My Aim

To analyse the *quantifiers*
My Aim

To analyse the quantifiers (including their interactions with modals)
My Aim

To analyse the quantifiers (including their interactions with modals) using the tools of proof theory
My Aim

To analyse the quantifiers (including their interactions with modals) using the tools of proof theory in order to better understand quantification, existence and identity.
My Aim for This Talk

Understanding the interactions between quantifiers and modal operators.
Today's Plan

Sequents & Defining Rules

Hypersequents & Defining Rules

Quantification & the Barcan Formula

Positions & Models

Consequences & Questions
SEQUENTS & DEFINING RULES
Don’t assert each element of $\Gamma$ and deny each element of $\Delta$. 
Identity: $A \triangleright A$
Structural Rules

Identity: $A \vdash A$

Weakening:

\[
\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}
\]
**Structural Rules**

**Identity:** \( A \vdash A \)

**Weakening:**
\[
\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}
\]

**Contraction:**
\[
\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}
\]
Structural Rules

**Identity:** \( A \succ A \)

**Weakening:**
\[
\frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta} \quad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}
\]

**Contraction:**
\[
\frac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta} \quad \frac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}
\]

**Cut:**
\[
\frac{\Gamma \succ A, \Delta \quad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}
\]
Structural Rules

Identity: \[ A \vdash A \]

Weakening: \[
\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}
\]

Contraction: \[
\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}
\]

Cut: \[
\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}
\]

Structural rules govern declarative sentences as such.
Giving the Meaning of a Logical Constant

With Left/Right rules?

\[
\frac{\Gamma, A, B \triangleright \Delta}{\Gamma, A \land B \triangleright \Delta} \quad [\land L] \\
\frac{\Gamma \triangleright A, \Delta \quad \Gamma \triangleright B, \Delta}{\Gamma \triangleright A \land B, \Delta} \quad [\land R]
\]
Giving the Meaning of a Logical Constant

With Left/Right rules?

\[
\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta} \quad \text{[\&L]}
\]

\[
\frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \land B, \Delta} \quad \text{[\&R]}
\]

\[
\frac{\Gamma, B \succ \Delta}{\Gamma, A \ tonk B \succ \Delta} \quad \text{[tonkL]}
\]

\[
\frac{\Gamma \succ A, \Delta}{\Gamma \succ A \ tonk B, \Delta} \quad \text{[tonkR]}
\]
What is involved in going from $\mathcal{L}$ to $\mathcal{L}'$?

Use $\succ \mathcal{L}$ to define $\succ \mathcal{L}'$. 
What is involved in going from $\mathcal{L}$ to $\mathcal{L}'$?

Use $\models_\mathcal{L}$ to define $\models_{\mathcal{L}'}$.

Desideratum #1: $\models_{\mathcal{L}'}$ is conservative: $(\models_{\mathcal{L}'})|_{\mathcal{L}}$ is $\models_{\mathcal{L}}$. 
What is involved in going from $\mathcal{L}$ to $\mathcal{L}'$?

Use $\not\in_{\mathcal{L}}$ to define $\not\in_{\mathcal{L}'}$.

**Desideratum #1:** $\not\in_{\mathcal{L}'}$ is conservative: $(\not\in_{\mathcal{L}'})|_{\mathcal{L}}$ is $\not\in_{\mathcal{L}}$.

**Desideratum #2:** Concepts are defined uniquely.
A Defining Rule

\[
\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta} \quad [\land \text{Df}]
\]
A Defining Rule

\[ \frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta} \] \[ \land Df \]

Fully specifies norms governing conjunctions on the left in terms of simpler vocabulary.
A Defining Rule

\[
\Gamma, A, B \succ \Delta \\
\frac{\Gamma, A \land B \succ \Delta}{\Gamma, A \land B \succ \Delta} \quad [\land Df]
\]

Fully specifies norms governing conjunctions on the \textit{left} in terms of simpler vocabulary.

\textit{Identity} and \textit{Cut} determine the behaviour of conjunctions on the \textit{right}.
From $\land Df$ to $\land L/R$

\[
\begin{align*}
\Gamma \succ A, \Delta & \quad \Gamma, A \succ A \land B, \Delta \\
\Gamma \succ B, \Delta & \quad A, B \succ A \land B \\
\end{align*}
\]

$\land Df$  \quad [\text{Cut}]

\[
A \land B \succ A \land B
\]

[Id]

\[
\begin{align*}
\Gamma \succ & A \land B, \Delta \\
\end{align*}
\]

$\land L/R$  \quad [\text{Cut}]
From $[\land Df]$ to $[\land L/R]$

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \rightharpoonup \quad \Gamma \vdash A \land B, \Delta \\
A, B \vdash A \land B, \Delta & \quad [\land Df] \\
\Gamma \vdash B, \Delta & \quad [Id] \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \rightharpoonup \quad \Gamma, A \vdash A \land B, \Delta \\
A \land B \vdash A \land B, \Delta & \quad [Cut] \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \rightharpoonup \quad \Gamma \vdash A \land B, \Delta \\
\end{align*}
\]

[Cut]
From $[\land Df]$ to $[\land L/R]$

\[
\Gamma \vdash A, \Delta \\
\frac{A \land B \vdash A \land B}{[\land Df]} \\
\frac{\Gamma \vdash B, \Delta}{A, B \vdash A \land B} \\
\frac{\Gamma \vdash A \land B, \Delta}{[\land Df]} \\
\frac{\Gamma \vdash A, \Delta}{[Cut]} \\
\frac{\Gamma \vdash A \land B, \Delta}{[Cut]}
\]

[Greg Restall]
Generality & Existence II
11 of 60
From \(\land Df\) to \(\land L/R\)

\[
\frac{\Gamma \not\vDash B, \Delta}{\Gamma \vDash A, \Delta} \quad \frac{\Gamma \not\vDash A, \Delta}{\Gamma, A \not\vDash A \land B, \Delta} \quad [\text{Cut}]
\]

\[
\frac{\Gamma \not\vDash B, \Delta}{\Gamma \not\vDash A \land B, \Delta} \quad \frac{\Gamma \not\vDash A \land B, \Delta}{A \land B \not\vDash A \land B} \quad [\text{Id}]
\]

\[
\frac{\Gamma \not\vDash A \land B, \Delta}{\Gamma, A \not\vDash A \land B, \Delta} \quad [\text{Cut}]
\]
From $[\land Df]$ to $[\land L/R]$

\[
\frac{A \land B \succ A \land B}{\Gamma \succ B, \Delta} \quad \frac{A, B \succ A \land B}{\Gamma \succ A \land B, \Delta} \quad [\land Df]
\]

\[
\frac{\Gamma \succ A, \Delta}{Gamma \succ A \land B, \Delta} \quad [\land L/R]
\]

\[
\frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \land B, \Delta} \quad [Cut]
\]

\[
\frac{\Gamma \succ A, \Delta \quad \Gamma \succ A \land B, \Delta}{\Gamma \succ A \land B, \Delta} \quad [Cut]
\]
\[ \frac{A \entails A \quad B \entails B}{A, B \entails A \land B}[^{\land R}] \quad \frac{\Gamma, A \land B \entails \Delta}{\Gamma, A, B \entails \Delta}[^{Cut}] \]
This works for more than the classical logical constants

I want to see how this works for modal operators, and examine their interaction with the quantifiers.
Explaining *why* the modal operators have the logical properties they exhibit is an open question.
... possible worlds, in the sense of possible states of affairs are not *really* individuals (just as numbers are not *really* individuals).

To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case ‘in’ a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i.e. if something else were the case ...

We understand ‘truth in states of affairs’ because we understand ‘necessarily’; not *vice versa*.

— “Worlds, Times and Selves” (1969)
I agree with Prior...

... but a Priorean about possibility and worlds must address these issues:
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... but a Priorean about possibility and worlds must address these issues:

▶ Why is it that modal concepts (which are conceptually prior to worlds) have a structure that fits possible worlds models?
I agree with Prior...

... but a Priorean about possibility and worlds must address these issues:

- Why is it that modal concepts (which are conceptually prior to worlds) have a structure that fits possible worlds models?

- (Why does possibility distribute of disjunction, necessity over disjunction? Why do the modalities work like normal modal logics?)
... but a Priorean about possibility and worlds must address these issues:

- Why is it that modal concepts (which are conceptually prior to worlds) have a structure that fits possible worlds models?
- (Why does possibility distribute of disjunction, necessity over disjunction? Why do the modalities work like normal modal logics?)
- If modality is primitive we have no explanation.
... but a Priorean about possibility and worlds must address these issues:

- Why is it that modal concepts (which are conceptually prior to worlds) have a structure that fits possible worlds models?

- (Why does possibility distribute of disjunction, necessity over disjunction? Why do the modalities *work* like normal modal logics?)

- If modality is *primitive* we have no explanation.

- If modality is governed by the rules introduced here, then we can see why possible worlds are useful, and model the behaviour of modal concepts.
HYPERSEQUENTS & DEFINING RULES
Suppose it’s necessary that p and necessary that q.
Suppose it’s necessary that \( p \) and necessary that \( q \). Is it necessary that both \( p \) and \( q \)?
Modal Reasoning involves *Shifts*

Suppose it’s necessary that $p$ and necessary that $q$. Is it necessary that both $p$ and $q$?

Could we *avoid* $p$ and $q$?
Modal Reasoning involves *Shifts*

Suppose it’s necessary that $p$ and necessary that $q$. Is it necessary that both $p$ and $q$?

*Could we* avoid $p$ and $q$?

Consider any way it could go:
Modal Reasoning involves *Shifts*

Suppose it’s necessary that $p$ and necessary that $q$. Is it necessary that both $p$ and $q$?

*Could we avoid* $p$ and $q$?

Consider any way it could go:

Since it’s necessary that $p$, here we have $p$. Since it’s necessary that $q$, here we have $q$. So, we have both $p$ and $q$. So, no matter how things go, we have $p$ and $q$. So the conjunction $p$ and $q$ is necessary.
Suppose it’s necessary that $p$ and necessary that $q$. Is it necessary that both $p$ and $q$?

Could we *avoid* $p$ and $q$?

Consider any way it could go:
Since it’s necessary that $p$, here we have $p$.
Since it’s necessary that $q$, here we have $q$.
Suppose it’s necessary that p and necessary that q. Is it necessary that both p and q?

Could we *avoid* p and q?

Consider any way it could go:
Since it’s necessary that p, here we have p.
Since it’s necessary that q, here we have q.

So, we have both p and q.
Modal Reasoning involves *Shifts*

Suppose it’s necessary that $p$ and necessary that $q$.
Is it necessary that both $p$ and $q$?

Could we *avoid* $p$ and $q$?

Consider any way it could go:
Since it’s necessary that $p$, here we have $p$.
Since it’s necessary that $q$, here we have $q$.

So, we have both $p$ and $q$.

So, no matter how things go, we have $p$ and $q$.
Modal Reasoning involves *Shifts*

Suppose it’s necessary that $p$ and necessary that $q$. Is it necessary that both $p$ and $q$?

Could we *avoid* $p$ and $q$?

Consider any way it could go:
Since it’s necessary that $p$, here we have $p$.
Since it’s necessary that $q$, here we have $q$.
So, we have both $p$ and $q$.
So, no matter how things go, we have $p$ and $q$.
So the conjunction $p$ and $q$ is necessary.
Exposing the Structure of that Deduction

\[
\begin{align*}
\Box p, \Box q \rightarrow \Box p & \quad [\Box Df] \\
\Box p, \Box q \rightarrow & \quad [\Box Df] \\
\Box p, \Box q \rightarrow \Box (p \land q) & \quad [\Box Df] \\
\Box p \land \Box q \rightarrow \Box (p \land q) & \quad [\Box Df]
\end{align*}
\]
Exposing the Structure of that Deduction

\[
\begin{align*}
\square p, \square q & \rightarrow \square p & \quad \square p, \square q & \rightarrow \square q \\
\square p, \square q & \rightarrow p & \quad \square p, \square q & \rightarrow q \\
\square p, \square q & \rightarrow p \land q & \quad \square p, \square q & \rightarrow \square (p \land q) \\
\square p \land \square q & \rightarrow \square (p \land q)
\end{align*}
\]
Exposing the Structure of that Deduction

\[
\begin{align*}
\Box p, \Box q & \rightarrow \Box p \quad [\Box Df] \\
\Box p, \Box q & \rightarrow \Box q \quad [\Box Df] \\
\Box p, \Box q & \rightarrow \Box (p \land q) \quad [\Box Df] \\
\Box p \land \Box q & \rightarrow \Box (p \land q) \quad [\land Df] \\
\end{align*}
\]
Exposing the Structure of that Deduction

\[\begin{align*}
\Box p, \Box q & \rightarrow \Box p & \Box p, \Box q & \rightarrow \Box q \\
\Box p, \Box q & \rightarrow p & \Box p, \Box q & \rightarrow q \\
\Box p, \Box q & \rightarrow p \land q & \Box p, \Box q & \rightarrow \Box (p \land q) \\
\Box p \land \Box q & \rightarrow \Box (p \land q)
\end{align*}\]
Exposing the Structure of that Deduction

\[
\begin{align*}
\allowdisplaybreaks
\Box p, \Box q & \vdash \Box p \\
\Box p, \Box q & \vdash \Box p & [\Box Df] \\
\Box p & \vdash \Box (p \land q) & [\Box Df] \\
\Box p \land \Box q & \vdash \Box (p \land q) & [\Box Df]
\end{align*}
\]
Exposing the Structure of that Deduction

\[
\begin{align*}
\Box p, \Box q & \vdash \Box p & [\Box Df] \\
\Box p, \Box q & \vdash \Box q & [\Box Df] \\
\Box p, \Box q & \vdash p & [\Box Df] \\
\Box p, \Box q & \vdash q & [\Box Df] \\
\Box p, \Box q & \vdash p \land q & [\land R] \\
\Box p, \Box q & \vdash (p \land q) & [\Box Df] \\
\Box (p \land q) & \vdash (p \land q) & [\Box Df] \\
\Box p \land \Box q & \vdash (p \land q) & [\Box Df]
\end{align*}
\]
Exposing the Structure of that Deduction

\[
\begin{align*}
\Box p, \Box q & \rightarrow \Box p & \Box p, \Box q & \rightarrow \Box q \\
\Box p, \Box q & \rightarrow p & \Box p, \Box q & \rightarrow q \\
\Box p, \Box q & \rightarrow p \land q & \Box p, \Box q & \rightarrow \Box (p \land q) \\
\Box p \land \Box q & \rightarrow \Box (p \land q)
\end{align*}
\]
Exposing the Structure of that Deduction

\[
\begin{align*}
\Box p, \Box q & \rightarrow \Box p \\
\Box p, \Box q & \rightarrow \Box p & \Box p, \Box q & \rightarrow \Box q
\end{align*}
\]

\[\Box Df\]

\[
\begin{align*}
\Box p, \Box q & \rightarrow \Box p \\
\Box p, \Box q & \rightarrow \Box q & \Box p, \Box q & \rightarrow \Box (p \land q)
\end{align*}
\]

\[\Box Df\]

\[
\begin{align*}
\Box p, \Box q & \rightarrow \Box p \land q \\
\Box p, \Box q & \rightarrow \Box (p \land q)
\end{align*}
\]

\[\Box Df\]

\[
\begin{align*}
\Box p \land \Box q & \rightarrow \Box (p \land q)
\end{align*}
\]

\[\Box Df\]

\[\wedge R\]
Don’t assert $\Box p$ and $\Box q$ in one ‘zone’ and deny $p \land q$ in another.
Don’t assert each member of $\Gamma$ and deny each member of $\Delta$ in one ‘zone’ and assert each member of $\Gamma'$ and deny each member of $\Delta'$ in another.
Two Kinds of Zone Shift

**INDICATIVE:** suppose I’m wrong and that...

**SUBJUNCTIVE:** suppose things go differently.

or *had gone* differently.
Two Kinds of Zone Shift
Two Kinds of Zone Shift

▶ Suppose Oswald didn’t shoot JFK.

The New York Times, November 22, 1963:

KENNEDY IS KILLED BY SNIPER AS HE RIDES IN CAR IN DALLAS; JOHNSON SWORN IN ON PLANE

Gov. Connally Shot; Mrs. Kennedy Safe

Why America Weeps

TULSA FINISH UNITY

President J. F. Kennedy was killed by a sniper as he rode in a motorcade in Dallas. His vice president, Lyndon B. Johnson, has been sworn in as President.

Johnson is the first southern President since Reconstruction.

President Johnson was sworn in as President in the State Capitol in Dallas, where Kennedy fell.

Why America Weeps

Kennedy was shot in the head by a man who was standing on the grass and fired two shots.

The man was dressed in a raincoat and had a scarf over his face. He was later identified as Lee Harvey Oswald.

Johnson was sworn in as President as the nation mourned the loss of its beloved leader.

Johnson has called for a moment of silence to remember Kennedy.

The nation is in deep mourning.

Two Kinds of Zone Shift

- Suppose Oswald *didn’t* shoot JFK.
- Suppose Oswald *hadn’t* shot JFK.
STEREOSCOPIC VISION:
Persons, Freedom, and Two Spaces of Material Inference

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<www.philosophersimprint.org/007004/>

What is a person, as opposed to a non-person? One might begin to address the question by appealing to a second distinction: between agents, characterized by the ability to act freely and intentionally, and mere patients, caught up in events but in no sense authors of the happenings involving them. An alternative way to address the question appeals to a third distinction: between subjects — bearers of rights and responsibilities, commitments and entitlements, makers of claims, thinkers of thoughts, issuers of orders, and posers of questions — and mere objects, graspable or evaluable by subjects but not themselves graspers or evaluators.

We take it as a methodological point of departure that these three distinctions are largely coextensive, indeed coextensive in conceptually central cases. Granted, these distinctions can come apart. One might think that ‘person’ applies to anything that is worthy of a distinctive sort of moral respect and think this applicable to some fetuses or the deeply infirm elderly. Even if the particular respect due such beings is importantly different from “what we owe each other”, such respect could still be thought to be of the kind distinctively due people, and think this even while holding that such people lack agentive or subjective capacity. Similarly, one might think dogs or various severely impaired humans to be attenuated subjects but not agents.

Without taking any particular stand on such examples, our methodological hypothesis is that such cases, if they exist, are understood as persons (agents, subjects) essentially by reference to paradigm cases and, indeed, to a single paradigm within which person/non-person, subject/object, and agent/patient are conceptually connected.1 Stated
We are social creatures, who act on the basis of views

- **DISAGREEMENT**: We disagree. We have reason to come to shared positions.
We are social creatures, who act on the basis of views.

- **DISAGREEMENT**: We disagree. We have reason to come to shared positions.

- **PLANNING**: We plan. We have reason to consider options (prospectively) or replay scenarios (retrospectively).
We are *social* creatures, who *act* on the basis of *views*

- **DISAGREEMENT**: We disagree. We have reason to come to shared positions.

- **PLANNING**: We *plan*. We have reason to consider options (prospectively) or replay scenarios (retrospectively).

- We do *many different and strange things* in our messy zone-shifting practices, but we can isolate a particular convention or practice, idealise it, to see what we could do following *those* rules.
We are social creatures, who act on the basis of views

- **DISAGREEMENT**: We disagree. We have reason to come to shared positions.

- **PLANNING**: We plan. We have reason to consider options (prospectively) or replay scenarios (retrospectively).

- We do many different and strange things in our messy zone-shifting practices, but we can isolate a particular convention or practice, idealise it, to see what we could do following those rules.

  - (Analogy: ∀x from first order logic and natural language’s ‘all.’ Frictionless planes. etc.)
Oswald *did* shoot JFK, but suppose he *hadn’t*? How would history have gone differently then?
Oswald *did* shoot JFK, but suppose he *hadn’t*? How would history have gone differently then?

\[ [\text{oS}k : @] | [\text{@oS}k : \text{oSk}] \]
Oswald *did* shoot JFK, but suppose he *hadn’t*? How would history have gone differently then?

\[
[\text{oSk} : ~ @ ~ | ~ [\text{@oSk} : \text{oSk}]
\]

We open up a zone for consideration, in which we deny oSk, while keeping track of the initial zone where we assert it.

(And if we like, we can assert @oSk in the zone under the counterfactual supposition.)
I think that Oswald shot JFK, but you don’t.

I consider what it would mean for you to be right.

If you’re right, Oswald *actually didn’t* shoot JFK.
I think that Oswald shot JFK, but you don’t.

I consider what it would mean for you to be right.

If you’re right, Oswald actually didn’t shoot JFK.

*Epistemic* alternatives interact differently with actuality.
I think that Oswald shot JFK, but you don’t.

I consider what it would mean for you to be right.

If you’re right, Oswald actually didn’t shoot JFK.

*Epistemic alternatives interact differently with actuality.*

\[ [oSk : ] @ \quad \rightarrow \quad [ \ : oSk ] @ \]
I think that Hesperus is Phosphorous, but I recognise that you don’t. I don’t take you to be inconsistent or misusing names.
I think that Hesperus is Phosphorous, but I recognise that you don’t. I don’t take you to be inconsistent or misusing names.

We don’t have this:

\[ \alpha = \beta \not\implies \Phi \alpha \not\implies \Phi \beta \]

It’s coherent for you to assert \( \Phi \alpha \) and deny \( \Phi \beta \) even if I take it that \( \alpha = \beta \), and it’s coherent for me to consider an alternative in which \( \alpha \neq \beta \) even if I don’t agree.
Let’s take any zone found by indicatively shifting from here (or from anywhere from here, etc.) to be an alternative indicative context from any other context indicatively shifted from here.
Let’s take any zone found by indicatively shifting from here (or from anywhere from here, etc.) to be an alternative indicative context from any other context indicatively shifted from here.

(And each are actual zones.)
Let's take any zone found by indicatively shifting from here (or from anywhere from here, etc.) to be an alternative indicative context from any other context indicatively shifted from here.

(And each are actual zones.)

This is as liberal as possible about what counts as an alternative from any alternative zone.
Let’s take any zone found by indicatively shifting from here (or from anywhere from here, etc.) to be an alternative indicative context from any other context indicatively shifted from here.

(And each are actual zones.)

This is as liberal as possible about what counts as an alternative from any alternative zone.

This gives us a motivation for a richer family of hypersequents.
Two Dimensional Hypersequents

\[ X^1_1 \vdash Y^1_1 \ | \ X^1_2 \vdash Y^1_2 \ | \ \cdots \ | \ X^1_{m_1} \vdash Y^1_{m_1} \ ||
\]
\[ X^2_1 \vdash Y^2_1 \ | \ X^2_2 \vdash Y^2_2 \ | \ \cdots \ | \ X^2_{m_2} \vdash Y^2_{m_2} \ ||
\]
\[ \vdots \ | \ \vdots \ \vdots \ |
\]
\[ X^n_1 \vdash Y^n_1 \ | \ X^n_2 \vdash Y^n_2 \ | \ \cdots \ | \ X^n_{m_n} \vdash Y^n_{m_n} \]
Two Dimensional Hypersequents

\[
X_1^1 \succ @ Y_1^1 \mid X_2^1 \succ Y_2^1 \mid \cdots \mid X_{m_1}^1 \succ Y_{m_1}^1 \parallel \\
X_1^2 \succ @ Y_1^2 \mid X_2^2 \succ Y_2^2 \mid \cdots \mid X_{m_2}^2 \succ Y_{m_2}^2 \parallel \\
\vdots \mid \vdots \mid \vdots \\
X_1^n \succ @ Y_1^n \mid X_2^n \succ Y_2^n \mid \cdots \mid X_{m_n}^n \succ Y_{m_n}^n
\]

Think of these as scorecards, keeping track of assertions and denials.
\[ H[\Gamma > \Delta] \]
Notation

$\mathcal{H}[\Gamma' \supset \Delta']$
\[ \mathcal{H}[\Gamma' \succ \Delta'] \]

\[ \mathcal{H}[\Gamma \succ \Delta \mid \Gamma' \succ \Delta'] \]
\[ \mathcal{H}[\Gamma' \succ \Delta'] \]

\[ \mathcal{H}[\Gamma \succ \Delta \mid \Gamma' \succ \Delta'] \]

\[ \mathcal{H}[\Gamma \succ \Delta \parallel \Gamma' \succ \Delta'] \]
Defining Rule for □

\[
\frac{\mathcal{H}[\Gamma \vdash \Delta \mid \vdash A]}{\mathcal{H}[\Gamma \vdash \Box A, \Delta]} [\Box Df]
\]
Defining Rule for $\@$

\[
\frac{\mathcal{H}[\Gamma, A \vdash \Delta \mid \Gamma' \vdash \Delta']}{\mathcal{H}[\Gamma \vdash \Delta \mid \Gamma', \@A \vdash \Delta']} \quad [\@Df]
\]
Defining Rule for \([e]\)

\[
\frac{\mathcal{H}[\Gamma \triangleright \Delta \parallel \triangleright @ A]}{\mathcal{H}[\Gamma \triangleright [e]A, \Delta]} \quad [[e]Df]
\]
Example Derivation

\[
\begin{align*}
\Gamma & \vdash [e]A \supset [e]A & \text{[[e]Df]} \\
\Gamma & \vdash [e]A \supset \top & \vdash \top & \vdash @A & \text{[@Df]} \\
\Gamma & \vdash [e]A \supset \top & \vdash @A & \vdash [e]@A & \text{[[e]Df]} \\
\Gamma & \vdash \top & \vdash [e]A \supset [e]@A & \text{[[e]Df]} \\
\Gamma & \vdash [e]A \supset [e]@A & \vdash [e]A \supset [e]@A & \vdash [[e]A \supset [e]@A] & \text{[[e]Df]} \\
\end{align*}
\]
QUANTIFICATION & THE BARCAN FORMULA
The Standard Quantifier Rules

\[
\begin{align*}
\Gamma \vdash A(n), \Delta & \quad \Rightarrow \quad \Gamma \vdash (\forall x)A(x), \Delta \quad [\forall Df] \\
\Gamma, A(n) \vdash \Delta & \quad \Rightarrow \quad \Gamma, (\exists x)A(x) \vdash \Delta \quad [\exists Df]
\end{align*}
\]
Deriving the Barcan Formula

\[
\begin{align*}
(\forall x)\Box Fx &\Rightarrow (\forall x)\Box Fx \\
(\forall x)\Box Fx &\Rightarrow \Box F n \\
(\forall x)\Box F x &\Rightarrow \Box (\forall x) F x \\
\Rightarrow (\forall x)\Box F x &\subseteq \Box (\forall x) F x
\end{align*}
\]
Where the derivation breaks down

\[
(\forall x) \Box Fx \triangleright (\forall x) \Box Fx
\]

\[
\frac{(\forall x) \Box Fx \triangleright \Box F_n}{\Box F_n} [\Box Df]
\]

\[
(\forall x) \Box Fx \triangleright F_n
\]

\[
\frac{(\forall x) \Box Fx \triangleright F_n}{(\forall x) \Box Fx \triangleright (\forall x) Fx} [\forall Df]
\]

\[
(\forall x) \Box Fx \triangleright (\forall x) Fx
\]

\[
\frac{(\forall x) \Box Fx \triangleright (\forall x) Fx}{\Box (\forall x) F} [\Box Df]
\]

\[
\triangleright (\forall x) \Box Fx \sqsubseteq \Box (\forall x) Fx
\]

\[
(\forall x) \Box Fx \triangleright (\forall x) \Box Fx
\]
Where the derivation breaks down

\[
(\forall x) \square Fx \supset (\forall x) \square Fx \\
\frac{(\forall x) \square Fx \supset \square Fn}{(\forall x) \square Fx \supset \square Fn} \quad [\square Df] \\
\frac{(\forall x) \square Fx \supset \supset \supset Fn}{(\forall x) \square Fx \supset \supset (\forall x) Fx} \quad [\forall Df] \\
\frac{(\forall x) \square Fx \supset \square (\forall x) Fx}{\supset (\forall x) \square Fx \supset \square (\forall x) Fx} \quad [\supset Df]
\]
Where the derivation breaks down

\[
\begin{align*}
(\forall x) \square \neg Fx & > (\forall x) \square Fx \\
\hline
(\forall x) \square Fx & > \square \neg Fx & [\square Df]
\end{align*}
\]

\[
\begin{align*}
(\forall x) \square Fx & > \neg Fx \\
\hline
(\forall x) \square Fx & > (\forall x) Fx & [\forall Df]
\end{align*}
\]

\[
\begin{align*}
(\forall x) \square Fx & > (\forall x) Fx \\
\hline
(\forall x) \square Fx & > \square (\forall x) Fx & [\forall Df]
\end{align*}
\]

\[
\begin{align*}
(\forall x) \square Fx & > (\forall x) Fx \\
\hline
(\forall x) \square Fx & > (\forall x) Fx & [\forall Df]
\end{align*}
\]

\[
\begin{align*}
(\forall x) \square Fx & > (\forall x) Fx \\
\hline
(\forall x) \square Fx & > (\forall x) Fx & [\forall Df]
\end{align*}
\]
To rule a term *in* is to take it as suitable to substitute into a quantifier, i.e., to take the term to *denote*.

To rule a term *out* is to take it as unsuitable to substitute into a quantifier, i.e., to take the term to *not denote*.
Pro and Con attitudes to Terms

To rule a term \textit{in} is to take it as suitable to substitute into a quantifier, i.e., to take the term to \textit{denote}.

To rule a term \textit{out} is to take it as unsuitable to substitute into a quantifier, i.e., to take the term to \textit{not denote}.

We add terms to the LHS and RHS of sequents $\Gamma \rightarrow \Delta$. 
Structural Rules remain as before

**Identity:** \( \text{X} \succ \text{X} \)

**Weakening:**

\[
\frac{\mathcal{H}[\Gamma \succ \Delta]}{\mathcal{H}[\Gamma, \text{X} \succ \Delta]} \quad \frac{\mathcal{H}[\Gamma \succ \Delta]}{\mathcal{H}[\Gamma \succ \text{X}, \Delta]}
\]

**Contraction:**

\[
\frac{\mathcal{H}[\Gamma, \text{X}, \text{X} \succ \Delta]}{\mathcal{H}[\Gamma, \text{X} \succ \Delta]} \quad \frac{\mathcal{H}[\Gamma \succ \text{X}, \text{X}, \Delta]}{\mathcal{H}[\Gamma \succ \text{X}, \Delta]}
\]

**Cut:**

\[
\frac{\mathcal{H}[\Gamma \succ \text{X}, \Delta]}{\mathcal{H}[\Gamma, \text{X} \succ \Delta]} \quad \frac{\mathcal{H}[\Gamma, \text{X} \succ \Delta]}{\mathcal{H}[\Gamma \succ \Delta]}
\]

Here \( \text{X} \) is either a sentence or a term.
...and there are some more

**Ext. Weak.:**

\[
\frac{\mathcal{H}[\Gamma > \Delta]}{\mathcal{H}[\Gamma > \Delta \mid \Gamma' > \Delta']} \quad \frac{\mathcal{H}[\Gamma > \Delta]}{\mathcal{H}[\Gamma > \Delta \parallel \Gamma' > \Delta']}
\]

**Ext. Contr.:**

\[
\frac{\mathcal{H}[\Gamma > \Delta \mid \Gamma > \Delta]}{\mathcal{H}[\Gamma > \Delta]} \quad \frac{\mathcal{H}[S \parallel S]}{\mathcal{H}[S]}
\]
Quantifier Rules, allowing for non-denoting terms

\[
\frac{\mathcal{H}[\Gamma, n \vdash A(n), \Delta]}{\mathcal{H}[\Gamma \vdash (\forall x)A(x), \Delta]} \quad [\forall Df]
\]

\[
\frac{\mathcal{H}[\Gamma, n \vdash A(n), \Delta]}{\mathcal{H}[\Gamma \vdash (\exists x)A(x), \Delta]} \quad [\exists Df]
\]
Now you can't derive the Barcan Formula

$$(\forall x)\Box Fx \to \Box(\forall x)Fx$$
Now you *can't* derive the Barcan Formula

\[(\forall x)\Box Fx \rightarrow \Box(\forall x)Fx \quad \text{and} \quad \rightarrow (\forall x)Fx\]
Now you *can't* derive the Barcan Formula

\[(\forall x)\square Fx \rightarrow \square (\forall x)Fx \mid b \rightarrow Fb, (\forall x)Fx\]
Now you can't derive the Barcan Formula

\((\forall x) \Box Fx \rightarrow b, Fb, \Box (\forall x) Fx \mid b \rightarrow Fb, (\forall x) Fx\)
Now you *can't* derive the Barcan Formula

\[ a, (\forall x)\Box Fx \rightarrow b, Fb, \Box (\forall x)Fx \mid a, b \rightarrow Fb, (\forall x)Fx \]
Now you *can't* derive the Barcan Formula

\[ a, \Box Fa, (\forall x)\Box Fx \models b, Fb, \Box (\forall x)Fx \ | \ a, b \models Fb, (\forall x)Fx \]
Now you can't derive the Barcan Formula

\[ a, F_a, \Box F_a, (\forall x)\Box F_x \rightarrow b, F_b, \Box (\forall x)F_x \mid a, b, F_a \rightarrow F_b, (\forall x)F_x \]
Now you can't derive the Barcan Formula

\[ a, Fa, \Box Fa, (\forall x)\Box Fx \rightarrow b, Fb, \Box (\forall x)Fx \] \[ a, b, Fa \rightarrow Fb, (\forall x)Fx \]
Now you *can't* derive the Barcan Formula

\[
\begin{align*}
a, Fa, □Fa, (∀x)□Fx & \succ b, Fb, □ (∀x)Fx & | & a, b, Fa \succ Fb, (∀x)Fx \\
\end{align*}
\]

This hypersequent is underivable...
Now you can't derive the Barcan Formula

\[ a, Fa, \Box Fa, (\forall x) \Box Fx \Rightarrow b, Fb, \Box (\forall x) Fx \mid a, b, Fa \Rightarrow Fb, (\forall x) Fx \]

This hypersequent is underivable...

...and it's fully refined.
*Epistemic Barcan Formula*

\[(\forall x)[e]Fx \supset [e](\forall x)Fx\]
**Epistemic Barcan Formula**

\[
(\forall x)[e]Fx \succ [e](\forall x)Fx \\
\langle e\rangle(\exists x)Fx \succ (\exists x)\langle e\rangle Fx
\]
Suppose $\langle e \rangle (\exists x)(\exists y)(Mx \& Ey \& x \neq y)$.
Suppose $\langle e \rangle (\exists x)(\exists y)(Mx \& Ey \& x \neq y)$.

Do we have $(\exists x)\langle e \rangle ((\exists y)(Mx \& Ey \& x \neq y))$?
Suppose $\langle e \rangle (\exists x) (\exists y) (Mx \& Ey \& x \neq y)$.

Do we have $(\exists x) \langle e \rangle ((\exists y) (Mx \& Ey \& x \neq y))$?

And $(\exists x) (\exists y) \langle e \rangle (Mx \& Ey \& x \neq y)$?
Suppose $\langle e \rangle (\exists x)(\exists y) (Mx \& Ey \& x \neq y)$.

Do we have $(\exists x)\langle e \rangle ((\exists y) (Mx \& Ey \& x \neq y))$?

And $(\exists x)(\exists y) \langle e \rangle (Mx \& Ey \& x \neq y)$?

What could such $x$ and $y$ be?
POSITIONS & MODELS
A finite position is an underivable hypersequent.
A finite position is an underivable hypersequent.

An arbitrary position is a set (indicative alternatives) of sets (subjunctive alternatives) of pairs of sets of formulas or terms (components), where one component in each indicative alternative is marked with an @.
A position *fully refined* if it is closed downward under the evaluation conditions for the connectives and modal operators.
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For example:

- If $A \land B$ is in the LHS of a component, so are $A$ and $B$. 

A position *fully refined* if it is closed downward under the evaluation conditions for the connectives and modal operators.

For example:
- If $A \land B$ is in the LHS of a component, so are $A$ and $B$.
- If $A \land B$ is in the RHS of a component, so is one of $A$ and $B$. 
A position **fully refined** if it is closed downward under the evaluation conditions for the connectives and modal operators.

For example:

- If $A \land B$ is in the LHS of a component, so are $A$ and $B$.
- If $A \land B$ is in the RHS of a component, so is one of $A$ and $B$.
- If $(\forall x)A(x)$ is in the LHS of a component, so is $A(t)$ for every term $t$ in the LHS of that component.
A position *fully refined* if it is closed downward under the evaluation conditions for the connectives and modal operators.

For example:

- If $A \land B$ is in the LHS of a component, so are $A$ and $B$.
- If $A \land B$ is in the RHS of a component, so is one of $A$ and $B$.
- If $(\forall x)A(x)$ is in the LHS of a component, so is $A(t)$ for every term $t$ in the LHS of that component.
- If $(\forall x)A(x)$ is in the RHS of a component, so is $A(t)$ for some term $t$ in the LHS of that component.
A position fully refined if it is closed downward under the evaluation conditions for the connectives and modal operators.

For example:

- If $A \land B$ is in the LHS of a component, so are $A$ and $B$.
- If $A \land B$ is in the RHS of a component, so is one of $A$ and $B$.
- If $(\forall x)A(x)$ is in the LHS of a component, so is $A(t)$ for every term $t$ in the LHS of that component.
- If $(\forall x)A(x)$ is in the RHS of a component, so is $A(t)$ for some term $t$ in the LHS of that component.
- If $\Box A$ is in the LHS of a component, $A$ is in the LHS of every subjunctive alternative of that component.
A position fully refined if it is closed downward under the evaluation conditions for the connectives and modal operators.

For example:

- If $A \land B$ is in the LHS of a component, so are $A$ and $B$.
- If $A \land B$ is in the RHS of a component, so is one of $A$ and $B$.
- If $(\forall x)A(x)$ is in the LHS of a component, so is $A(t)$ for every term $t$ in the LHS of that component.
- If $(\forall x)A(x)$ is in the RHS of a component, so is $A(t)$ for some term $t$ in the LHS of that component.
- If $\Box A$ is in the LHS of a component, $A$ is in the LHS of every subjunctive alternative of that component.
- If $\Box A$ is in the RHS of a component, $A$ is in the RHS of some subjunctive alternative of that component.
Models

*Fully refined positions* are examples of *models*, with variable domains and the expected truth conditions for the connectives, quantifiers and modal operators.
Soundness and Completeness

- Any derivable hypersequent (using Cut) holds in all models.
Soundness and Completeness

- Any derivable hypersequent (using *Cut*) holds in all models.
- Any hypersequent that cannot be derived (without *Cut*) can be extended into a fully refined position.
Soundness and Completeness

- Any derivable hypersequent (using Cut) holds in all models.
- Any hypersequent that cannot be derived (without Cut) can be extended into a fully refined position.
- That fully refined position determines a model in which the hypersequent does not hold.
Soundness and Completeness

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- So the models are adequate for the logic.
Soundness and Completeness

- Any derivable hypersequent (using Cut) holds in all models.
- Any hypersequent that cannot be derived (without Cut) can be extended into a fully refined position.
- That fully refined position determines a model in which the hypersequent does not hold.
- So the models are adequate for the logic.
- And in the logic, the cut rule is admissible in the cut-free system.
CONSEQUENCES & QUESTIONS
Principled, Motivated Ersatzism

The structure of modal concepts is explained in terms of the rules for their use.
Principled, Motivated Ersatzism

The structure of modal concepts is explained in terms of the rules for their use.

Worlds (and their domains) are idealised positions.
Coherent, Well Behaved Contingentism
‘Outer’ quantification is an issue for contingentism. On most approaches to contingentism, it can be \textit{defined}.
'Outer' quantification is an issue for contingentism. On most approaches to contingentism, it can be defined. This proof theoretical semantics is no different in that regard....
We have *Outer* Quantification

\[
\begin{align*}
\mathcal{H}(n > | \Gamma > A(n), \Delta) & \quad [\forall \Diamond Df] \\
\mathcal{H}(\Gamma > (\forall \Diamond x)A(x), \Delta) & \\
\mathcal{H}(n > | \Gamma, A(n) > \Delta) & \quad [\exists \Diamond Df] \\
\mathcal{H}(\Gamma, (\exists \Diamond x)A(x) > \Delta)
\end{align*}
\]

for which the substituted term need be defined in *some* zone.
The Barcan Formula is Derivable

\[
\begin{align*}
(\forall^\Diamond x) \Box A(x) & > (\forall^\Diamond x) \Box A(x) \\
\left[\forall^\Diamond Df\right] \\
\left[\Box Df\right] \\
n > (\forall^\Diamond x) \Box A(x) & > \Box A(n) \\
\left[\forall^\Diamond Df\right] \\
\left[\Box Df\right] \\
(\forall^\Diamond x) \Box A(x) & > \Box (\forall^\Diamond x) A(x)
\end{align*}
\]
But we also have *Way Out Quantification*

\[
\frac{\mathcal{H}(\Gamma \vdash A(n), \Delta)}{\mathcal{H}(\Gamma \vdash (\Pi x)A(x), \Delta)} \quad [\Pi Df] \\
\frac{\mathcal{H}(\Gamma, A(n) \vdash \Delta)}{\mathcal{H}(\Gamma, (\Sigma x)A(x) \vdash \Delta)} \quad [\Sigma Df]
\]

for which the term need not be defined *anywhere*.
Higher Order Contingentism?

\[ \forall X \Box \phi(X) \supset \Box \forall X \phi(X) \]
Higher Order Contingentism?

\[ \forall X \Box \phi(X) \supset \Box \forall X \phi(X) \]

What could it mean to rule a *predicate* in or out?
Identity!
THANK YOU!

http://consequently.org/presentation/2015/generality-and-existence-2-arche

@consequently on Twitter