsubseteq \).
Collection Relations

$R z \quad x \ R \ z \quad x y \ R \ z$
$X \mathbin{R} z$

$X$ is a finite *collection* of elements of $P$; $z$ is in $P$. 
What kind of finite collection?

Leaf-Labelled Trees  Lists  Multisets  Sets  more…
What kind of finite collection?

Leaf-Labelled Trees  Lists  Multisets  Sets  more…

\[ R_{xyz} \iff R_{yxz} \]
\[ R^2(xy)zw \iff R'^2x(yz)w \]
What kind of finite collection?

Leaf-Labelled Trees  Lists  Multisets  Sets  more …

\[ R_{xyz} \iff R_{yxz} \]

\[ R^2(xyzw) \iff R'2x(yzw) \]
MULTISET RELATIONS
(Finite) Multisets

\[
\begin{array}{cccc}
[1, 2] & [1, 1, 2] & [1, 2, 1] & [1] \\
\end{array}
\]
Finding our Target

\[ R \subseteq \mathcal{M}(P) \times P \]
Finding our Target

$R \subseteq \mathcal{M}(P) \times P$

$R$ generalises $\sqsubseteq$. 
Finding our Target

\[ R \subseteq \mathcal{M}(P) \times P \]

\[ R \text{ generalises } \sqsubseteq. \]

So, it should satisfy analogues of \textit{reflexivity} and \textit{transitivity}. 
Reflexivity

\[ [\chi] \ R \chi \]
Generalised Transitivity

\[ X \mathcal{R} x \quad [x] \cup Y \mathcal{R} y \]
Generalised Transitivity

\[ X R x \quad [x] \cup Y R y \quad X \cup Y R y \]
Generalised Transitivity

\[(X \ R \ x \land [x] \cup Y \ R \ y) \Rightarrow X \cup Y \ R \ y\]
Generalised Transitivity

\[(X R x \land [x] \cup Y R y) \Rightarrow X \cup Y R y\]

\[X \cup Y R y\]
Generalised Transitivity

\[(X \mathcal{R} x \land [x] \cup Y \mathcal{R} y) \Rightarrow X \cup Y \mathcal{R} y\]

\[X \cup Y \mathcal{R} y \quad X \mathcal{R} x\]
Generalised Transitivity

\[(X \, R \, x \land [x] \cup Y \, R \, y) \Rightarrow X \cup Y \, R \, y\]

\[X \cup Y \, R \, y\]  \[X \, R \, x\]  \[[x] \cup Y \, R \, y\]
Generalised Transitivity

\[(X \, R \, x \, \land \, [x] \, \cup \, Y \, R \, y) \Rightarrow X \cup Y \, R \, y\]

\[X \cup Y \, R \, y \Rightarrow (\exists x)(X \, R \, x \, \land \, [x] \, \cup \, Y \, R \, y)\]
Generalised Transitivity

\((\exists x)(X \mathcal{R} x \land [x] \cup Y \mathcal{R} y) \iff X \cup Y \mathcal{R} y\)
Left to Right

\[ x \rightarrow y \]

\[ y \rightarrow x \]
Right to Left

\[
\begin{array}{c}
  \text{x} \\
  \text{y}
\end{array}
\]
Compositional Multiset Relations

\[ R \subseteq \mathcal{M}(P) \times P \text{ is } \textit{compositional} \text{ iff for each } X, Y \in \mathcal{M}(P) \text{ and } y \in P \]

- \([y] R y\)
- \( \exists x (X R x \land [x] \cup Y R y) \iff X \cup Y R y \)
Examples on $\mathcal{M}(\omega) \times \omega$

$X \ R \ y \ \text{iff...}$

\[ \text{sum} \ y = \Sigma X \ (\text{where} \ \Sigma[\ ] = 0) \]
Examples on $\mathcal{M}(\omega) \times \omega$

$X \ R \ y$ iff...

**SUM** $\ y = \sum X \ (\text{where } \sum[\ ] = 0)$

**PRODUCT** $\ y = \prod X \ (\text{where } \prod[\ ] = 1)$
Examples on $\mathcal{M}(\omega) \times \omega$

$X \ R \ y$ iff...

**SUM** $y = \Sigma X$ (where $\Sigma[] = 0$)

**PRODUCT** $y = \Pi X$ (where $\Pi[] = 1$)

**SOME SUM** for some $X' \leq X$, $y = \Sigma X'$
Examples on $\mathcal{M}(\omega) \times \omega$

$X R y$ iff…

**SUM**  $y = \Sigma X$ (where $\Sigma[] = 0$)

**PRODUCT**  $y = \Pi X$ (where $\Pi[] = 1$)

**SOME SUM**  for some $X' \leq X$, $y = \Sigma X'$

**SOME PROD**.  for some $X' \leq X$, $y = \Pi X'$
Examples on $\mathcal{M}(\omega) \times \omega$

$X \mathrel{R} y$ iff...

**SUM** \quad y = \Sigma X \ (\text{where } \Sigma[] = 0)

**PRODUCT** \quad y = \Pi X \ (\text{where } \Pi[] = 1)

**SOME SUM** \quad \text{for some } X' \leq X, \ y = \Sigma X'

**SOME PROD.** \quad \text{for some } X' \leq X, \ y = \Pi X'

**MAXIMUM** \quad y = \max(X) \ (\text{where } \max[] = 0)
Sum

\[ X \, R \, y \iff y = \Sigma X \]
\[ X \ R \ y \ \text{iff} \ y = \Sigma X \]

**REFL.** \[ n = \Sigma [n] \]
$\Sigma X \text{ R } y \text{ iff } y = \Sigma X$

**REFL.** $n = \Sigma[n]$

**TRANS.** $y = \Sigma(X \cup Y) = \Sigma X + \Sigma Y = \Sigma(\Sigma X \cup Y)$. 
Some Product

\[ X \mathrel{R} y \text{ iff for some } X' \leq X, \ y = \Pi X' \]
Some Product

\[ X \mathbin{R} y \text{ iff for some } X' \leq X, \, y = \Pi X' \]

**REFL.** \( n = \Pi [n] \)
Some Product

\[ X \text{ R } y \text{ iff for some } X' \leq X, y = \Pi X' \]

**REFL.** \( n = \Pi [n] \)

**TRANS.** \( Z \leq X \cup Y \text{ iff for some } X' \leq X \text{ and } Y' \leq Y, Z = X' \cup Y', \)
Some Product

\[ X \; \mathcal{R} \; y \; \text{iff for some } X' \leq X, \; y = \Pi X' \]

**REFL.** \( n = \Pi[n] \)

**TRANS.** \( Z \leq X \cup Y \; \text{iff for some } X' \leq X \text{ and } Y' \leq Y, \; Z = X' \cup Y', \)
so \( X \cup Y \; \mathcal{R} \; y \; \text{iff for some } X' \leq X \text{ and } Y' \leq Y, \; y = \Pi(X' \cup Y'). \)

But \( \Pi(X' \cup Y') = \Pi X' \times \Pi Y' = \Pi((\Pi X') \cup Y'), \) and \( X \; \mathcal{R} \; \Pi X'. \)
Membership?

\[ X \mathrel{R} y \iff y \in X \]
Member?

\[ X R y \iff y \in X \]

\textbf{REFL.} \quad n \in [n]
Membership?

\[ X R y \iff y \in X \]

**REFL.** \( n \in [n] \)

**TRANS.** Left to right: If \( x \in X \) and \( y \in ([x] \cup Y) \), then \( y \in X \cup Y \).
Membership?

\[ X \mathrel{R} y \text{ iff } y \in X \]

**REFL.** \( n \in [n] \)

**TRANS.** Left to right: If \( x \in X \) and \( y \in ([x] \cup Y) \), then \( y \in X \cup Y \).

Right to left: Suppose \( y \in X \cup Y \). Is there some \( x \in X \) where \( y \in [x] \cup Y \)?
Membership?

\[ X R y \text{ iff } y \in X \]

**REFL.** \( n \in [n] \)

**TRANS.** Left to right: If \( x \in X \) and \( y \in ([x] \cup Y) \), then \( y \in X \cup Y \).

Right to left: Suppose \( y \in X \cup Y \). Is there some \( x \in X \) where \( y \in [x] \cup Y \)?

If \( X \) is non-empty, sure: pick \( y \) if \( y \in X \), and an arbitrary member otherwise.
**Membership?**

\[ X R y \iff y \in X \]

**REFL.** \( n \in [n] \)

**TRANS.** Left to right: If \( x \in X \) and \( y \in ([x] \cup Y) \), then \( y \in X \cup Y \).

Right to left: Suppose \( y \in X \cup Y \). Is there some \( x \in X \) where \( y \in [x] \cup Y \)?

If \( X \) is non-empty, sure: pick \( y \) if \( y \in X \), and an arbitrary member otherwise.

But this fails when \( X = [\ ] \).
$X R y$ iff $y \in X$  

**REFL.** $n \in [n]$ 

**TRANS.** Left to right: If $x \in X$ and $y \in ([x] \cup Y)$, then $y \in X \cup Y$.

Right to left: Suppose $y \in X \cup Y$. Is there some $x \in X$ where $y \in [x] \cup Y$?

If $X$ is non-empty, sure: pick $y$ if $y \in X$, and an arbitrary member otherwise.

But this fails when $X = [\ ]$.

Membership is a compositional relation on $\mathcal{M}'(\omega) \times \omega$, on *non-empty* multisets.
Between?

\[ \min (X) \leq y \leq \max (X) \]
$\min (X) \leq y \leq \max (X)$

This is also compositional on $\mathcal{M}'(\omega) \times \omega$. 
MULTISET FRAMES
Consider the binary relation $\sqsubseteq$ on $P$ given by setting $x \sqsubseteq y$ iff $[x] R y$.

This is a preorder on $P$. 
Consider the binary relation $\sqsubseteq$ on $P$ given by setting $x \sqsubseteq y$ iff $[x] R y$.

This is a preorder on $P$.

$[x] R x$
Consider the binary relation $\sqsubseteq$ on $P$ given by setting $x \sqsubseteq y$ iff $[x] \mathrel{R} y$.

This is a preorder on $P$.

$[x] \mathrel{R} x$

If $[x] \mathrel{R} y$ and $[y] \mathrel{R} z$, then since $[x] \mathrel{R} y$ and $[y] \cup [\ ] \mathrel{R} z$, we have $[x] \mathrel{R} z$, as desired.
R respects order

\[ X \; R \; \overline{y} \]
Propositions

If $x \models p$ and $[x] R y$ then $y \models p$
Truth Conditions

- $x \vdash A \land B$ iff $x \vdash A$ and $x \vdash B$. 
Truth Conditions

- $\phi \vdash A \land B$ iff $\phi \vdash A$ and $\phi \vdash B$.
- $\phi \vdash A \lor B$ iff $\phi \vdash A$ or $\phi \vdash B$.
Truth Conditions

- $x \models A \land B$ iff $x \models A$ and $x \models B$.
- $x \models A \lor B$ iff $x \models A$ or $x \models B$.
- $x \models A \rightarrow B$ iff for each $y, z$ where $[x, y]Rz$, if $y \models A$ then $z \models B$. 

Truth Conditions

- $x \models A \land B$ iff $x \models A$ and $x \models B$.
- $x \models A \lor B$ iff $x \models A$ or $x \models B$.
- $x \models A \rightarrow B$ iff for each $y, z$ where $[x, y]Rz$, if $y \models A$ then $z \models B$.
- $x \models A \circ B$ iff for some $y, z$ where $[y, z]Rx$, both $y \models A$ and $z \models B$. 
Truth Conditions

- \( x \vDash A \land B \) iff \( x \vDash A \) and \( x \vDash B \).
- \( x \vDash A \lor B \) iff \( x \vDash A \) or \( x \vDash B \).
- \( x \vDash A \rightarrow B \) iff for each \( y, z \) where \([x, y]Rz\), if \( y \vDash A \) then \( z \vDash B \).
- \( x \vDash A \circ B \) iff for some \( y, z \) where \([y, z]Rx\), both \( y \vDash A \) and \( z \vDash B \).
- \( x \vDash t \) iff \([ ]Rx\).
Truth Conditions

- $x \vdash A \land B$ iff $x \vdash A$ and $x \vdash B$.
- $x \vdash A \lor B$ iff $x \vdash A$ or $x \vdash B$.
- $x \vdash A \rightarrow B$ iff for each $y, z$ where $[x, y]Rz$, if $y \vdash A$ then $z \vdash B$.
- $x \vdash A \circ B$ iff for some $y, z$ where $[y, z]Rx$, both $y \vdash A$ and $z \vdash B$.
- $x \vdash t$ iff $[ ]Rx$.

This models the logic $RW^+$.
Truth Conditions

- \( x \models A \land B \) iff \( x \models A \) and \( x \models B \).
- \( x \models A \lor B \) iff \( x \models A \) or \( x \models B \).
- \( x \models A \rightarrow B \) iff for each \( y, z \) where \([x, y]Rz\), if \( y \models A \) then \( z \models B \).
- \( x \models A \circ B \) iff for some \( y, z \) where \([y, z]Rx\), both \( y \models A \) and \( z \models B \).
- \( x \models t \) iff \([ ]Rx\).

This models the logic \( RW^+ \).

Our frames \textit{automatically} satisfy the \( RW^+ \) conditions:
Truth Conditions

\[ x \models A \land B \iff x \models A \text{ and } x \models B. \]

\[ x \models A \lor B \iff x \models A \text{ or } x \models B. \]

\[ x \models A \rightarrow B \iff \text{for each } y, z \text{ where } [x, y]Rz, \text{ if } y \models A \text{ then } z \models B. \]

\[ x \models A \circ B \iff \text{for some } y, z \text{ where } [y, z]Rx, \text{ both } y \models A \text{ and } z \models B. \]

\[ x \models t \iff [ ]Rx. \]

This models the logic \( RW^+ \).

Our frames automatically satisfy the \( RW^+ \) conditions:

\[[x, y]Rz \iff [y, x]Rz\]
Truth Conditions

- \( \models x \iff A \land B \iff x \models A \text{ and } x \models B \).
- \( \models x \iff A \lor B \iff x \models A \text{ or } x \models B \).
- \( \models x \iff A \rightarrow B \iff \text{for each } y, z \text{ where } [x, y]Rz, \text{ if } y \models A \text{ then } z \models B \).
- \( \models x \iff A \circ B \iff \text{for some } y, z \text{ where } [y, z]Rx, \text{ both } y \models A \text{ and } z \models B \).
- \( \models x \iff t \iff [\ ]Rx \).

This models the logic \( RW^+ \).

Our frames automatically satisfy
the \( RW^+ \) conditions:

\[
[x, y]Rz \iff [y, x]Rz
\]

\[
(\exists v)([x, y]Rv \land [v, z]Rw) \iff (\exists u)([y, z]Ru \land [x, u]Rw)
\]
Ternary Relational Frames for RW$^+$

$\langle P, N, \sqsubseteq, R \rangle$

- $P$: a non-empty set
- $N \subseteq P$
- $\sqsubseteq \subseteq P \times P$
- $R \subseteq P \times P \times P$

1. $N$ is non-empty.
2. $\sqsubseteq$ is a partial order (or preorder).
3. $R$ is downward preserved in the its two positions and upward preserved in the third.
4. $y \sqsubseteq y'$ iff $(\exists x)(N x \land R x y y')$.
5. $R x y z \iff R x y z$
6. $(\exists v)(R x y v \land R v z w) \iff (\exists u)(R y z u \land R x u w)$
Multiset Frames for $\text{RW}^+$

$\langle P, R \rangle$

- $P$: a non-empty set
- $R \subseteq \mathcal{M}(P) \times P$

1. $R$ is compositional. That is, $[x] \ R x$ and $(\exists x)(X \ R x \land [x] \cup Y \ R y) \iff X \cup Y \ R y$
SOUNDNESS
Soundness Proof

Standard argument, by induction on the length of a proof.
It is straightforward in a natural deduction sequent system for $\text{RW}^+$. 
Soundness Proof

Standard argument, by induction on the length of a proof.

It is straightforward in a natural deduction sequent system for $\mathbf{RW}^+$. Show that if $\Gamma \vdash A$ is derivable, then for any model, if $x \models \Gamma$ then $x \models A$. 
Soundness Proof

Standard argument, by induction on the length of a proof.

It is straightforward in a natural deduction sequent system for RW$^+$. Show that if $\Gamma \vdash \Lambda$ is derivable, then for any model, if $x \models \Gamma$ then $x \models \Lambda$.

Extend $\models$ to structures by setting

\[
\begin{align*}
    x \models \varepsilon & \iff [\varepsilon] R x \\
    x \models \Gamma, \Gamma' & \iff x \models \Gamma \text{ and } x \models \Gamma'
\end{align*}
\]

\[
    x \models \Gamma; \Gamma' \iff \text{for some } y, z \text{ where } [y, z] R x, y \models \Gamma \text{ and } y \models \Gamma'
\]
COMPLETENESS
The canonical $RW^+$ frame is a multiset frame.
BEYOND MULTISETS
Non-Empty Multisets

Membership, Betweenness, . . .
Non-Empty Multisets

Membership, Betweenness, …

$$(\exists x)(X R x \land [x] \cup Y R y) \iff X \cup Y R y$$
Non-Empty Multisets

Membership, Betweenness, …

$$(\exists x)(X \mathcal{R} x \land [x] \cup [ ] \mathcal{R} y) \iff X \cup [ ] \mathcal{R} y$$
Non-Empty Multisets

Membership, Betweenness, . . .

\[(\exists x) (X R x \land Y(x) R y) \iff Y(X) R y\]
Non-Empty Multisets

Membership, Betweenness, . . .

$(\exists x)(X R x \land Y(x) R y) \iff Y(X) R y$

If $Y(x)$ is a multiset containing $x$ and $X$ is a multiset, $Y(X)$ is the multiset found by replacing $x$ in $Y(x)$ by $X$, in the natural way.

e.g., if $Y(x)$ is $[1, 2, 3, x]$ then $Y([3, 4])$ is $[1, 2, 3, 3, 4]$. 
Frames on non-empty multisets model $\text{RW}^+$ without $t$. There are *no* normal points.
Frames on non-empty multisets model $\text{RW}^+$ without $t$.

There are no normal points.

They model *entailment* but not *logical truth*.

(Sequents $\Gamma \Rightarrow A$ with a non-empty right hand side.)
Sets

\[ R \subseteq P^{\text{fin}}(P) \times P \]
\( R \subseteq \mathcal{P}^{\text{fin}}(P) \times P \)

\( \{x\} R x \)
Sets

\[ R \subseteq \mathcal{P}^{\text{fin}}(P) \times P \]

\{x\} R x

\((\exists x)(X R x \land Y(x) R y) \iff Y(X) R y\)
Since $\{x\} R x$, we have $\{x, x\} R x$. 
Since $\{x\} R x$, we have $\{x, x\} R x$.

Set frames are models of $R^+$. 
Since \( \{x\} \mathcal{R} x \), we have \( \{x, x\} \mathcal{R} x \).

Set frames are models of \( \mathbb{R}^+ \).

**Open Question**: Is the logic of set frames *exactly* \( \mathbb{R}^+ \)?
Lists, Trees

We can take collections to be \textit{lists} (order matters) or \textit{leaf-labelled binary trees} (associativity matters), and the generalisation works well.

We can model the Lambek Calculus (lists), or the basic substructural logic $B^+$ (trees).
Lists, Trees

We can take collections to be \textit{lists} (order matters) or \textit{leaf-labelled binary trees} (associativity matters), and the generalisation works well.

We can model the Lambek Calculus (lists), or the basic substructural logic $\mathbf{B}^+$ (trees).

The \textit{empty list} is straightforward and natural.

The \textit{empty tree} is less straightforward.

(To get the logic $\mathbf{B}^+$ take the empty tree to be a \textit{left} but not a \textit{right} identity.)
There is a general mathematical theory of finite structures.
(The theory of *species.*)
There is a general mathematical theory of finite structures. (The theory of *species*)

What other finite structures give rise to natural logics like these?
The collection of conditions on \( N, \sqsubseteq, R \) in ternary frames are not *ad hoc*, but arise out of a single underlying phenomenon, the *compositional relation*. 
The Upshot

- The collection of conditions on $N$, $\sqsubseteq$, $R$ in ternary frames are not *ad hoc*, but arise out of a single underlying phenomenon, the *compositional relation*.

- Identifying compositional relations on structures is a way to look for *natural* models of substructural logics.
The collection of conditions on $\mathbb{N}$, $\sqsubseteq$, $R$ in ternary frames are not *ad hoc*, but arise out of a single underlying phenomenon, the *compositional relation*.

Identifying compositional relations on structures is a way to look for *natural* models of substructural logics.

Different logics are found by varying the *collections* being related, whether sets, multisets, lists, leaf-labelled binary trees or something else.
THANK YOU!