Exploring Three-Valued Models for Identity

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Bochum ⋆ 20 July 2023

MY PLAN

1. Traditional LP Models for Identity
2. LP, K3, ST... & Classical Logic
3. The 'Weakest' Rules for Identity
4. Example Three-Valued Models
5. Strengthening the rules.

1. Traditional LP Models for Identity

$A \wedge B$ is TRUE iff A is TRUE & B is TRUE

$A \wedge B$ is FALSE iff A is FALSE or B is FALSE

$\neg A$ is TRUE iff A is FALSE

$\neg A$ is FALSE iff A is TRUE

$\forall x\, A(x)$ is TRUE iff $A(d)$ is TRUE for every $d \in D$

$\forall x\, A(x)$ is FALSE iff $A(d)$ is FALSE for some $d \in D$

extension

anti-extension

$$[\![P]\!] = (P^+, P^-) \quad \text{where} \quad P^+ \cup P^- = D^n$$

$Pt_1 \dots t_n$ is TRUE iff $\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle \in P^+$

$Pt_1 \dots t_n$ is FALSE iff $\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle \in P^-$

$$X \models_{LP} Y$$

iff whenever each member of X is TRUE, some member of Y is TRUE.

NEAT FACT: $\models_{LP} Y$ iff $\models_{CL} Y$

The final part of first order machinery, identity, can be simply accommodated. We merely take '$=$' to be a particular two-place predicate such that

$$d^+(=) = \{<x, x> | x \in D\}.$$

$d^-(=)$ is arbitrary, except that $d^+(=) \cup d^-(=) = D^2$. (There may be philosophical arguments for placing other constraints on $\mathrm{d}^-(=)$, but they need not concern us here.) We can now state the final Fact.

Graham Priest, *In Contradiction*, §5.4

$$[\![=]\!] = (id_D, N)$$

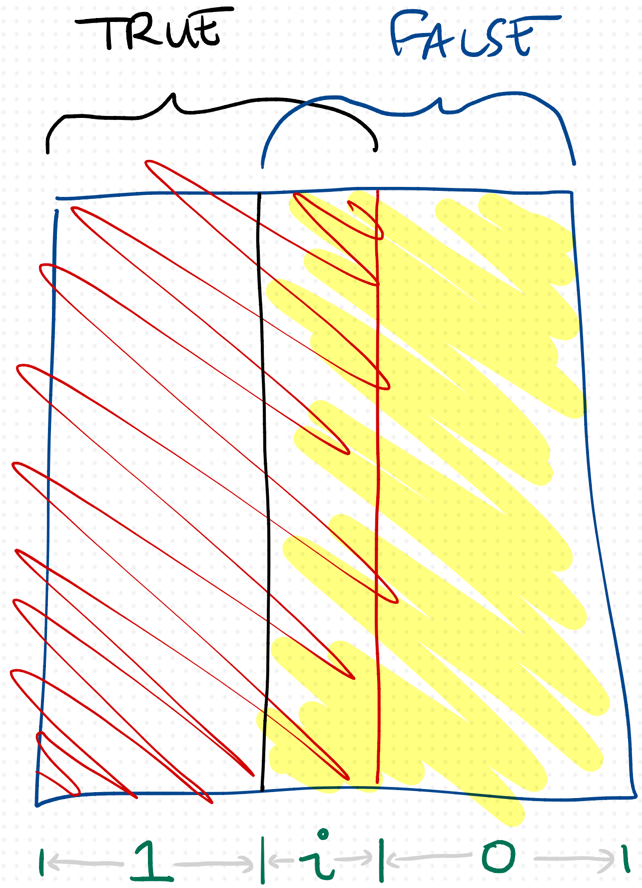
$$id_D = \{<x, x> | x \in D\}$$

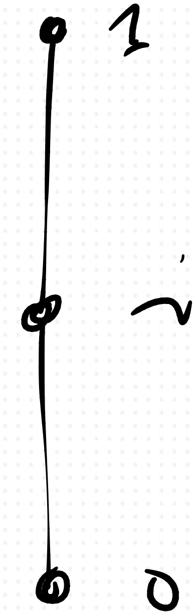
$$id_D \cup N = D^2$$

This seems rather CONSTRAINED

but it does have the virtue of making LP-valid every validity of classical first-order logic with identity.

2. LP, K3, ST... & Classical logic

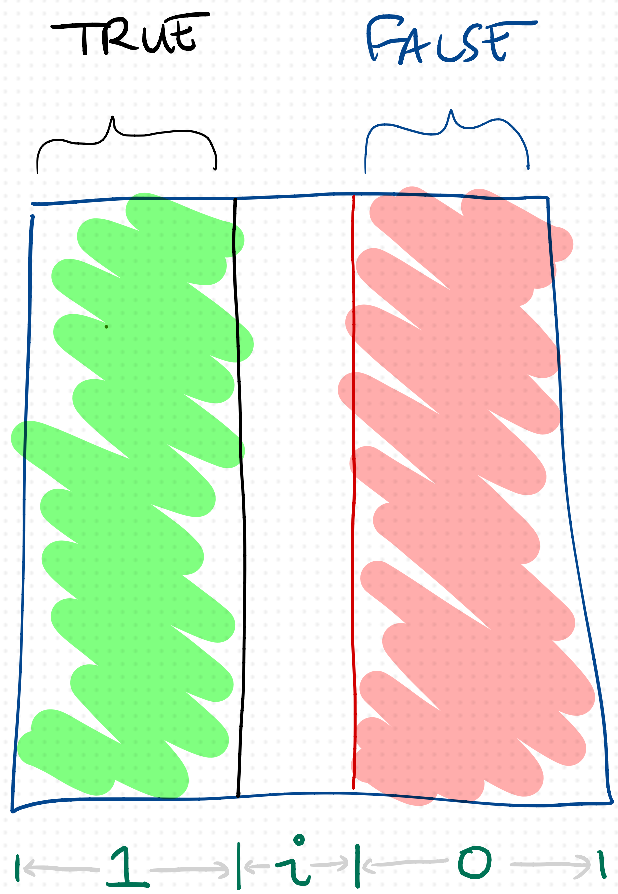




$$A \models_{LP} B$$

$$[\![A]\!] = 1 \text{ or } i \Rightarrow [\![B]\!] = 1 \text{ or } i$$

$$\neg\left([\![A]\!] = 1 \text{ or } i \;\&\; [\![B]\!] = 0\right)$$



$A \vDash_{LP} B$

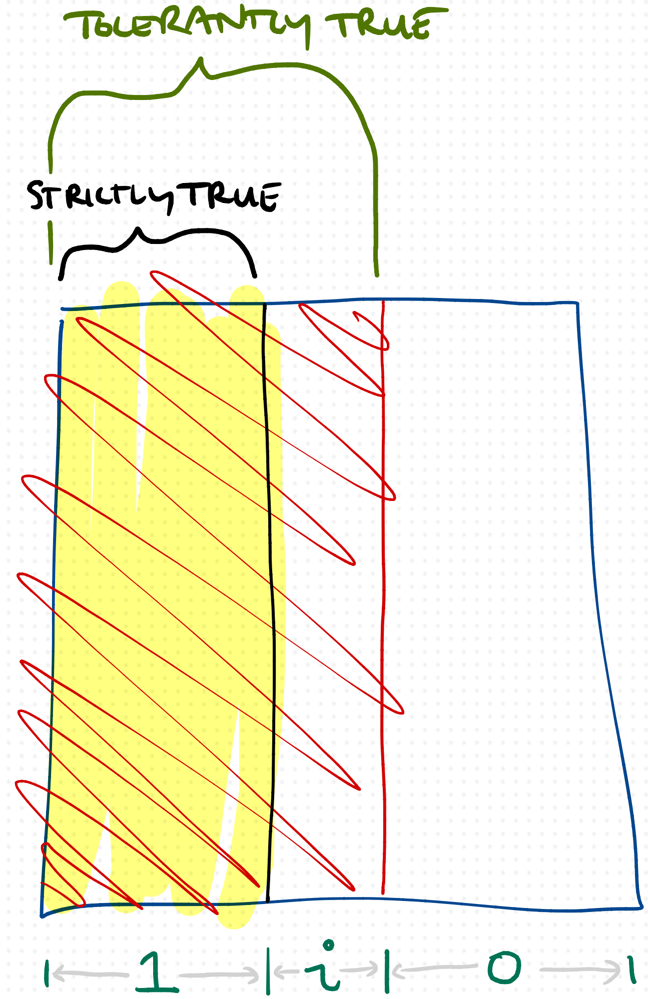
$[\![A]\!] = 1 \text{ or } i \Rightarrow [\![B]\!] = 1 \text{ or } i$

$\neg([\![A]\!] = 1 \text{ or } i \;\&\; [\![B]\!] = 0)$

$A \vDash_{K3} B$

$[\![A]\!] = 1 \Rightarrow [\![B]\!] = 1$

$\neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0 \text{ or } i)$



$$A \vDash_{LP} B$$

$$[\![A]\!] = 1 \text{ or } i \Rightarrow [\![B]\!] = 1 \text{ or } i$$

$$\neg([\![A]\!] = 1 \text{ or } i \;\&\; [\![B]\!] = 0)$$

$$A \vDash_{ST} B$$

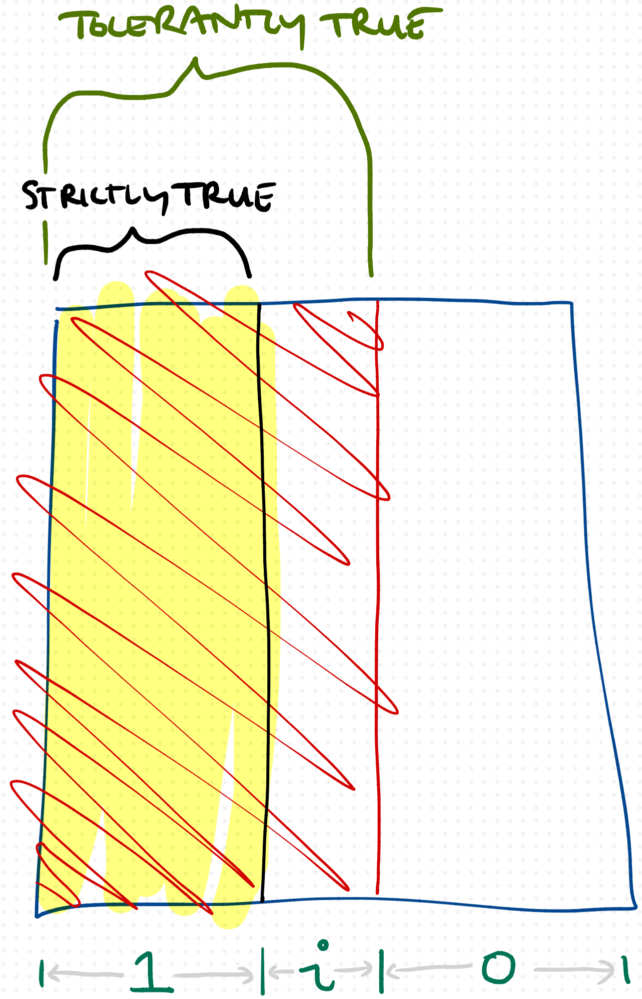
$$[\![A]\!] = 1 \Rightarrow [\![B]\!] = 1 \text{ or } i$$

$$\neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0)$$

$$A \vDash_{K3} B$$

$$[\![A]\!] = 1 \Rightarrow [\![B]\!] = 1$$

$$\neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0 \text{ or } i)$$



$A \vDash_{LP} B$

$[\![A]\!] = 1 \text{ or } i \Rightarrow [\![B]\!] = 1 \text{ or } i$

$\neg([\![A]\!] = 1 \text{ or } i \;\&\; [\![B]\!] = 0)$

$A \vDash_{ST} B$

$[\![A]\!] = 1 \Rightarrow [\![B]\!] = 1 \text{ or } i$

$\neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0)$

$[\![A \to B]\!] = 1 \text{ or } i$

$[\![A \wedge \neg B]\!] = 0 \text{ or } i$

$A \vDash_{K3} B$

$[\![A]\!] = 1 \Rightarrow [\![B]\!] = 1$

$\neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0 \text{ or } i)$

$\to$	0	i	1
0	1	1	1
i	i	i	1
1	0	i	1

$\leftrightarrow$	0	i	1
0	1	i	0
i	i	i	i
1	0	i	1

It is useful to have notions of closeness & apartness for these three values, corresponding to the biconditional connective

$\approx$	0	i	1
0	+	+	−
i	+	+	+
1	−	+	+

$\not\approx$	0	i	1
0	−	−	+
i	−	−	−
1	+	−	−

$0 \approx i \qquad i \approx 1$

$0 \not\approx 1$

$A \vDash_{LP} B$

$\neg([\![A]\!] = 1 \text{ or } i \ \& \ [\![B]\!] = 0)$

$A \vDash_{ST} B$

$\neg([\![A]\!] = 1 \ \& \ [\![B]\!] = 0)$

$A \vDash_{K3} B$

$\neg([\![A]\!] = 1 \ \& \ [\![B]\!] = 0 \text{ or } i)$

$A \vDash_{ST} B$ iff $A \vDash_{CL} B$

$A \models_{LP} B$

$\neg(\llbracket A \rrbracket = 1 \text{ or } i \ \& \ \llbracket B \rrbracket = 0)$

$\models_{LP} B$ iff $\models_{CL} B$

$A \models_{ST} B$

$\neg(\llbracket A \rrbracket = 1 \ \& \ \llbracket B \rrbracket = 0)$

$A \models_{ST} B$ iff $A \models_{CL} B$

$A \models_{K3} B$

$\neg(\llbracket A \rrbracket = 1 \ \& \ \llbracket B \rrbracket = 0 \text{ or } i)$

$A \models_{K3}$ iff $A \models_{CL}$

$$A \vDash_{LP} B \qquad A \vDash_{ST} B \qquad A \vDash_{K3} B$$

$$\neg([\![A]\!] = 1 \text{ or } i \;\&\; [\![B]\!] = 0) \qquad \neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0) \qquad \neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0 \text{ or } i)$$

$$\vDash_{LP} B \text{ iff } \vDash_{CL} B \qquad A \vDash_{ST} B \text{ iff } A \vDash_{CL} B \qquad A \vDash_{K3} \text{ iff } A \vDash_{CL}$$

$$\frac{A \vDash_{ST} B \qquad B \vDash_{ST} C}{A \vDash_{ST} C}$$

Admissible for the logical vocabulary

$A \vDash_{LP} B$

$\neg([\![A]\!] = 1 \text{ or } i \ \& \ [\![B]\!] = 0)$

$\vDash_{LP} B$ iff $\vDash_{CL} B$

$A \vDash_{ST} B$

$\neg([\![A]\!] = 1 \ \& \ [\![B]\!] = 0)$

$A \vDash_{ST} B$ iff $A \vDash_{CL} B$

$A \vDash_{K3} B$

$\neg([\![A]\!] = 1 \ \& \ [\![B]\!] = 0 \text{ or } i)$

$A \vDash_{K3}$ iff $A \vDash_{CL}$

$$\frac{A \vDash_{ST} B \qquad B \vDash_{ST} C}{A \vDash_{ST} C}$$

Admissible for the logical vocabulary

Extend the language with a formula λ whose $[\![\lambda]\!] = i$

$\top \vDash_{ST_\lambda} \lambda \qquad \lambda \vDash_{ST_\lambda} \bot$

$\top \nvDash_{ST_\lambda} \bot$

But <u>not</u> a principle for all ST theories!

THESE ARE <u>ALL</u> ST-Valid INFERENCE PRINCIPLES

$$X, A \succ A, Y$$

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ \wedge L \qquad \frac{X \succ A, Y}{X, \neg A \succ Y}\ \neg L \qquad \frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \vee B, Y}\ \vee L$$

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \wedge B, Y}\ \wedge R \qquad \frac{X, A \succ Y}{X \succ \neg A, Y}\ \neg R \qquad \frac{X \succ A, B, Y}{X \succ A \vee B, Y}\ \vee R$$

$$\frac{X, A(t) \succ Y}{X, \forall x A(x) \succ Y}\ \forall L \qquad \frac{X \succ A(t), Y}{X \succ \exists x A(x), Y}\ \exists R$$

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\ \forall R^{*} \qquad \frac{X, A(n) \succ Y}{X, \exists x A(x) \succ Y}\ \exists L^{*}$$

* n must be fresh.

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y} \wedge L$$

If $X, A \wedge B \not\models_{ST} Y$ then $X, A, B \not\models_{ST} Y$

If $[\![A \wedge B]\!] = 1$ then $[\![A]\!] = [\![B]\!] = 1$.

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ \wedge L$$

If $X, A \wedge B \not\models_{ST} Y$ then $X, A, B \not\models_{ST} Y$

If $[\![A \wedge B]\!] = 1$ then $[\![A]\!] = [\![B]\!] = 1$.

$$\frac{X \succ A, Y \qquad X \succ B, Y}{X \succ A \wedge B, Y}\ \wedge R$$

If $X \not\models_{ST} A \wedge B, Y$ then $X \not\models_{ST} A, Y$ or $X \not\models_{ST} B, Y$

If $[\![A \wedge B]\!] = 0$ then $[\![A]\!] = 0$ or $[\![B]\!] = 0$.

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ \wedge L$$

If $X, A \wedge B \nvDash_{ST} Y$ then $X, A, B \nvDash_{ST} Y$

If $[\![A \wedge B]\!] = 1$ then $[\![A]\!] = [\![B]\!] = 1$.

$$\frac{X \succ A, Y \qquad X \succ B, Y}{X \succ A \wedge B, Y}\ \wedge R$$

If $X \nvDash_{ST} A \wedge B, Y$ then $X \nvDash_{ST} A, Y$ or $X \nvDash_{ST} B, Y$

If $[\![A \wedge B]\!] = 0$ then $[\![A]\!] = 0$ or $[\![B]\!] = 0$.

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\ \forall R^{*}$$

If $X \nvDash_{ST} \forall x A(x), Y$ then $X \nvDash_{ST} A(n), Y$

If $[\![\forall x A(x)]\!] = 0$ choose $d \in D$ & $[\![n]\!] = d$ where $[\![A(n)]\!] = 0$.

What about Identity?

Why not take classical sequent rules for identity & see what these mean for ST-models?

Which rules?

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a=b, Y} = Df$$

Which rules?

$$\cancel{\dfrac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a=b, Y}\;=\mathrm{Df}}$$

$$\dfrac{X \succ A(a), Y \qquad X, A(b) \succ Y}{X, a=b \succ Y}\;=L$$

$$\dfrac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a=b, Y}\;=R$$

$$\dfrac{X \succ A(b), Y \qquad X, A(a) \succ Y}{X, a=b \succ Y}\;=L$$

Which rules?

$$\cancel{\dfrac{X, Fa \vdash Fb, Y \qquad X, Fb \vdash Fa, Y}{X \vdash a=b, Y}} = Df$$

$$\cancel{\dfrac{X, Fa \vdash Fb, Y \qquad X, Fb \vdash Fa, Y}{X \vdash a=b, Y}} = R$$

$$\cancel{\dfrac{X \vdash A(a), Y \qquad X, A(b) \vdash Y}{X, a=b \vdash Y}} = L$$

$$\cancel{\dfrac{X \vdash A(b), Y \qquad X, A(a) \vdash Y}{X, a=b \vdash Y}} = L$$

$$\vdash a=a \quad (Refl)$$

$$\dfrac{X \vdash A(a), Y}{X, a=b \vdash A(b), Y} = L \qquad \dfrac{X, A(a) \vdash Y}{X, a=b, A(b) \vdash Y} = L$$

3. The "Weakest" Rules for Identity

Identity Axioms

$\succ a=a$ $\qquad$ $a=b, Fa \succ Fb$ $\qquad$ $a=b, Fb \succ Fa$

Here, F is any predicate of any arity

Identity Axioms

$\vdash a=a$ $\qquad$ $a=b, Fa \vdash Fb$ $\qquad$ $a=b, Fb \vdash Fa$

let Fx be $x=a$.

$$\dfrac{\vdash a=a \qquad a=b,\ a=a \vdash b=a}{a=b \vdash b=a}\ \text{cut}$$

Identity Axioms

$$\vdash a=a \qquad a=b, Fa \vdash Fb \qquad a=b, Fb \vdash Fa$$

$$\dfrac{\vdash a=a \qquad a=b, a=a \vdash b=a}{a=b \vdash b=a}\ \text{cut}$$

$$\dfrac{a=b, Fa \vdash Fb \qquad \dfrac{b=c, Fb \vdash Fc \qquad d=c, Fc \vdash Fd}{b=c, d=c, Fb \vdash Fd}\ \text{cut}}{a=b, b=c, d=c, Fa \vdash Fd}\ \text{cut}$$

$$X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y$$

I^a_b is any set of identity statements linking a to b.

$$\boxed{X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y}$$

I^a_b is any set of identity statements linking a to b.

- $\emptyset$ links a to a for all a.
- If X links a to b, $a=c, X$ & $c=a, X$ links b to c,
  $b=c, X$ & $c=b, X$ links a to c,
  (as well as linking all pairs linked by X)

$$\boxed{X, I^a_b \vdash a=b, Y \qquad X, I^a_b, Fa \vdash Fb, Y}$$

I^a_b is any set of identity statements linking a to b.

- These axioms are classically valid.
- If you add them to the sequent rules for first order predicate logic, the resulting system is complete & cut is admissible.

$$\boxed{X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y}$$

I^a_b is any set of identity statements linking a to b.

What do ST-models for these axioms look like?

$$\boxed{X, I^a_b \vdash a = b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y}$$

I^a_b is any set of identity statements linking a to b.

What do ST-models for these axioms look like?

★ $[\![a]\!]$ & $[\![b]\!]$ are strictly connected iff either $[\![a]\!] = [\![b]\!]$, or some sequence of identity statements linking a & b are strictly true.

Ax 1

$$X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y$$

I^a_b is any set of identity statements linking a to b.

What do ST-models for these axioms look like?

★ $[\![a]\!]$ & $[\![b]\!]$ are strictly connected iff either $[\![a]\!]=[\![b]\!]$, or some sequence of identity statements linking a & b are strictly true.

Ax1 • If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.

Ax1

$$X, I^a_b \vdash a=b, Y$$

Ax2

$$X, I^a_b, Fa \vdash Fb, Y$$

I^a_b is any set of identity statements linking a to b.

What do ST-models for these axioms look like?

★ $[\![a]\!]$ & $[\![b]\!]$ are strictly connected iff either $[\![a]\!] = [\![b]\!]$, or some sequence of identity statements linking a & b are strictly true.

Ax1 • If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.

Ax2 • If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

What is the logic of such models?

$$X \models_{ST^=} Y$$

iff

$$X \models_{CL^=} Y$$

What is the logic of such models?

$\models_{LP^=} Y$

iff

$\models_{CL^=} Y$

$X \models_{ST^=} Y$

iff

$X \models_{CL^=} Y$

$X \models_{K3^=}$

iff

$X \models_{CL^=}$

4. Example Three-Valued Models

- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.
- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

Strict Identity Models

$[\![=]\!]$	d_1	d_2	$\cdots$	d_i	$\cdots$
d_1	1	0	$\cdots$	0	$\cdots$
d_2	0	1	$\ddots$	$\vdots$	
$\vdots$	$\vdots$		$\ddots$	$\vdots$	
d_i	0	$\cdots$	$\cdots$	1	$\cdots$
	$\vdots$			$\vdots$	$\ddots$

- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.
- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

Lax Identity Models

$[\![=]\!]$	d_1	d_2	$\cdots$	d_i	$\cdots$
d_1	i	i	$\cdots$	i	$\cdots$
d_2	i	i		$\vdots$	
$\vdots$	$\vdots$		$\ddots$	$\vdots$	
d_i	i	$\cdots$	$\cdots$	i	$\cdots$
	$\vdots$			$\vdots$	$\ddots$

- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a = b]\!] \neq 0$.
- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

In General...

$[\![=]\!]$	d_1	d_2	$\cdots$	d_i	$\cdots$
d_1	$i/1$	?	$\cdots$	?	$\cdots$
d_2	?	$i/1$			
$\vdots$	$\vdots$		$\ddots$	$\vdots$	
d_i	?	$\cdots$	$\cdots$	$i/1$	$\cdots$
	$\vdots$			$\vdots$	$\ddots$

- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a = b]\!] \neq 0$.
- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
$[\![F]\!]$	1	1	1	i	i	i	0	0	0

$[\![=]\!]$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	1								
a_2		1							
a_3			1						
a_4				1					
a_5					1				
a_6						1			
a_7							1		
a_8								1	
a_9									1

Symmetry "failures"

$[\![=]\!]$	a	b
a	1	i
b	0	1

$a = b \models_{ST} b = a$

$a = b \not\models_{LP} b = a$

Compatible with any predicates on $D = \{a, b\}$

Symmetry "failures"

$[\![=]\!]$	a	b
a	1	i
b	0	1

$[\![=]\!]$	a	b
a	1	1
b	i	1

Compatible with any predicate on $D = \{a, b\}$

Requires $[\![Fa]\!] \approx [\![Fb]\!]$ for every predicate F.

5. Strengthening the rules.

Stronger Indiscernibility Rules

$$\frac{X,\ Fa \succ Y}{X, a=b, Fb \succ Y} =L$$

If $[\![a=b]\!]=1$ & $[\![Fb]\!]=1$ then $[\![Fa]\!]=1$

Stronger Indiscernibility Rules

$$\frac{X, Fa \succ Y}{X, a=b, Fb \succ Y} =L$$

If $[\![a=b]\!]=1$ & $[\![Fb]\!]=1$ then $[\![Fa]\!]=1$

$$\frac{X, Fb \succ Y}{X, a=b, Fa \succ Y} =L$$

If $[\![a=b]\!]=1$ & $[\![Fa]\!]=1$ then $[\![Fb]\!]=1$

$$\frac{X \succ Fa, Y}{X, a=b \succ Fb, Y} =L$$

If $[\![a=b]\!]=1$ & $[\![Fb]\!]=0$, then $[\![Fa]\!]=0$

$$\frac{X \succ Fb, Y}{X, a=b \succ Fa, Y} =L$$

If $[\![a=b]\!]=1$ & $[\![Fa]\!]=0$, then $[\![Fb]\!]=0$

Symmetry

$$\frac{X, a=b \vdash Y}{X, b=a \vdash Y} = \mathrm{SwapL}$$

If $[\![b=a]\!]=1$ then $[\![a=b]\!]=1$

$$\frac{X \vdash a=b, Y}{X \vdash b=a, Y} = \mathrm{SwapR}$$

If $[\![b=a]\!]=0$ then $[\![a=b]\!]=0$

LP-style Indiscernibility

$$\frac{X \vdash a=b, Y \qquad X \vdash Fa, Y}{X \vdash Fb, Y} = LP$$

If $[\![Fb]\!] = 0$ then either $[\![a=b]\!] = 0$ or $[\![Fa]\!] = 0$

LP-style Indiscernibility

$$\frac{X \vdash a=b, Y \qquad X \vdash Fa, Y}{X \vdash Fb, Y} = LPI$$

If $[\![Fb]\!] = 0$ then either $[\![a=b]\!] = 0$ or $[\![Fa]\!] = 0$

If $[\![a=b]\!] = 1$ or i & $[\![Fa]\!] = 1$ or i, then $[\![Fb]\!] = 1$ or i.

A "Drop" Rule

$$\frac{X, a=a \vdash Y}{X \vdash Y} = \text{Drop}$$

$$[\![a=a]\!] = 1$$

There is plenty more here for you to explore. The logic-agnostic (or pluralist) perspective on models gives us a number of new tools for developing distinctive three-valued models for identity.

Thanks!