Generality & Existence I
Quantifiers & Identity

Greg Restall

THE UNIVERSITY OF
MELBOURNE

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To analyse the quantifiers
My Aim

To analyse the quantifiers (including their interactions with modals)
My Aim

To analyse the *quantifiers* (including their interactions with *modals*) using the tools of *proof theory*
My Aim

To analyse the quantifiers (including their interactions with modals) using the tools of proof theory in order to better understand quantification, existence and identity.
My Aim for Today

Understanding the quantifier rules.
Today's Plan
SEQUENTS & DEFINING RULES
Don’t assert each element of $\Gamma$ and deny each element of $\Delta$. 

$$\Gamma \succ \Delta$$
Structural Rules

Identity: \( A \models A \)
Structural Rules

Identity: \( A \succ A \)

Weakening:

\[
\frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta} \quad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}
\]
Structural Rules

Identity: $A \vdash A$

Weakening: $\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}$

Contraction: $\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}$
Identity: \( A \not\rightarrow A \)

Weakening: \[
\frac{\Gamma \not\rightarrow \Delta}{\Gamma, A \not\rightarrow \Delta}
\frac{\Gamma \not\rightarrow \Delta}{\Gamma \not\rightarrow A, \Delta}
\]

Contraction: \[
\frac{\Gamma, A, A \not\rightarrow \Delta}{\Gamma, A \not\rightarrow \Delta}
\frac{\Gamma \not\rightarrow A, A, \Delta}{\Gamma \not\rightarrow A, \Delta}
\]

Cut: \[
\frac{\Gamma \not\rightarrow A, \Delta}{\Gamma \not\rightarrow A, \Delta}
\frac{\Gamma \not\rightarrow A, \Delta}{\Gamma \not\rightarrow \Delta}
\]
Structural Rules

**Identity:** \[ A \to A \]

**Weakening:** \[
\frac{\Gamma \to \Delta}{\Gamma, A \to \Delta} \quad \frac{\Gamma \to \Delta}{\Gamma \to A, \Delta}
\]

**Contraction:** \[
\frac{\Gamma, A, A \to \Delta}{\Gamma, A \to \Delta} \quad \frac{\Gamma \to A, A, \Delta}{\Gamma \to A, \Delta}
\]

**Cut:** \[
\frac{\Gamma \to A, \Delta \quad \Gamma, A \to \Delta}{\Gamma \to \Delta}
\]

Structural rules govern declarative sentences *as such.*
Extending a Language with Specific Vocabulary

With Left/Right rules?

\[
\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta} \quad [\land L]
\]

\[
\frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \land B, \Delta} \quad [\land R]
\]
Extending a Language with Specific Vocabulary

With Left/Right rules?

\[
\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta} \quad \text{[\land_L]} \quad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \land B, \Delta} \quad \text{[\land_R]}
\]

\[
\frac{\Gamma, B \succ \Delta}{\Gamma, A \text{ tonk} B \succ \Delta} \quad \text{[tonk_L]} \quad \frac{\Gamma \succ A, \Delta}{\Gamma \succ A \text{ tonk} B, \Delta} \quad \text{[tonk_R]}
\]
What is involved in going from $\mathcal{L}$ to $\mathcal{L}'$?

Use $\succ_{\mathcal{L}}$ to define $\succ_{\mathcal{L}'}$. 
What is involved in going from $\mathcal{L}$ to $\mathcal{L}'$?

Use $\triangleright_{\mathcal{L}}$ to define $\triangleright_{\mathcal{L}'}$.

*Desideratum #1:* $\triangleright_{\mathcal{L}'}$ is conservative: $(\triangleright_{\mathcal{L}'})|_{\mathcal{L}}$ is $\triangleright_{\mathcal{L}}$. 
What is involved in going from $\mathcal{L}$ to $\mathcal{L}'$?

Use $\not\in_\mathcal{L}$ to define $\not\in_{\mathcal{L}'}$.

**Desideratum #1:** $\not\in_{\mathcal{L}'}$ is conservative: $(\not\in_{\mathcal{L}'})|_{\mathcal{L}}$ is $\not\in_{\mathcal{L}}$.

**Desideratum #2:** Concepts are defined uniquely.
A Defining Rule

\[
\frac{\Gamma, A, B \triangleright \Delta}{\Gamma, A \land B \triangleright \Delta} \quad [\land Df]
\]
\[\Gamma, A, B \succ \Delta \quad \frac{}{\Gamma, A \land B \succ \Delta} \quad [\land Df]\]

Fully specifies norms governing conjunctions on the left in terms of simpler vocabulary.
A Defining Rule

\[
\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta} [\land Df]
\]

Fully specifies norms governing conjunctions on the left in terms of simpler vocabulary.

Identity and Cut determine the behaviour of conjunctions on the right.
From $\lnot Df$ to $\lnot L/R$

$$
\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta} \quad \frac{A, B \vdash A \land B}{A \land B \vdash A \land B} \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta} \quad \frac{\Gamma, A \vdash A \land B, \Delta}{\Gamma \vdash A \land B, \Delta}
$$

[Id]

$[\land Df]$

$[Cut]$

$[Cut]$
From $[\land Df]$ to $[\land L/R]$
From $\land \text{Df}$ to $\land L/R$

- $\Gamma \vdash A \land B, \Delta$
- $\Gamma \vdash B, \Delta$
- $\Gamma \vdash A, \Delta$

$\frac{A \land B \vdash A \land B}{A, B \vdash A \land B}$ [Id]

$\frac{A \land B \vdash A \land B}{A, B \vdash A \land B}$ [Df]

$\frac{A \land B \vdash A \land B}{A, B \vdash A \land B}$ [Cut]

$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta}$

$\frac{\Gamma, A \vdash A \land B, \Delta}{\Gamma, A \vdash A \land B, \Delta}$ [Cut]

$\frac{\Gamma \vdash A \land B, \Delta}{\Gamma \vdash A \land B, \Delta}$ [Cut]
From $[\land Df]$ to $[\land L/R]$

\[
\frac{
\Gamma \vdash B, \Delta \\
\Gamma \vdash A, \Delta
}
{\Gamma \vdash A \land B, \Delta} \quad [\text{Cut}]
\]

\[
\frac{
A \land B \vdash A \land B
}
{A, B \vdash A \land B} \quad [\text{Id}]
\]

\[
\frac{
A, B \vdash A \land B
}
{\Gamma, A \vdash A \land B, \Delta} \quad [\text{Cut}]
\]
From \(\land Df\) to \(\land L/R\)

\[
\Gamma \vdash B, \Delta \quad A, B \vdash A \land B \quad \text{[\(\land Df\)]}
\]

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \Gamma, A \vdash A \land B, \Delta \\
\hline
\Gamma \vdash A \land B, \Delta
\end{align*} \quad \text{[\texttt{Cut}]}
\]

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \Gamma \vdash B, \Delta \\
\hline
\Gamma \vdash A \land B, \Delta
\end{align*} \quad \text{[\(\land R\)]}
\]
And Back

\[
\begin{align*}
A & \succ A & B & \succ B \\
\hline
A, B & \succ A \land B & [\land R] & \Gamma, A \land B & \succ \Delta \\
\hline
\Gamma, A, B & \succ \Delta & [Cut]
\end{align*}
\]
This generalises: \( \land Df, Cut \equiv \land L/R, Cut \)
Equivalence

\[ \mathcal{L}[\wedge Df, \text{Cut}] = \mathcal{L}[\wedge L/R, \text{Cut}] = \mathcal{L}[\wedge L/R] \]
Equivalence

\[ \mathcal{L}[\land Df, Cut] = \mathcal{L}[\land L/R, Cut] = \mathcal{L}[\land L/R] \]

This generalises: \( \land, \lor, \supset, \neg \) work in the same way.
Equivalence

\[ \mathcal{L}[\land Df, \text{Cut}] = \mathcal{L}[\land L/R, \text{Cut}] = \mathcal{L}[\land L/R] \]

This generalises: \( \land, \lor, \supset, \neg \) work in the same way.

I want to see how this works for quantifiers.
GENERALITY & CLASSICAL QUANTIFIERS
The Rules

\[ \Gamma \vdash A(n), \Delta \quad [\forall Df] \quad \Gamma, A(n) \vdash \Delta \quad [\exists Df] \]

(\text{where } n \text{ is not present in the bottom sequent of both rules})
The Rules

\[
\Gamma \vdash A(n), \Delta \\
\hline
\Gamma \vdash (\forall x)A(x), \Delta
\]

[∀Df]

\[
\Gamma, A(n) \vdash \Delta \\
\hline
\Gamma, (\exists x)A(x) \vdash \Delta
\]

[∃Df]

(where \(n\) is not present in the bottom sequent of both rules)

For this to work as expected, \(n\) must be *deductively general*. 
(where \( n \) is not present in the bottom sequent of both rules)

For this to work as expected, \( n \) must be *deductively general*.

Function terms are not deductively general:

\[
(\forall x)(0 \neq x') \not\vdash 0 \neq 1, \text{ but } (\forall x)(0 \neq x') \not\vdash (\forall x)(0 \neq x).
\]
A term $n$ is *deductively general* for the category $\mathcal{T}$ iff the rule of *specification* is admissible for each term $t$ of category $\mathcal{T}$.

$$
\Gamma \vdash \Delta \\
\Gamma[n := t] \vdash \Delta[n := t] \quad [\text{Spec}^n_t]
$$
Generality and Specification

A term \( n \) is *deductively general* for the category \( \mathcal{T} \) iff the rule of *specification* is admissible for each term \( t \) of category \( \mathcal{T} \).

\[
\Gamma \vdash \Delta \\
\Gamma[n := t] \vdash \Delta[n := t] \quad [Spec^n_t]
\]

In classical first order predicate logic, names are deductively general.
\[
\begin{align*}
(\forall x)Fx & \supset (\forall x)Fx \\
\frac{}{(\forall x)Fx \supset Fn} & \quad [\forall Df]
\end{align*}
\]
How can we derive \((\forall x)Fx \supset Ft\)?
\[ (\forall x)Fx \succ (\forall x)Fx \quad [\forall Df] \]

\[ (\forall x)Fx \succ F_n \]

How can we derive \((\forall x)Fx \succ Ft\)?

We must make explicit use of specification.

\[ (\forall x)Fx \succ (\forall x)Fx \quad [\forall Df] \]

\[ (\forall x)Fx \succ F_n \quad [\forall Df] \]

\[ (\forall x)Fx \succ Ft \quad [Spec^n] \]
From $[\forall Df]$ to $[\forall L]$

$$
\frac{(\forall x)\, A(x) \supset (\forall x)\, \overline{A}(x)}{[Id]}
\frac{(\forall x)\, A(x) \supset A(n)}{[\forall Df]} \frac{\Gamma, A(n) \supset \Delta}{[Cut]}
\frac{\Gamma, (\forall x)\, A(x) \supset \Delta}{[Cut]}
$$
From $[\forall Df]$ to $[\forall L]$
From $\forall Df$ to $\forall L$

\[
\frac{(\forall x)\overline{A}(x) \succ (\forall x)\overline{A}(x)}{[\text{Id}]}
\frac{(\forall x)\overline{A}(x) \succ A(n)}{[\forall Df]} \frac{\Gamma, A(n) \succ \Delta}{{\text{Cut}}} \frac{\Gamma, (\forall x)\overline{A}(x) \succ \Delta}{[\text{Cut}]}\]
From $[\forall Df]$ to $[\forall L]$

$\frac{(\forall x)A(x) > (\forall x)\bar{A}(x)}{[Id]}$

$\frac{(\forall x)A(x) > A(n)}{[\forall Df]}$

$\frac{\Gamma, A(n) > \Delta}{[Cut]}$

$\Gamma, (\forall x)\bar{A}(x) > \Delta$
From $[\forall Df]$ to $[\forall L]$

\[
\frac{(\forall x)A(x) \succ (\forall x)A(x)}{[Id]} \quad \frac{(\forall x)A(x) \succ A(n)}{[\forall Df]} \quad \frac{\Gamma, A(n) \succ \Delta}{[Cut]} \quad \Gamma, (\forall x)A(x) \succ \Delta
\]

The rule that results no longer has the side condition for $n$, because the premise sequent $\Gamma, A(n) \succ \Delta$ is arbitrary.

\[
\frac{\Gamma, A(n) \succ \Delta}{[\forall L: for names]} \quad \frac{\Gamma, (\forall x)A(x) \succ \Delta}{[Cut]}
\]
From \([\forall Df]\) to \([\forall L]\)

\[
\frac{(\forall x)A(x) \to (\forall x)A(x)}{[Id]} \quad (\forall x)A(x) \to A(n) \quad [\forall Df] \\
\frac{\Gamma, A(n) \to \Delta}{\Gamma, (\forall x)A(x) \to \Delta} \quad [Cut]
\]

The rule that results no longer has the side condition for \(n\), because the premise sequent \(\Gamma, A(n) \to \Delta\) is arbitrary.

\[
\frac{\Gamma, A(n) \to \Delta}{\Gamma, (\forall x)A(x) \to \Delta} \quad [\forall L: \text{for names}]
\]

However, it applies only to names, not terms.
From $[\forall Df]$ to $[\forall L]$, cont.

\[
\frac{(\forall x) \mathcal{A}(x) \supset (\forall x) \mathcal{A}(x)}{[\forall Df]} \quad \frac{(\forall x) \mathcal{A}(x) \supset \mathcal{A}(n)}{[Spec^n_t]} \quad \frac{(\forall x) \mathcal{A}(x) \supset \mathcal{A}(t)}{[Cut]} \quad \frac{\Gamma, \mathcal{A}(t) \supset \Delta}{\Gamma, (\forall x) \mathcal{A}(x) \supset \Delta}
\]
From $[\forall Df]$ to $[\forall L]$, cont.

\[
\frac{(\forall x)\mathcal{A}(x) \succ (\forall x)\mathcal{A}(x)}{[Id]}
\]

\[
\frac{(\forall x)\mathcal{A}(x) \succ \mathcal{A}(n)}{[\forall Df]}
\]

\[
\frac{(\forall x)\mathcal{A}(x) \succ \mathcal{A}(t)}{[Spec^n_t]} \quad \Gamma, \mathcal{A}(t) \succ \Delta
\]

\[
\frac{\Gamma, (\forall x)\mathcal{A}(x) \succ \Delta}{[Cut]}
\]

\[
\frac{\Gamma, \mathcal{A}(t) \succ \Delta}{[\forall L]}
\]

\[
\Gamma, (\forall x)\mathcal{A}(x) \succ \Delta
\]
Equivalence

\[ \mathcal{L}[\forall \text{Df}, \text{Spec}, \text{Cut}] = \mathcal{L}[\forall \text{L/R}, \text{Spec}, \text{Cut}] \]
Equivalence

\[ \mathcal{L}[\forall \text{Df}, \text{Spec}, \text{Cut}] = \mathcal{L}[\forall \text{L}/\text{R}, \text{Spec}, \text{Cut}] = \mathcal{L}[\forall \text{L}/\text{R}, \text{Cut}] \]
Equivalence

QUANTIFIERS & NON-DENOTING TERMS
Non-Denoting Terms

\[ \lim_{x \to 0} \frac{\sin x}{x} \quad \sum_{n=0}^{\infty} f(n) \]
Non-Denoting Terms

\[
\frac{1}{0} \quad \lim_{x \to 0} \frac{\sin x}{x} \quad \sum_{n=0}^{\infty} f(n) \quad Pegasus
\]
Non-Denoting Terms

\[
\begin{align*}
\frac{1}{0} & \quad \lim_{x \to 0} \frac{\sin x}{x} & \quad \sum_{n=0}^{\infty} f(n) & \quad \text{Pegasus}
\end{align*}
\]

It is difficult to eliminate non-denoting terms as a matter of syntax.
Non-Denoting Terms

\[
\begin{align*}
\frac{1}{0} \quad \lim_{x \to 0} \frac{\sin x}{x} \quad \sum_{n=0}^{\infty} f(n) \quad Pegasus
\end{align*}
\]

It is difficult to eliminate non-denoting terms as a matter of syntax.

\[(\forall x)(x < 0 \lor x = 0 \lor x > 0) \not\equiv (\frac{1}{0} < 0 \lor \frac{1}{0} = 0 \lor \frac{1}{0} > 0)\]
It is difficult to eliminate non-denoting terms as a matter of *syntax*.

$$(\forall x)(x < 0 \lor x = 0 \lor x > 0) \not\equiv (\frac{1}{0} < 0 \lor \frac{1}{0} = 0 \lor \frac{1}{0} > 0)$$

How can we modify the quantifier rules to allow for non-denoting terms?
To rule a term *in* is to take it as suitable to substitute into a quantifier, i.e., to take the term to *denote*.

To rule a term *out* is to take it as unsuitable to substitute into a quantifier, i.e., to take the term to *not denote*. 
*Pro and Con* attitudes to Terms

To rule a term *in* is to take it as suitable to substitute into a quantifier, i.e., to take the term to *denote*.

To rule a term *out* is to take it as unsuitable to substitute into a quantifier, i.e., to take the term to *not denote*.

We add terms to the LHS and RHS of sequents $\Gamma \rightarrow \Delta$. 
Structural Rules remain as before

**Identity:** \[ X \vdash X \]

**Weakening:**
\[
\begin{array}{c}
\Gamma \vdash \Delta \\
\hline
\Gamma, X \vdash \Delta \\
\end{array}
\quad
\begin{array}{c}
\Gamma \vdash \Delta \\
\hline
\Gamma \vdash X, \Delta \\
\end{array}
\]

**Contraction:**
\[
\begin{array}{c}
\Gamma, X, X \vdash \Delta \\
\hline
\Gamma, X \vdash \Delta \\
\end{array}
\quad
\begin{array}{c}
\Gamma \vdash X, X, \Delta \\
\hline
\Gamma \vdash X, \Delta \\
\end{array}
\]

**Cut:**
\[
\begin{array}{c}
\Gamma \vdash X, \Delta \\
\hline
\Gamma, X \vdash \Delta \\
\end{array}
\quad
\begin{array}{c}
\Gamma, X \vdash \Delta \\
\hline
\Gamma \vdash \Delta \\
\end{array}
\]

Here \( X \) is either a sentence or a term.
Quantifier Rules, allowing for non-denoting terms

\[\frac{\Gamma, n \vdash A(n), \Delta}{\Gamma \vdash (\forall x)A(x), \Delta} [\forall Df]\]

\[\frac{\Gamma, n, A(n) \vdash \Delta}{\Gamma, (\exists x)A(x) \vdash \Delta} [\exists Df]\]
From $[\forall Df]$ to $[\forall L]$

\[
\begin{align*}
(\forall x)A(x) & \succ (\forall x)A(x) \quad [Id] \\
(\forall x)A(x), n \succ A(n) & \quad [\forall Df] \\
(\forall x)A(x), t \succ A(t) & \quad [Spec^n] \\
\Gamma, (\forall x)A(x), t \succ \Delta & \quad \Gamma, A(t) \succ \Delta \quad [Cut] \\
\Gamma, (\forall x)A(x), t \succ \Delta & \quad \Gamma \succ t, \Delta \quad [Cut] \\
\Gamma, (\forall x)A(x) & \succ \Delta \quad [Cut]
\end{align*}
\]
From $[\forall Df]$ to $[\forall L]$

\[
\begin{align*}
(\forall x)A(x) & \vdash (\forall x)A(x) & [Id] \\
(\forall x)A(x), n & \succ A(n) & [\forall Df] \\
(\forall x)A(x), t & \succ A(t) & [Spec^n] \\
\Gamma, (\forall x)A(x), t & \succ \Delta & \Gamma, A(t) \succ \Delta & [Cut] \\
\Gamma, (\forall x)A(x), t & \succ \Delta & \Gamma \succ t, \Delta & [Cut] \\
\Gamma, (\forall x)A(x) & \succ \Delta & \Gamma, (\forall x)A(x) \succ \Delta & [\forall L]
\end{align*}
\]

This results in a two-premise rule:

\[
\begin{align*}
\Gamma, A(t) & \succ \Delta & \Gamma \succ t, \Delta & [\forall L] \\
\Gamma, (\forall x)A(x) & \succ \Delta & \Gamma, (\forall x)A(x) \succ \Delta & [\forall L]
\end{align*}
\]
From $[\exists Df]$ to $[\exists R]$
From $[\exists Df]$ to $[\exists R]$

\[
\frac{(\exists x)A(x) \succ (\exists x)A(x)}{[Id]}
\]

\[
\frac{A(n), n \succ (\exists x)A(x)}{[\exists Df]}
\]

\[
\frac{\Gamma \succ A(t), \Delta \quad \Gamma, t \succ (\exists x)A(x)}{[\text{Spec}^n]}\]

\[
\frac{\Gamma, t \succ (\exists x)A(x), \Delta}{[\text{Cut}]}
\]

\[
\frac{\Gamma \succ (\exists x)A(x), \Delta}{[\text{Cut}]}
\]

This gives a two-premise $[\exists R]$ rule:

\[
\frac{\Gamma \succ t, \Delta \quad \Gamma \succ A(t), \Delta}{[\exists R]}\]

\[
\frac{\Gamma \succ (\exists x)A(x), \Delta}{[\exists R]}
\]
\[ \Gamma, t \succ \Delta \quad \frac{\Gamma, t \downarrow \succ \Delta}{\Gamma, t \downarrow \succ \Delta} \quad \text{[↓}Df\text{]} \]
Making Denotation Explicit

\[
\frac{\Gamma, t \gg \Delta}{\Gamma, t \Downarrow \gg \Delta} \quad [\downarrow Df]
\]

This results in the obvious [$\downarrow R$] rule.

\[
\frac{\Gamma \gg t, \Delta}{t \Downarrow \gg t \Downarrow} \quad [Id]
\]

\[
\frac{t \Downarrow \gg t \Downarrow}{\Gamma \gg t \Downarrow, \Delta} \quad [\downarrow Df]
\]

\[
\frac{\Gamma \gg t \Downarrow, \Delta}{\Gamma \gg t, \Delta} \quad [\downarrow R]
\]

\[
\frac{\Gamma \gg t \Downarrow, \Delta}{\Gamma \gg t, \Delta} \quad [\text{Cut}]
\]
ABSTRACT. Questions of definedness are ubiquitous in mathematics. Informally, these involve reasoning about expressions which may or may not have a value. This paper surveys work on logics in which such reasoning can be carried out directly, especially in computational contexts. It begins with a general logic of “partial terms”, continues with partial combinatory and lambda calculi, and concludes with an expressively rich theory of partial functions and polymorphic types, where termination of functional programs can be established in a natural way.

Definedness, function terms and predicates

\[ \frac{t_i, \Gamma \triangleright \Delta}{f(t_1, \ldots, t_n), \Gamma \triangleright \Delta} [fL] \]

\[ \frac{t_i, \Gamma \triangleright \Delta}{Ft_1 \cdots t_n, \Gamma \triangleright \Delta} [FL] \]
DERIVATIONS
& SYSTEMS
Structural Rules

Identity: \( X \vdash X \)

Weakening: \[
\begin{align*}
\frac{\Gamma \vdash \Delta}{\Gamma, X \vdash \Delta} & \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash X, \Delta} \\
\frac{\Gamma, X, X \vdash \Delta}{\Gamma, X \vdash \Delta} & \quad \frac{\Gamma \vdash X, X, \Delta}{\Gamma \vdash X, \Delta}
\end{align*}
\]

Contraction: \[
\begin{align*}
\frac{\Gamma \vdash X, \Delta}{\Gamma \vdash \Delta} & \quad \frac{\Gamma, X \vdash \Delta}{\Gamma \vdash \Delta}
\end{align*}
\]

Cut: \[
\frac{\Gamma, X, \Delta \quad \Gamma, X \vdash \Delta}{\Gamma \vdash \Delta}
\]
Predicate and Function Rules

\[
\frac{t_i, \Gamma \succ \Delta}{f(t_1, \ldots, t_n), \Gamma \succ \Delta} \quad [fL]
\]

\[
\frac{t_i, \Gamma \succ \Delta}{F t_1 \cdots t_n, \Gamma \succ \Delta} \quad [FL]
\]
\[
\frac{\Gamma \succ \Delta}{\Gamma[n := t] \succ \Delta[n := t]} \quad [Spec^n_t]
\]
Defining Rules

![Formulas and definitions of logical rules]

\[\Gamma, A, B \vdash \Delta \quad \text{[\(\wedge Df\)]} \]
\[\Gamma \vdash A \wedge B, \Delta \quad \text{[\(\forall Df\)]} \]

\[\Gamma, A \vdash B, \Delta \quad \text{[\(\exists Df\)]} \]
\[\Gamma \vdash A \supset B, \Delta \quad \text{[\(\forall Df\)]} \]

\[\Gamma, n \vdash A(n), \Delta \quad \text{[\(\forall Df\)]} \]
\[\Gamma \vdash (\forall x)A(x), \Delta \quad \text{[\(\forall Df\)]} \]

\[\Gamma, t \vdash \Delta \quad \text{[\(\exists Df\)]} \]
\[\Gamma, t \downarrow \vdash \Delta \quad \text{[\(\downarrow Df\)]} \]
The System

DL[Df, Cut, Spec]
Example Derivation

\[(\forall x)(Fx \supset Gx) \supset (\forall x)(Fx \supset Gx)\]  

\[(\forall x)(Fx \supset Gx), n \supset Fn \supset Gn\]  

\[Fn \supset Gn \supset Fn \supset Gn\]  

\[(\exists x)Gx \supset (\exists x)Gx\]  

\[\forall x \supset \exists x\]  

\[n, Gn \supset (\exists x)Gx\]  

\[Fn \supset Gn, n, Fn \supset (\exists x)Gx\]  

\[\exists x \supset (\exists x)\]  

\[(\forall x)(Fx \supset Gx), (\exists x)Fx \supset (\exists x)Gx\]  

\[\exists x \supset (\exists x)\]  

\[(\forall x)(Fx \supset Gx) \supset (\exists x)Fx \supset (\exists x)Gx\]
Eliminating *Spec*

Replace the quantifier rules by these *generalised* defining rules:

\[
\frac{\Gamma, n \vdash A(n), \Delta}{\Gamma \vdash (\forall x)A(x), \Delta} \quad \text{[\(\forall Df\downarrow\)]} \quad \frac{\Gamma, t \vdash A(t), \Delta}{\Gamma \vdash (\forall x)A(x), \Delta} \quad \text{[\(\forall Df\uparrow\)]}
\]

\[
\frac{\Gamma, n, A(n) \vdash \Delta}{\Gamma, (\exists x)A(x) \vdash \Delta} \quad \text{[\(\exists Df\downarrow\)]} \quad \frac{\Gamma, (\exists x)A(x) \vdash \Delta}{\Gamma, t, A(t) \vdash \Delta} \quad \text{[\(\exists Df\uparrow\)]}
\]
Eliminating Spec

Replace the quantifier rules by these *generalised* defining rules:

\[
\begin{align*}
\Gamma, n & \vdash A(n), \Delta \\
\Gamma & \vdash (\forall x)A(x), \Delta & [\forall Df] \\
\Gamma, n, A(n) & \vdash \Delta \\
\Gamma, (\exists x)A(x) & \vdash \Delta & [\exists Df] \\
\Gamma, t & \vdash A(t), \Delta & [\forall Df'] \\
\Gamma, (\exists x)A(x) & \vdash \Delta \\
\Gamma, t, A(t) & \vdash \Delta & [\exists Df'] \\
\end{align*}
\]

\[\text{DL}[GDf, Cut]\]
Theorem

A derivation of a sequent \( \Gamma \vdash \Delta \) in DL\([Df, \text{Cut}, \text{Spec}]\) can be systematically transformed into a derivation of that sequent in DL\([GDf, \text{Cut}]\), and vice versa.

Proof.

All of the rules in DL\([GDf, \text{Cut}]\), are closed under specification. Take a derivation in DL\([Df, \text{Cut}, \text{Spec}]\), and systematically replace each derivation leading up to the first use of a \( \text{Spec}_n \) rule by transforming that derivation by replacing \( n \) by \( t \) throughout.

Conversely, the \( GDf \) rules are a composition of \( Df \) rules and \( \text{Spec} \), so a DL\([GDf, \text{Cut}]\) derivation can be transformed into a DL\([Df, \text{Cut}, \text{Spec}]\) derivation.
Left/Right Rules for Connectives

\[
\frac{\Gamma, A, B \not\vdash \Delta}{\Gamma, A \land B \not\vdash \Delta} \quad \text{[\land L]}
\]

\[
\frac{\Gamma, A \not\vdash \Delta \quad \Gamma, B \not\vdash \Delta}{\Gamma, A \lor B \not\vdash \Delta} \quad \text{[\lor L]}
\]

\[
\frac{\Gamma, A \not\vdash \Delta \quad \Gamma, B \not\vdash \Delta}{\Gamma, A \supset B \not\vdash \Delta} \quad \text{[\sup L]}
\]

\[
\frac{\Gamma \not\vdash A, \Delta \quad \Gamma, B \not\vdash \Delta}{\Gamma \not\vdash A \lor B, \Delta} \quad \text{[\lor R]}
\]

\[
\frac{\Gamma, A \not\vdash \Delta \quad \Gamma, B \not\vdash \Delta}{\Gamma, A \supset B \not\vdash \Delta} \quad \text{[\sup R]}
\]

\[
\frac{\Gamma \not\vdash A, \Delta}{\Gamma, \neg A \not\vdash \Delta} \quad \text{[\neg L]}
\]

\[
\frac{\Gamma \not\vdash A, \Delta}{\Gamma \not\vdash \neg A, \Delta} \quad \text{[\neg R]}
\]
Left/Right Rules for Quantifiers and Definedness

\[
\begin{align*}
\Gamma, A(t) & \succ \Delta \quad \Gamma \succ t, \Delta \quad [\forall L] \\
\Gamma & \succ (\forall x)A(x), \Delta \\
\Gamma, n, A(n) & \succ \Delta \quad [\exists L] \\
\Gamma & \succ (\exists x)A(x), \Delta \\
\Gamma, t & \succ \Delta \quad [\downarrow L] \\
\Gamma & \succ t\downarrow, \Delta \\
\Gamma, n & \succ A(n), \Delta \\
\Gamma & \succ (\forall x)A(x), \Delta \\
\Gamma, t & \succ A(t), \Delta \\
\Gamma & \succ (\exists x)A(x), \Delta \\
\Gamma & \succ t\downarrow, \Delta \\
\end{align*}
\]
Left/Right Rules for Quantifiers and Definedness

\[
\frac{\Gamma, A(t) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta} [\forall L] \quad \frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} [\forall R]
\]

\[
\frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} [\exists L] \quad \frac{\Gamma \succ t, \Delta}{\Gamma \succ (\exists x)A(x), \Delta} [\exists R]
\]

\[
\frac{\Gamma \succ t, \Delta}{\Gamma \succ t \perp, \Delta} [\perp L] \quad \frac{\Gamma \succ t \perp, \Delta}{\Gamma \succ t \perp, \Delta} [\perp R]
\]

DL[L/R, Cut]
Theorem

A derivation of a sequent $\Gamma \vdash \Delta$ in $\text{DL}[\text{GDf, Cut}]$ can be systematically transformed into a derivation of that sequent in $\text{DL}[\text{L/R, Cut}]$, and vice versa.

Proof.

Using $\text{Cut}$ and $\text{Id}$, each (generalised) defining rule can mimic a Left/Right pair, and vice versa.
These systems satisfy Desideratum #2

\[
\begin{align*}
\frac{(\forall x)A(x) \supset (\forall x)A(x)}{(\forall x)A(x), n \supset A(n)} & \quad [\forall Df] \\
\frac{(\forall x)A(x) \supset (\forall^\prime x)A(x)}{(\forall^\prime x)A(x), n \supset A(n)} & \quad [\forall^\prime Df] \\
\frac{(\forall^\prime x)A(x) \supset (\forall x)A(x)}{(\forall^\prime x)A(x) \supset (\forall x)A(x)} & \quad [\forall Df]
\end{align*}
\]
For Desideratum #1 we eliminate Cut

To show that L/R rules are conservative additions, we eliminate Cut, since the other rules do not introduce new connectives, quantifiers or predicates.

Then, any derivation of a sequent $\Gamma \vdash \Delta$ in a system will use only the rules involving the connectives, quantifiers and predicates in that sequent.
POSITIONS & MODELS
Positions

\[ [\Gamma : \Delta] \]

A pair of sets, $\Gamma$ and $\Delta$ where for no $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ do we have $\Gamma' \rhd \Delta'$.
[\Gamma : \Delta]

A pair of sets, \(\Gamma\) and \(\Delta\) where for no \(\Gamma' \subseteq \Gamma\) and \(\Delta' \subseteq \Delta\) do we have \(\Gamma' \succ \Delta'\).

Here \(\Gamma\) and \(\Delta\) can be infinite, unlike sequents.
[Γ_2 : Δ_2] is a refinement of [Γ_1 : Δ_1] iff Γ_1 ⊆ Γ_2 and Δ_1 ⊆ Δ_2.
Refinement for Conjunction

\[
\frac{\Gamma, A \land B, A, B \not\vdash \Delta \quad [\land Df]}{\Gamma, A \land B, A \land B \not\vdash \Delta \quad [W]}
\]

\[
\frac{\Gamma, A \land B \not\vdash \Delta}{[\land W]}
\]
Refinement for Conjunction

\[
\frac{\Gamma, A \land B, A, B \succ \Delta}{\Gamma, A \land B, A \land B \succ \Delta} \quad [\land Df]
\]

\[
\frac{\Gamma, A \land B \succ \Delta}{\Gamma, A \land B \succ \Delta} \quad [W]
\]

If \([\Gamma, A \land B : \Delta]\) is a position, so is \([\Gamma, A \land B, A, B : \Delta]\).
Refinement for Conjunction

\[
\frac{\Gamma, A \land B, A, B \triangleright \Delta}{\Gamma, A \land B, A \land B \triangleright \Delta} \quad [\land Df]
\]

\[
\frac{\Gamma \triangleright A, A \land B, \Delta \quad \Gamma \triangleright B, A \land B, \Delta}{\Gamma \triangleright A \land B, A \land B, \Delta} \quad [\land R]
\]

\[
\frac{\Gamma \triangleright A \land B, A \land B, \Delta}{\Gamma \triangleright A \land B, \Delta} \quad [W]
\]

If \([\Gamma, A \land B : \Delta]\) is a position, so is \([\Gamma, A \land B, A, B : \Delta]\).
If $[\Gamma, A \land B : \Delta]$ is a position, so is $[\Gamma, A \land B, A, B : \Delta]$.

If $[\Gamma : A \land B, \Delta]$ is a position, so is (at least) one of $[\Gamma : A, A \land B, \Delta]$ and $[\Gamma : B, A \land B, \Delta]$. 
## Refinement for Connectives

<table>
<thead>
<tr>
<th>POSITION</th>
<th>REFINEMENTS</th>
</tr>
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<tbody>
<tr>
<td>$[\Gamma, A \land B : \Delta]$</td>
<td>$[\Gamma, A \land B, A, B : \Delta]$</td>
</tr>
<tr>
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<td>One of $[\Gamma : A, A \land B, \Delta]$ and $[\Gamma : B, A \land B, \Delta]$</td>
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### Refinement for Connectives

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<td>[Γ, A ∧ B : Δ]</td>
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</tr>
<tr>
<td>[Γ, A ∨ B : Δ]</td>
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</tr>
<tr>
<td>[Γ : A ∨ B, Δ]</td>
<td>[Γ : A, B, A ∨ B, Δ]</td>
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<td>--------------------------------------------------</td>
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<tr>
<td>$[\Gamma : A \land B, \Delta]$</td>
<td>One of $[\Gamma : A, A \land B, \Delta]$ and $[\Gamma : B, A \land B, \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma, A \lor B : \Delta]$</td>
<td>One of $[\Gamma, A, A \lor B : \Delta]$ and $[\Gamma, B, A \lor B : \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma : A \lor B, \Delta]$</td>
<td>$[\Gamma : A, B, A \lor B, \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma, A \supset B : \Delta]$</td>
<td>One of $[\Gamma, A \supset B : A, \Delta]$ and $[\Gamma, B, A \supset B : \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma : A \supset B, \Delta]$</td>
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## Refinement for Connectives

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<td>One of $[\Gamma, A \supset B : A, \Delta]$ and $[\Gamma, B, A \supset B : \Delta]$</td>
</tr>
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<td>$[\Gamma : A \supset B, \Delta]$</td>
<td>$[\Gamma, A : B, A \lor B, \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma, \neg A : \Delta]$</td>
<td>$[\Gamma, \neg A : A, \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma : \neg A, \Delta]$</td>
<td>$[\Gamma, A : \neg A, \Delta]$</td>
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</table>
Refinement for Quantifiers, Predicates and Functions

<table>
<thead>
<tr>
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<th>REFINEMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\Gamma, Ft_1 \cdots t_n : \Delta]$</td>
<td>$[\Gamma, Ft_1 \cdots t_n, t_1, \ldots, t_n : \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma, f(t_1, \ldots, t_n) : \Delta]$</td>
<td>$[\Gamma, f(t_1, \ldots, t_n), t_1, \ldots, t_n : \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma, (\forall x)A(x) : \Delta]$</td>
<td>One of $[\Gamma, (\forall x)A(x), A(t) : \Delta], [\Gamma, (\forall x)A(x) : t, \Delta]$ for each term $t$ in $[\Gamma, (\forall x)A(x) : \Delta]$.</td>
</tr>
<tr>
<td>$[\Gamma : (\forall x)A(x), \Delta]$</td>
<td>$[\Gamma, n : A(n), (\forall x)A(x), \Delta]$, for some $n$.</td>
</tr>
<tr>
<td>$[\Gamma, (\exists x)A(x) : \Delta]$</td>
<td>$[\Gamma, (\exists x)A(x), A(n), n : \Delta]$, for some $n$.</td>
</tr>
<tr>
<td>$[\Gamma : (\exists x)A(x), \Delta]$</td>
<td>One of $[\Gamma : A(t), (\exists x)A(x), \Delta], [\Gamma : t, (\exists x)A(x), \Delta]$ for each term $t$ in $[\Gamma : (\exists x)A(x), \Delta]$.</td>
</tr>
<tr>
<td>$[\Gamma, t \downarrow : \Delta]$</td>
<td>$[\Gamma, t \downarrow, t : A, \Delta]$</td>
</tr>
<tr>
<td>$[\Gamma : t \downarrow, \Delta]$</td>
<td>$[\Gamma, A : t, t \downarrow, \Delta]$</td>
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A position \([\Gamma : \Delta]\) is FULLY Refined when it is closed under each of these conditions.
A position $[\Gamma : \Delta]$ is FULLY REFINED when it is closed under each of these conditions.

**Theorem**

Any DL[LR] position $[\Gamma : \Delta]$ is extended by some fully refined position.
A position \([\Gamma : \Delta]\) is **fully refined** when it is closed under each of these conditions.

**Theorem**

*Any DL[LR] position* \([\Gamma : \Delta]\) *is extended by some fully refined position.*

**Proof.**

Use the usual tableaux method.
Models

A model for the logic DL is a structure $\mathcal{M}$ consisting of

1. A domain $D$.
2. An $n$-ary predicate $F$ is interpreted as a subset $F^{\mathcal{M}}$ of $D^n$ (as usual).
3. An $n$-ary function symbol $f$ is interpreted as a partial function $f^{\mathcal{M}} : D^n \rightarrow D$. 
Assigning Values

- $\alpha$ is a (partial) assignment of values to variables.
- $[[x]]_{m,\alpha} = \alpha(x)$
- $[[f(t_1, \ldots, t_n)]]_{m,\alpha} = f^m([[t_1]]_{m,\alpha}, \ldots, [[t_n]]_{m,\alpha})$ if each $[[t_i]]_{m,\alpha}$ is defined, and $f^m$ is defined on the inputs $[[t_1]]_{m,\alpha}, \ldots, [[t_n]]_{m,\alpha}$. 
Interpreting a Language

- $M \models_\alpha t \downarrow$ iff $\llbracket t \rrbracket_{M,\alpha}$ is defined.
- $M \models_\alpha Ft_1 \cdots t_n$ iff for each $i$, the value $\llbracket t_i \rrbracket_{M,\alpha}$ is defined, and the $n$-tuple $\langle \llbracket t_n \rrbracket_{M,\alpha}, \ldots, \llbracket t_1 \rrbracket_{M,\alpha} \rangle \in F^M$.
- $M \models_\alpha A \land B$ iff $M \models_\alpha A$ and $M \models_\alpha B$.
- $M \models_\alpha A \lor B$ iff $M \models_\alpha A$ or $M \models_\alpha B$.
- $M \models_\alpha A \supset B$ iff $M \not\models_\alpha A$ or $M \models_\alpha B$.
- $M \models_\alpha \neg A$ iff $M \not\models_\alpha A$.
- $M \models_\alpha (\forall x)A(x)$ iff $M \models_\alpha[x:=d] A(x)$ for every $d$ in $D$.
- $M \models_\alpha (\exists x)A(x)$ iff $M \models_\alpha[x:=d] A(x)$ for some $d$ in $D$. 

Greg Restall
Generality & Existence I
M is a model of the position \([\Gamma : \Delta]\) iff every sentence in \(\Gamma\) is true in \(M\), every term in \(\Gamma\) is defined in \(M\), every sentence in \(\Delta\) is false in \(M\) and every term in \(\Delta\) is undefined in \(M\).
For any fully refined position $[\Gamma : \Delta]$ the model where

1. the domain $D$ is the set of terms in $\Gamma$
For any fully refined position $[\Gamma : \Delta]$ the model where

1. the domain $D$ is the set of terms in $\Gamma$,
2. the $n$-ary predicate $F$ is interpreted as the set of all $\langle t_1, \ldots, t_n \rangle$ where $Ft_1 \cdots t_n$ is in $\Gamma$
For any fully refined position $[\Gamma : \Delta]$ the model where

1. the domain $D$ is the set of terms in $\Gamma$,
2. the $n$-ary predicate $F$ is interpreted as the set of all $\langle t_1, \ldots, t_n \rangle$ where $F t_1 \cdot \cdot \cdot t_n$ is in $\Gamma$, and
3. the $n$-ary function symbol $f$ is interpreted by setting $f(t_1, \ldots, t_n)$ to be defined iff it is in $\Gamma$, and then it takes itself as its value.
For any fully refined position $[\Gamma : \Delta]$ the model where

(1) the domain $D$ is the set of terms in $\Gamma$,

(2) the $n$-ary predicate $F$ is interpreted as the set of all $\langle t_1, \ldots, t_n \rangle$ where $F t_1 \cdots t_n$ is in $\Gamma$, and

(3) the $n$-ary function symbol $f$ is interpreted by setting $f(t_1, \ldots, t_n)$ to be defined iff it is in $\Gamma$, and then it takes itself as its value

is said to be the model from $[\Gamma : \Delta]$. 

COMPLETENESS & CUT
Theorem

The model from a fully refined position is a model for that position.
Completeness

**Theorem**

The model from a fully refined position is a model for that position.

**Proof.**

Inspect the conditions for satisfaction in a model.
Completeness

**Theorem**

*The model from a fully refined position is a model for that position.*

**Proof.**

Inspect the conditions for satisfaction in a model.

**Corollary**

*Each position has some model.*
Completeness

Theorem

The model from a fully refined position is a model for that position.

Proof.
Inspect the conditions for satisfaction in a model.

Corollary

Each position has some model.

Proof.
Extend \([\Gamma : \Delta]\) into a fully refined position. Take the model from that position. It is a model for \([\Gamma : \Delta]\).
Admissibility of Cut

Theorem

If $\Gamma \vdash \Delta$ is derivable in DL[LR, Cut] then $[\Gamma : \Delta]$ has no model.
Admissibility of Cut

Theorem

If $\Gamma \vdash \Delta$ is derivable in $DL[LR, \text{Cut}]$ then $[\Gamma : \Delta]$ has no model.

Proof.
Induction on the length of the derivation. The special case is $\text{Cut}$: If $[\Gamma : X, \Delta]$ and $[\Gamma, X : \Delta]$ have no model, then neither does $[\Gamma : \Delta]$. 

Admissibility of Cut

Theorem

If $\Gamma \vdash \Delta$ is derivable in $\mathsf{DL}[LR, \text{Cut}]$ then $[\Gamma : \Delta]$ has no model.

Proof.

Induction on the length of the derivation. The special case is $\text{Cut}$: If $[\Gamma : X, \Delta]$ and $[\Gamma, X : \Delta]$ have no model, then neither does $[\Gamma : \Delta]$.

Corollary

If $\Gamma \vdash \Delta$ derivable in $\mathsf{DL}[LR, \text{Cut}]$, it derivable in $\mathsf{DL}[LR]$ too.
Theorem

If $\Gamma \vdash \Delta$ is derivable in $\text{DL}[LR, \text{Cut}]$ then $[\Gamma : \Delta]$ has no model.

Proof.

Induction on the length of the derivation. The special case is $\text{Cut}$: If $[\Gamma : X, \Delta]$ and $[\Gamma, X : \Delta]$ have no model, then neither does $[\Gamma : \Delta]$.

Corollary

If $\Gamma \vdash \Delta$ derivable in $\text{DL}[LR, \text{Cut}]$, it derivable in $\text{DL}[LR]$ too.

Proof.

If $\Gamma \vdash \Delta$ is not derivable in $\text{DL}[LR]$, then $[\Gamma : \Delta]$ has a model. So it is not derivable in $\text{DL}[LR, \text{Cut}]$ either.
CONSEQUENCES & QUESTIONS
Defining Rules, with Generality, give insight into the quantifiers.
If we allow for ‘non-denoting’ terms, defining rules for free logic are straightforward...
How wide is the category of terms?

If we allow for ‘non-denoting’ terms, defining rules for free logic are straightforward...

...and they have a ready interpretation in terms of rules governing our vocabulary without taking models as primary.
These are *also* defining rules:

\[
\frac{\Gamma \vdash A(n), \Delta}{\Gamma \vdash (\Pi x)A(x), \Delta} \quad [\Pi Df] \quad \frac{\Gamma, A(n) \vdash \Delta}{\Gamma, (\Sigma x)A(x) \vdash \Delta} \quad [\Sigma Df]
\]
These are also defining rules:

\[
\frac{\Gamma \vdash A(n), \Delta}{\Gamma \vdash (\Pi x)A(x), \Delta} \quad [\Pi Df] \quad \frac{\Gamma, A(n) \vdash \Delta}{\Gamma, (\Sigma x)A(x) \vdash \Delta} \quad [\Sigma Df]
\]

Are they meaningful?
Wider Quantifiers?

These are also defining rules:

\[
\frac{\Gamma \triangleright A(n), \Delta}{\Gamma \triangleright (\Pi x)A(x), \Delta} \quad [\Pi Df]
\]

\[
\frac{\Gamma, A(n) \triangleright \Delta}{\Gamma, (\Sigma x)A(x) \triangleright \Delta} \quad [\Sigma Df]
\]

Are they meaningful?

\[
\frac{(\Sigma x)\neg x \downarrow \triangleright (\Sigma x)\neg x \downarrow}{\neg n \downarrow \triangleright (\Sigma x)\neg x \downarrow} \quad [\Sigma Df]
\]

\[
\frac{\neg 1/0 \downarrow \triangleright (\Sigma x)\neg x \downarrow}{[Spec^n_{\uparrow/\downarrow}]}
\]
Moving Onward

Modality
Moving Onward

Modality

Identity
THANK YOU!

http://consequently.org/presentation/2015/generality-and-existence-1-arche

@consequently on Twitter