Isomorphisms in a Category of Proofs

Greg Restall

CUNY GC LOGIC & METAPHYSICS SEMINAR · 9 APRIL 2018
My Aim

To show how a category of propositions and classical proofs can give rise to finely grained hyperintensional notions of sameness of content.

One notion is very finely grained (distinguishing $p$ and $p \land p$) others are is less finely grained.

One of these notions amounts to equivalence in Richard B. Angell’s logic of analytic containment.
My Motivation

To apply distinctively proof theoretical methods to issues in philosophical logic.
Acknowledgements

Thanks to Rohan French, Lloyd Humberstone, Dave Ripley, Shawn Standefer & the Melbourne Logic Seminar for helpful feedback on this material.
My Plan

The Category of Classical Proofs

Isomorphisms

More Proofs from A to A

Matching & Logics of Analytic Containment

Matching as Isomorphism
THE CATEGORY OF CLASSICAL PROOFS
There can be different ways to prove the same thing

\[ p \land q \Rightarrow p \lor q \]
Four different derivations,

\[
\frac{p \Rightarrow p}{p \wedge q \Rightarrow p} \quad \wedge L
\]

\[
\frac{p \wedge q \Rightarrow p}{p \wedge q \Rightarrow p \lor q} \quad \lor R
\]

\[
\frac{q \Rightarrow q}{p \wedge q \Rightarrow q} \quad \wedge L
\]

\[
\frac{q \Rightarrow q}{p \wedge q \Rightarrow q} \quad \lor R
\]
Four different derivations, two \textit{proofs}

\[
\frac{p \rightarrow p}{p \land q \rightarrow p} \quad ^\land L \quad \frac{p \land q \rightarrow p}{p \land q \rightarrow p \lor q} \quad ^\lor R \quad \frac{p \land q}{p \lor q} \quad \approx \quad \frac{p}{p \lor q} \quad \approx \quad \frac{p \land q}{p \land q \rightarrow p \lor q} \quad ^\lor L
\]

\[
\frac{q \rightarrow q}{p \land q \rightarrow q} \quad ^\land L \quad \frac{p \land q}{p \land q \rightarrow p \lor q} \quad ^\lor R \quad \frac{q \rightarrow q}{p \land q \rightarrow q \lor q} \quad ^\lor L
\]
Motivating Idea

Proof terms are an invariant for derivations under rule permutation.

$\delta_1$ and $\delta_2$ have the same term iff some permutation sends $\delta_1$ to $\delta_2$. 
Four different derivations, two proof terms
Ingredients

\[ \text{\textit{\lambda} terms} \quad \bullet \quad \text{flow graphs} \quad \bullet \quad \text{proof nets} \]
A proof term for $\Sigma \vdash \Delta$
encodes the flow of information
in a proof of $\Sigma \vdash \Delta$. 
Proof Terms

\[ x \vdash p \land (q \lor r) \quad \Rightarrow \quad y : (p \land q) \lor (p \land r) \]
Proof Terms as Graphs on Sequents

\[\forall x \forall y \quad \forall x \forall y \quad \forall x \forall y \quad \forall x \forall y \quad \forall x \forall y\]

\[x : p \land (q \lor r) \; \vdash \; y : (p \land q) \lor (p \land r)\]
Finding a Proof Term from a Derivation

\[
\begin{align*}
    p & \to p \\
    q & \to q \\
    \frac{p, q \to p \land q}{p \land q} & \land R \\
    r & \to r \\
    \frac{p, r \to p \land r}{p \land r} & \land R \\
    \frac{p, q \lor r \to p \land q, p \land r}{(p \land q) \lor (p \land r)} & \lor R \\
    \frac{p \land (q \lor r) \to (p \land q) \lor (p \land r)}{(p \land q) \lor (p \land r)} & \lor L
\end{align*}
\]
Finding a Proof Term from a Derivation

\[
\begin{align*}
&\frac{p \rightarrow p \quad q \rightarrow q}{p, q \rightarrow p \land q} \quad \land R \quad \frac{p \rightarrow p \quad r \rightarrow r}{p, r \rightarrow p \land r} \quad \land R \\
&\quad \quad \frac{p, q \lor r \rightarrow p \land q, p \land r}{p, q \lor r \rightarrow (p \land q) \lor (p \land r)} \quad \lor R \quad \frac{p \land (q \lor r) \rightarrow (p \land q) \lor (p \land r)}{(p \land q) \lor (p \land r)}
\end{align*}
\]
Finding a Proof Term from a Derivation

\[\begin{align*}
p \vdash p & \quad q \vdash q \\
p, q \vdash p \land q & \quad p \vdash p \quad r \vdash r \\
p, r \vdash p \land r & \\
p, q \lor r \vdash p \land q, p \land r & \quad \vdash R \\
p, q \lor r \vdash (p \land q) \lor (p \land r) & \quad \lor L \\
p \land (q \lor r) \vdash (p \land q) \lor (p \land r) & \quad \lor R \\
\end{align*}\]
Finding a Proof Term from a Derivation

\[
\begin{align*}
\frac{p \to p \quad q \to q}{p, q \to p \land q} \quad \frac{p \to p \quad r \to r}{p, r \to p \land r} \\
\frac{p, q \lor r \to p \land q, p \land r}{p, q \lor r \to (p \land q) \lor (p \land r)} \quad \frac{p \land (q \lor r) \to (p \land q) \lor (p \land r)}
\end{align*}
\]
Finding a Proof Term from a Derivation

\[
\begin{align*}
p & \Rightarrow p & q & \Rightarrow q \\
p, q & \Rightarrow p \land q & p & \Rightarrow p \\
p, r & \Rightarrow p \land r & p, r & \Rightarrow p \land r
\end{align*}
\]

\[
\begin{align*}
p, q \lor r & \Rightarrow p \land q, p \land r \\
p \land (q \lor r) & \Rightarrow (p \land q) \lor (p \land r)
\end{align*}
\]

\[
p \land (q \lor r)
\]

\[
(p \land q) \lor (p \land r)
\]
Finding a Proof Term from a Derivation

\[
\begin{align*}
\frac{p \supset p \quad q \supset q}{p \land q \supset p \land q} \quad \frac{p \supset p \quad r \supset r}{p \land r \supset p \land r} \\
\frac{p, q \supset p \land q}{p, q \land r \supset p \land q, p \land r} \quad \frac{r \supset r}{p \land (q \lor r) \supset (p \land q) \lor (p \land r)}
\end{align*}
\]
More Flow Graphs

\[ p \supset q \]

\[ (q \supset r) \supset (p \supset (q \land r)) \]

\[ (p \land \neg p) \lor q \]

\[ q \lor r \]

\[ p \land q \]

\[ r \]
Links wholly internal to a *premise* or a *conclusion* are called *cups* (\(\mapsto\)) and *caps* (\(\mapsto\)).
Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.
Not every directed graph on occurrences of atoms in a sequent is a proof term.

- They typecheck. [An occurrence of \( p \) is linked only with an occurrence of \( p \).]

- They respect polarities. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are inputs. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are outputs.]

- They must satisfy an “enough connections” condition, amounting to a non-emptiness under every switching. [e.g. the obvious linking between premise \( p \rightarrow q \) and conclusion \( p \wedge q \) is not connected enough to be a proof term.]
Not every directed graph on occurrences of atoms in a sequent is a proof term.

- They *typecheck*. [An occurrence of \( p \) is linked only with an occurrence of \( p \).]
- They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]
Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

- They typecheck. [An occurrence of \( p \) is linked only with an occurrence of \( p \).]
- They respect polarities. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are \textit{inputs}. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are \textit{outputs}.]
- They must satisfy an “enough connections” condition, amounting to a non-emptiness under every \textit{switching}. [e.g. the obvious linking between premise \( p \lor q \) and conclusion \( p \land q \) is not connected enough to be a proof term.]
Cut is chaining of proof terms

\[(p \land q) \lor (p \land r)\]

\[p \land (q \lor r)\]
The cut formula is no longer a premise or a conclusion in the proof term.
Cut is chaining of proof terms

The *cut formula* is no longer a premise or a conclusion in the proof term.
Eliminating Cuts is Local

\[(p \land q) \lor (p \land r)\]

\[p \land (q \lor r)\]

\[(p \land q) \lor (p \land r)\]
Eliminating Cuts is Local

$(p \land q) \lor (p \land r)$

$(p \land q) \lor (p \land r)$
Eliminating Cuts is Local

\[(p \land q) \lor (p \land r)\]

\[
\begin{array}{ccc}
p & q & r \\
( p \land q ) & \lor & ( p \land r ) \\
\end{array}
\]
Eliminating Cuts is Local

\[(p \land q) \lor (p \land r)\]

\[(p \land q) \lor (p \land r)\]
Results

- Cut elimination is *confluent* and *terminating*.
Results

- Cut elimination is **confluent** and **terminating**.
  
  [So it can be understood as a kind of *evaluation*.]
Cut elimination is **confluent** and **terminating**.
[So it can be understood as a kind of *evaluation*.]
Cut elimination for proof terms is **local**.
Results

- Cut elimination is *confluent* and *terminating*.  
  [So it can be understood as a kind of *evaluation*.]
- Cut elimination for proof terms is *local*.  
  [So it is easily made parallel.]
Cuts with Caps and Cups

\[ p \land (p \Rightarrow q) \]
\[ q \land (r \lor \neg r) \]
\[ \neg p \lor (p \land q) \]
Cuts with Caps and Cups

\[ p \land (p \supset q) \]
\[ q \land (r \lor \neg r) \]
\[ \neg p \lor (p \land q) \]
\[ \mathcal{C} \text{ is the Category of Classical Proofs} \]

**OBJECTS**  Formulas — A, B, etc.

**ARROWS**  Cut-Free Proof Terms — \( \pi : A \rightarrow B \).

**COMPOSITION**  Composition of derivations with the elimination of \( \text{Cut} \) — If \( \pi : A \rightarrow B \) and \( \tau : B \rightarrow C \) then \( \tau \circ \pi : A \rightarrow C \).

**IDENTITY**  Canonical identity proofs — \( \text{Id}(A) : A \rightarrow A \).
Identity Proofs

\[
\frac{p \rightarrow p}{\neg L} \quad \frac{p \rightarrow p \quad p \rightarrow p}{\neg R} \quad \frac{\neg p \rightarrow \neg p}{\Rightarrow L} \quad \frac{p \supset p, p \rightarrow p}{\Rightarrow R} \quad \frac{p \supset p \rightarrow p \supset p}{\Rightarrow L} \quad \frac{\neg p \vee (p \supset p)}{\Rightarrow R} \quad \frac{\neg p \vee (p \supset p \supset p)}{\neg p \vee (p \supset p)}
\]
Identity Proofs

\[
\begin{array}{c}
p \nRightarrow p \\
\frac{p, \neg p}{\neg p} \quad \frac{p \nRightarrow p, p \nRightarrow p}{p \nRightarrow p} \quad \frac{p \nRightarrow p}{p} \\
\hline
\frac{\neg p \nRightarrow \neg p}{\neg p} \\
\frac{p \nRightarrow (p \nRightarrow p)}{p, p \nRightarrow p} \\
\frac{\neg p \nRightarrow (p \nRightarrow p)}{\neg p \nRightarrow (p \nRightarrow p)}
\end{array}
\]

\[
\frac{\neg p \nRightarrow (p \nRightarrow p)}{\neg p \nRightarrow (p \nRightarrow p)}
\]

\[
\neg p \nRightarrow (p \nRightarrow p)
\]
Identity Proofs

\[
\begin{align*}
\frac{p \rightarrow p}{p, \neg p \rightarrow \top} & \quad \frac{p \rightarrow p}{p \supset p, p \rightarrow p} \\
\frac{\neg p \rightarrow \bot}{p \rightarrow p} & \quad \frac{p \supset p}{p \supset p, p \supset p} \\
\frac{\neg p \supset (p \supset p) \rightarrow \bot, p \supset p}{p \supset (p \supset p) \rightarrow \bot} & \quad \frac{\neg p \supset (p \supset p)}{p \supset (p \supset p)}
\end{align*}
\]
Identity Proofs

\[
\frac{p \rightarrow p}{\neg L} \quad \frac{p \supset p \quad p \supset p}{\supset L} \quad \frac{p \supset p, p \supset p}{\supset R} \\
\frac{p \supset p, p \supset p}{\vee L} \quad \frac{\neg p \vee (p \supset p) \supset \neg p, p \supset p}{\vee R} \\
\frac{\neg p \vee (p \supset p) \supset \neg p \vee (p \supset p)}{
\neg L \quad \neg L \quad \neg R}
\]

\[
\neg p \vee (p \supset p)
\]

\[
\neg p \vee (p \supset p)
\]
Identity Proofs

\[
\begin{align*}
\frac{p \Rightarrow p}{p, \neg p \Rightarrow \neg p} & \quad \text{\(\neg L\)} \\
\frac{\neg p \Rightarrow \neg p}{\neg p, p \Rightarrow \neg p} & \quad \text{\(\neg R\)} \\
\frac{p \supset p, p \Rightarrow p}{p \supset p} & \quad \text{\(\Rightarrow L\)} \\
\frac{p \Rightarrow p, p \supset p}{p \supset p} & \quad \text{\(\Rightarrow R\)} \\
\frac{\neg p \lor (p \supset p) \Rightarrow \neg p, p \supset p}{\neg p \lor (p \supset p)} & \quad \text{\(\lor L\)} \\
\frac{\neg p \lor (p \supset p) \Rightarrow \neg p \lor (p \supset p)}{\neg p \lor (p \supset p)} & \quad \text{\(\lor R\)}
\end{align*}
\]
Identity Proofs

\[
\begin{align*}
\frac{p \rightarrow p}{p, \neg p \rightarrow \neg p} & \quad \neg L \\
\frac{p \rightarrow p \quad p \rightarrow p}{p \supset p \quad p \supset p} & \quad \supset L \\
\frac{p \supset p \quad p \supset p}{p \supset p \quad p \supset p} & \quad \supset R \\
\frac{\neg p \lor (p \supset p) \supset \neg p, p \supset p}{\neg p \lor (p \supset p) \lor \neg p \lor (p \supset p)} & \quad \lor R
\end{align*}
\]
Identity Proofs

\[
\frac{p \rightarrow p}{p, \neg p \rightarrow \neg p} \quad \frac{\neg L}{\neg p \rightarrow \neg \neg p} \\
\frac{\neg R}{p \supset p, p \rightarrow p} \quad \frac{\neg L}{p \supset p \rightarrow p 
\supset L} \\
\frac{\supset R}{p \supset p, p \supset p} \quad \frac{\vee L}{p \supset \neg p \rightarrow \neg p, p \supset p} \\
\frac{\vee R}{p \supset (p \supset p) \rightarrow \neg p \vee (p \supset p)} \\
\frac{\neg R}{p \supset (p \supset p)}
\]

\[
\neg p \vee (p \supset p)
\]
Identity Proofs

In the identity proof from $A$ to $A$,

- A *positive* occurrence of an atom in the premise linked *to* its mate in the conclusion.
- A *negative* occurrence of an atom in the premise is linked *from* its mate in the conclusion.
- There are no other links.
Identity and Composition in $\mathcal{C}$

\[
(p \land \neg q) \lor (p \land r)
\]

\[
p \land (\neg q \lor r)
\]

\[
p \land (\neg q \lor r)
\]

\[
(p \land \neg q) \lor (p \land r)
\]
Identity and Composition in $\mathcal{C}$

\[(p \land \neg q) \lor (p \land r)\]

\[p \land (\neg q \lor r)\]
Identity and Composition in $\mathcal{C}$

\[(p \land \neg q) \lor (p \land r)\]

\[\Downarrow\]

\[p \land (\neg q \lor r)\]

\[\Downarrow\]

\[p \land (\neg q \lor r)\]

\[\Downarrow\]

\[(p \land \neg q) \lor (p \land r)\]
... is symmetric monoidal and star autonomous

but not Cartesian,

with structural monoids and comonoids,

and is enriched in $SLat$ (the category of semilattices).
The Category $\mathcal{C}$ ...

- ... is symmetric monoidal and star autonomous
- but not *Cartesian*,
- with structural *monoids* and *comonoids*,
- and is enriched in $\text{SLat}$ (the category of semilattices).

Being enriched in $\text{SLat}$ means that proofs terms come ordered by $\subseteq$, and compose under $\cup$, and these interact sensibly with composition.
The Category $\mathcal{C} \ldots$

- ... is symmetric monoidal and star autonomous
- but not Cartesian,
- with structural monoids and comonoids,
- and is enriched in $SLat$ (the category of semilattices).

Being enriched in $SLat$ means that proofs terms come ordered by $\subseteq$, and compose under $\cup$, and these interact sensibly with composition.

\[
\pi \subseteq \pi' \quad \Rightarrow \quad \pi \circ \tau \subseteq \pi' \circ \tau
\]
The Category $\mathcal{C}$ ...

- ... is symmetric monoidal and star autonomous
- but not Cartesian,
- with structural monoids and comonoids,
- and is enriched in $\text{SLat}$ (the category of semilattices).

Being enriched in $\text{SLat}$ means that proofs terms come ordered by $\subseteq$, and compose under $\cup$, and these interact sensibly with composition.

\[
\pi \subseteq \pi' \quad \Rightarrow \quad \pi \circ \tau \subseteq \pi' \circ \tau
\]
\[
\tau \subseteq \tau' \quad \Rightarrow \quad \pi \circ \tau \subseteq \pi \circ \tau'
\]
... is symmetric monoidal and star autonomous

- but not Cartesian,
- with structural monoids and comonoids,
- and is enriched in $SLat$ (the category of semilattices).

Being enriched in $SLat$ means that proofs terms come ordered by $\subseteq$, and compose under $\cup$, and these interact sensibly with composition.

\[
\pi \subseteq \pi' \quad \Rightarrow \quad \pi \circ \tau \subseteq \pi' \circ \tau
\]

\[
\tau \subseteq \tau' \quad \Rightarrow \quad \pi \circ \tau \subseteq \pi \circ \tau'
\]

\[
\pi \circ (\tau \cup \tau') \quad = \quad (\pi \circ \tau) \cup (\pi \circ \tau')
\]
The Category $\mathcal{C}$ ...

- ... is symmetric monoidal and star autonomous
- but not Cartesian,
- with structural monoids and comonoids,
- and is enriched in $SLat$ (the category of semilattices).

Being enriched in $SLat$ means that proofs terms come ordered by $\subseteq$, and compose under $\cup$, and these interact sensibly with composition.

\[
\begin{align*}
\pi \subseteq \pi' & \implies \pi \circ \tau \subseteq \pi' \circ \tau \\
\tau \subseteq \tau' & \implies \pi \circ \tau \subseteq \pi \circ \tau'
\end{align*}
\]

\[
\begin{align*}
\pi \circ (\tau \cup \tau') & = (\pi \circ \tau) \cup (\pi \circ \tau') \\
(\pi \cup \pi') \circ \tau & = (\pi \circ \tau) \cup (\pi' \circ \tau)
\end{align*}
\]
\( \mathcal{C} \) is just *classical* propositional logic, in a categorical setting.
\(C\) is just *classical* propositional logic, in a categorical setting.

(The sequent calculus plays no essential role here. You can define proof terms on other proof systems, e.g. *natural deduction*, *Hilbert proofs*, *tableaux*, *resolution.*)
ISOMORPHISMS
f : A → B is an isomorphism in a category iff it has an inverse g : B → A, where
\[ g \circ f = id_A : A \to A \] and \[ f \circ g = id_B : B \to B. \]
Isomorphisms in Categories

\[ f : A \rightarrow B \text{ is an isomorphism in a category iff it has an inverse } g : B \rightarrow A, \text{ where} \]
\[ g \circ f = id_A : A \rightarrow A \text{ and } f \circ g = id_B : B \rightarrow B. \]

If \( g \) and \( g' \) are both inverses, we have
\[ g = id_A \circ g = (g' \circ f) \circ g = g' \circ (f \circ g) = g' \circ id_B = g', \]
so any inverse is unique. We can call it \( f^{-1} \).
Why Isomorphisms?

If $A$ and $B$ are isomorphic in a category $C$, then what we can do with $A$ (in $C$) we can do with $B$, too.
Why Isomorphisms?

If $A$ and $B$ are isomorphic in a category $\mathcal{C}$, then what we can do with $A$ (in $\mathcal{C}$) we can do with $B$, too.

If $A$ and $B$ are isomorphic in $\mathcal{C}$, then they agree not only on provability, but also, on proofs.
Why Isomorphisms?

If $A$ and $B$ are isomorphic in a category $C$, then what we can do with $A$ (in $C$) we can do with $B$, too.

If $A$ and $B$ are isomorphic in $C$, then they agree not only on provability, but also, on proofs.

The distinctions drawn when you analyse how something is proved (from premises), are not far from what you want to understand when you ask how something is made true.
\[ p \land q \cong q \land p \]
Isomorphisms in $\mathcal{C}$

$p \land q \cong q \land p$
Isomorphisms in $\mathcal{C}$

$$p \land q \cong q \land p$$

Diagram:

```
\begin{array}{c}
\begin{array}{c}
p \land q \\
\downarrow \\
q \land p \\
\downarrow \\
p \land q
\end{array} & \begin{array}{c}
p \land q \\
\downarrow \\
p \land q
\end{array}
\end{array}
```
$p \lor q \cong q \lor p$
Isomorphisms in $\mathcal{C}$

\[ p \lor q \cong q \lor p \]
Isomorphisms in $\mathcal{C}$

$p \lor q \cong q \lor p$
\[ p \land (q \land r) \cong (p \land q) \land r \]
Isomorphisms in $\mathcal{C}$

\[ p \land (q \land r) \cong (p \land q) \land r \]
Isomorphisms in $\mathcal{C}$

$p \land (q \land r) \cong (p \land q) \land r$

\[
\begin{array}{ccc}
p \land (q \land r) & \xrightarrow{\cong} & (p \land q) \land r \\
(p \land q) \land r & \xrightarrow{\cong} & p \land (q \land r) \\
p \land (q \land r) & \xrightarrow{\cong} & p \land (q \land r)
\end{array}
\]
\[-(p \land q) \cong \neg p \lor \neg q\]
Isomorphisms in $\mathcal{C}$

$$\neg(p \land q) \cong \neg p \lor \neg q$$
\[ \neg(p \land q) \cong \neg p \lor \neg q \]
Isomorphisms in $\mathcal{C}$

$\neg\neg p \cong p$
Isomorphisms in $\mathcal{C}$

$\neg\neg p \cong p$
Isomorphisms in $\mathcal{C}$

$\neg \neg p \cong p$
Non-isomorphisms in $\mathcal{C}$

$p \land (q \lor \neg q) \not\cong p$
Non-isomorphisms in $\mathcal{C}$

\[ p \land (q \lor \neg q) \not\equiv p \]

\[ p \land (q \lor \neg q) \]
Non-isomorphisms in $\mathcal{C}$

\[ p \land (q \lor \neg q) \not\equiv p \]

\[ p \land (q \lor \neg q) \quad p \land (q \lor \neg q) \]

\[ p \quad p \]

\[ p \land (q \lor \neg q) \quad p \land (q \lor \neg q) \]
Non-isomorphisms in $\mathcal{C}$

\[ p \land (q \lor \neg q) \not\equiv p \]

\[ p \land (q \lor \neg q) \quad \quad \quad p \land (q \lor \neg q) \]

\[ p \quad \quad \quad \quad \quad \quad \quad \quad \quad p \]

\[ p \land (q \lor \neg q) \quad \quad \quad p \land (q \lor \neg q) \]

\[ p \land (q \lor \neg q) \quad \quad \quad p \land (q \lor \neg q) \]
Non-isomorphisms in $C$

\[ p \land (q \lor \neg q) \not\cong p \]

\[ p \land (q \lor \neg q) \]

\[ p \]

\[ p \land (q \lor \neg q) \]

\[ p \land (q \lor \neg q) \]

\[ p \land (q \lor \neg q) \]
Non-isomorphisms in $\mathcal{C}$

$p \land (q \lor \neg q) \not\cong p$
Non-isomorphisms in \( \mathcal{C} \)

\[ p \land (q \lor \lnot q) \not\cong p \]
Non-isomorphisms in $\mathfrak{C}$

$p \land p \not\approx p$
Non-isomorphisms in $\mathcal{C}$

$p \land p \not\cong p$

$p$  

$p \land p$  

$p \land p$
Non-isomorphisms in $\mathcal{C}$

$p \land p \not\cong p$

Diagram:

```
p \land p
  |     |
  v     v
  p     p
  |     |
  v     v
p \land p
```
Non-isomorphisms in $\mathcal{C}$

$p \land p \nLeftrightarrow p$

\[
\begin{array}{cc}
p \land p & p \land p \\
p & p \\
p \land p & p \land p
\end{array}
\]
Non-isomorphisms in $C$

$p \land p \not\cong p$

\[
\begin{array}{c}
p \\
\downarrow \\
p \\
\downarrow \\
p \land p \\
\end{array}
\quad
\begin{array}{c}
p \\
\downarrow \\
p \\
\downarrow \\
p \land p \\
\end{array}
\]
Non-isomorphisms in $\mathcal{C}$

\[ p \land (q \lor r) \nleq (p \land q) \lor (p \land r) \]
Non-isomorphisms in $\mathcal{C}$

\[ p \land (q \lor r) \not\equiv (p \land q) \lor (p \land r) \]

\[ (p \land q) \lor (p \land r) \]

\[ p \land (q \lor r) \]

\[ (p \land q) \lor (p \land r) \]
Non-isomorphisms in $\mathcal{C}$

\[ p \land (q \lor r) \nless\neq (p \land q) \lor (p \land r) \]

\[ (p \land q) \lor (p \land r) \]

\[ p \land (q \lor r) \]

\[ (p \land q) \lor (p \land r) \]
Non-isomorphisms in $\mathcal{C}$

\[ p \land (q \lor r) \not\equiv (p \land q) \lor (p \land r) \]

Diagram:

\[ \begin{array}{ccc}
(p \land q) \lor (p \land r) & \rightarrow & p \land (q \lor r) \\
\downarrow & & \downarrow \\
(p \land q) \lor (p \land r) & \rightarrow & p \land (q \lor r) \\
\end{array} \]
Non-isomorphisms in $\mathfrak{C}$

$p \land (q \lor r) \nRightarrow (p \land q) \lor (p \land r)$

Greg Restall
Isomorphisms in a Category of Proofs
Non-isomorphisms in $\mathcal{C}$

$$p \land (p \lor q) \not\equiv p \lor (p \land q)$$
Occurrence Polarity Condition

If $A$ is isomorphic to $B$ in $\mathcal{C}$
then each variable occurs positively [negatively] in $A$
exactly as many times as it occurs positively [negatively] in $B$. 
Occurrence Polarity Condition

If $A$ is isomorphic to $B$ in $\mathcal{C}$ then each variable occurs positively [negatively] in $A$ exactly as many times as it occurs positively [negatively] in $B$.

(This condition is *necessary*, not *sufficient*: $p \land (p \lor q) \not\equiv p \lor (p \land q)$.)
Characterising Isomorphisms

A is isomorphic to B
iff A and B are equivalent
in the following calculus:

\[ A \land B \leftrightarrow B \land A, \quad A \land (B \land C) \leftrightarrow (A \land B) \land C. \]
\[ A \lor B \leftrightarrow B \lor A, \quad A \lor (B \lor C) \leftrightarrow (A \lor B) \lor C. \]
\[ \neg(A \lor B) \leftrightarrow \neg A \land \neg B, \quad \neg(A \land B) \leftrightarrow \neg A \lor \neg B. \]
\[ \neg\neg A \leftrightarrow A. \quad A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B). \]
A is *isomorphic* to B
iff A and B are equivalent
in the following calculus:

\[
\begin{align*}
A \land B & \leftrightarrow B \land A, & A \land (B \land C) & \leftrightarrow (A \land B) \land C. \\
A \lor B & \leftrightarrow B \lor A, & A \lor (B \lor C) & \leftrightarrow (A \lor B) \lor C. \\
\neg (A \lor B) & \leftrightarrow \neg A \land \neg B, & \neg (A \land B) & \leftrightarrow \neg A \lor \neg B. \\
\neg \neg A & \leftrightarrow A. & A & \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B).
\end{align*}
\]

This allows for a *negation normal form*, but not DNF or CNF.
Proof Sketch (Došen and Petrić, 2012)

- If $A \leftrightarrow B$ holds in the calculus, $A$ and $B$ are isomorphic.
Proof Sketch (Došen and Petrić, 2012)

- If $A \leftrightarrow B$ holds in the calculus, $A$ and $B$ are isomorphic.
- $A$ is isomorphic to $B$ iff there are diversified $A'$ and $B'$ where $A'$ and $B'$ are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution $\sigma$. 

Replace $l \land m$ by a new atom in both $A$ and $B$, and repeat. 

This shows how to reconstruct a proof of equivalence for $A$ and $B$ in the syntactic calculus for isomorphic formulas.
Proof Sketch (Došen and Petrić, 2012)

- If $A \leftrightarrow B$ holds in the calculus, $A$ and $B$ are isomorphic.
- $A$ is isomorphic to $B$ iff there are diversified $A'$ and $B'$ where $A'$ and $B'$ are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution $\sigma$.
- $A$ is isomorphic to $B$ iff their negation normal forms are isomorphic. (If $A$ is diversified, so is its negation normal form.)
Proof Sketch (Došen and Petrič, 2012)

- If $A \leftrightarrow B$ holds in the calculus, $A$ and $B$ are isomorphic.

- $A$ is isomorphic to $B$ iff there are diversified $A'$ and $B'$ where $A'$ and $B'$ are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution $\sigma$.

- $A$ is isomorphic to $B$ iff their negation normal forms are isomorphic. (If $A$ is diversified, so is its negation normal form.)

- If $A$ and $B$ are diversified, isomorphic, and in negation normal form, if $l \land m$ is a conjunction in $A$ ($l$ and $m$, literals), a substitution argument (substituting $\top$ and $\bot$ for the other atoms) shows that $l$ and $m$ must be conjunctively joined in $B$, too. The same goes for $l \lor m$. 

- Replace $l \land m$ by a new atom in both $A$ and $B$, and repeat.

- This shows how to reconstruct a proof of equivalence for $A$ and $B$ in the syntactic calculus for isomorphic formulas.
Proof Sketch (Došen and Petrić, 2012)

- If $A \leftrightarrow B$ holds in the calculus, $A$ and $B$ are isomorphic.
- $A$ is isomorphic to $B$ iff there are diversified $A'$ and $B'$ where $A'$ and $B'$ are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution $\sigma$.
- $A$ is isomorphic to $B$ iff their negation normal forms are isomorphic. (If $A$ is diversified, so is its negation normal form.)
- If $A$ and $B$ are diversified, isomorphic, and in negation normal form, if $l \land m$ is a conjunction in $A$ ($l$ and $m$, literals), a substitution argument (substituting $\top$ and $\bot$ for the other atoms) shows that $l$ and $m$ must be conjunctively joined in $B$, too. The same goes for $l \lor m$.
- Replace $l \land m$ by a new atom in both $A$ and $B$, and repeat.
Proof Sketch (Došen and Petrić, 2012)

- If $A \leftrightarrow B$ holds in the calculus, $A$ and $B$ are isomorphic.

- $A$ is isomorphic to $B$ iff there are diversified $A'$ and $B'$ where $A'$ and $B'$ are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution $\sigma$.

- $A$ is isomorphic to $B$ iff their negation normal forms are isomorphic. (If $A$ is diversified, so is its negation normal form.)

- If $A$ and $B$ are diversified, isomorphic, and in negation normal form, if $l \land m$ is a conjunction in $A$ ($l$ and $m$, literals), a substitution argument (substituting $\top$ and $\bot$ for the other atoms) shows that $l$ and $m$ must be conjunctively joined in $B$, too. The same goes for $l \lor m$.

- Replace $l \land m$ by a new atom in both $A$ and $B$, and repeat.

- This shows how to reconstruct a proof of equivalence for $A$ and $B$ in the syntactic calculus for isomorphic formulas.
If A and B are isomorphic, they can play *essentially* the same role in proof.
This is a very tight constraint

- If $A$ and $B$ are isomorphic, they can play *essentially* the same role in proof.

- Replacing $A$ by $B$ in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*. 
If A and B are isomorphic, they can play *essentially* the same role in proof.

Replacing A by B in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*.

Not even A and A ∧ A are equivalent in *this* sense.
If A and B are isomorphic, they can play *essentially* the same role in proof.

Replacing A by B in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*.

Not even A and A ∨ A are equivalent in *this* sense.

Yet, A and A ∨ A seem to have identical *subject matter* (insofar as we understand that notion).
If $A$ and $B$ are isomorphic, they can play *essentially* the same role in proof.

Replacing $A$ by $B$ in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*.

Not even $A$ and $A \land A$ are equivalent in *this* sense.

Yet, $A$ and $A \land A$ seem to have identical *subject matter* (insofar as we understand that notion).

Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?
MORE PROOFS FROM A TO A
In $Id(A)$, each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion. Different occurrences of atoms in $A$ are treated differently.
In $Id(A)$, each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion. Different occurrences of atoms in $A$ are treated differently.

In $Hz(A)$, each positive [negative] occurrence of an atom in the premise is linked with every positive [negative] occurrence in the conclusion. We treat occurrences of an atom in $A$—with the same polarity—equally.
In $Id(A)$, each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion. Different occurrences of atoms in $A$ are treated differently.

In $Hz(A)$, each positive [negative] occurrence of an atom in the premise is linked with every positive [negative] occurrence in the conclusion. We treat occurrences of an atom in $A$—with the same polarity—equally.

In $Mx(A)$, each syntactically possible linking is included. We treat all occurrences of an atom in $A$ equally.
Note: $Hz(A)$ is $Mx(A)$ with the caps and cups removed.
Let’s look at relations like isomorphism, but which erase distinctions, up to $Hz$ or $Mx$. 
Let’s say that $A$ and $B$ \textit{Hz-match}, when there are proofs $\pi : A \to B$ and $\pi' : B \to A$ where $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$. 

\textbf{Hz-Matching}
Let’s say that $A$ and $B$ \textit{Hz-match}, when there are proofs $\pi : A \rightarrow B$ and $\pi' : B \rightarrow A$ where $\pi' \circ \pi = \text{Hz}(A)$ and $\pi \circ \pi' = \text{Hz}(B)$.

We write “$\approx_{\text{Hz}}$” for the Hz-matching relation, and we write “$\pi, \pi' : A \approx_{\text{Hz}} B$” to say that $\pi : A \rightarrow B$ and $\pi' : B \rightarrow A$ define a Hz-match between $A$ and $B$. 
Let’s say that $A$ and $B$ $Mx$-match, when there are proofs $\pi : A \twoheadrightarrow B$ and $\pi' : B \twoheadrightarrow A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$. 

$Mx$-Matching
**Mx-Matching**

Let’s say that A and B *Mx-match*, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$.

We write “$\approx_{Mx}$” for the Mx-matching relation, and we write “$\pi, \pi' : A \approx_{Mx} B$” to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a Mx-match between A and B.
Isomorphism $\subseteq Hz$-Matching

If $\pi : A \rightarrow B$ and $\pi^{-1} : B \rightarrow A$, then consider $\pi' = Hz(B) \circ \pi \circ Hz(A)$ and $\tau' = Hz(A) \circ \pi^{-1} \circ Hz(B)$.

These satisfy the $Hz$-matching criteria, $\tau' \circ \pi' = Hz(A)$ and $\pi' \circ \tau' = Hz(B)$. 
Proof

\[ Hz(A) = Id(A) \circ Id(A) \circ Hz(A) \]
\[ \subseteq Hz(A) \circ Id(A) \circ Hz(A) \]
\[ = Hz(A) \circ (\pi^{-1} \circ \pi) \circ Hz(A) \]
\[ = Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \]
\[ \subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \]
\[ = (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \]
\[ = \tau' \circ \pi' \]
\[ \subseteq Hz(A) \]
Proof

\[ Hz(A) = Id(A) \circ Id(A) \circ Hz(A) \]
\[ \subseteq Hz(A) \circ Id(A) \circ Hz(A) \]
\[ = Hz(A) \circ (\pi^{-1} \circ \pi) \circ Hz(A) \]
\[ = Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \]
\[ \subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \]
\[ = (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \]
\[ = \tau' \circ \pi' \]
\[ \subseteq Hz(A) \]

...and similarly, \( Hz(B) \subseteq \pi' \circ \tau' \subseteq Hz(B) \)
If $\pi, \pi' : A \cong_{Hz} B$, then consider

$$\tau = Mx(B) \circ \pi \circ Mx(A)$$

and

$$\tau' = Mx(A) \circ \pi' \circ Mx(B).$$

These satisfy the $Mx$-matching criteria,

$$\tau' \circ \pi' = Mx(A) \text{ and } \pi' \circ \tau' = Mx(B).$$
Proof

\[ Mx(A) = Id(A) \circ Id(A) \circ Mx(A) \]
\[ \subseteq Mx(A) \circ Hz(A) \circ Mx(A) \]
\[ = Mx(A) \circ (\pi' \circ \pi) \circ Mx(A) \]
\[ = Mx(A) \circ (\pi' \circ Id(B) \circ Id(B) \circ \pi) \circ Mx(A) \]
\[ \subseteq Mx(A) \circ (\pi' \circ Mx(B) \circ Mx(B) \circ \pi) \circ Mx(A) \]
\[ = (Mx(A) \circ \pi' \circ Mx(B)) \circ (Mx(B) \circ \pi \circ Mx(A)) \]
\[ = \tau' \circ \tau \]
\[ \subseteq Mx(A) \]
Proof

\[ Mx(A) = Id(A) \circ Id(A) \circ Mx(A) \]
\[ \subseteq Mx(A) \circ Hz(A) \circ Mx(A) \]
\[ = Mx(A) \circ (\pi' \circ \pi) \circ Mx(A) \]
\[ = Mx(A) \circ (\pi' \circ Id(B) \circ Id(B) \circ \pi) \circ Mx(A) \]
\[ \subseteq Mx(A) \circ (\pi' \circ Mx(B) \circ Mx(B) \circ \pi) \circ Mx(A) \]
\[ = (Mx(A) \circ \pi' \circ Mx(B)) \circ (Mx(B) \circ \pi \circ Mx(A)) \]
\[ = \tau' \circ \tau \]
\[ \subseteq Mx(A) \]

...and similarly, \( Mx(B) \subseteq \pi' \circ \tau' \subseteq Mx(B) \)
If \( A \approx_{M_x} B \) then there are proofs \( \pi : A \rightarrow B \) and \( \tau : B \rightarrow A \).
Matching Relations are Equivalence Relations

**REFLEXIVE** \( \text{Hz}(A), \text{Hz}(A) : A \cong_{\text{Hz}} A. \)

\( \text{Mx}(A), \text{Mx}(A) : A \cong_{\text{Mx}} A. \)
Matching Relations are Equivalence Relations

**REFLEXIVE** \(Hz(A), Hz(A) : A \approx_{Hz} A.\)
\[\]
\(Mx(A), Mx(A) : A \approx_{Mx} A.\)

**SYMMETRIC**

If \(\pi, \pi' : A \approx_{Hz} B,\) then \(\pi', \pi : B \approx_{Hz} A.\)

If \(\pi, \pi' : A \approx_{Mx} B,\) then \(\pi', \pi : B \approx_{Mx} A.\)
Matching Relations are Equivalence Relations

**REFLEXIVE**

\[ Hz(A), Hz(A) : A \approx_{Hz} A. \]

\[ Mx(A), Mx(A) : A \approx_{Mx} A. \]

**SYMMETRIC**

If \( \pi, \pi' : A \approx_{Hz} B \), then \( \pi', \pi : B \approx_{Hz} A \).

If \( \pi, \pi' : A \approx_{Mx} B \), then \( \pi', \pi : B \approx_{Mx} A \).

**TRANSITIVE**

If \( \pi, \pi' : A \approx_{Hz} B \) and \( \tau, \tau' : B \approx_{Hz} C \), then

\( (\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Hz} C \).

If \( \pi, \pi' : A \approx_{Mx} B \) and \( \tau, \tau' : B \approx_{Mx} C \), then

\( (\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Mx} C \).
More Matchings

\[ p \lor p \cong_{Hz} p \cong_{Hz} p \land p \]

\[ \begin{array}{c}
\downarrow \\
p
\uparrow \\
\land
\end{array} \]

\[ \begin{array}{c}
\downarrow \\
p
\uparrow \\
\lor
\end{array} \]

\[ \begin{array}{c}
\downarrow \\
p
\uparrow \\
\lor
\end{array} \]

\[ \begin{array}{c}
\downarrow \\
p
\uparrow \\
\lor
\end{array} \]

\[ \begin{array}{c}
\downarrow \\
p
\uparrow \\
\lor
\end{array} \]
More Matchings

\[ p \lor p \cong_{Hz} p \cong_{Hz} p \land p \]
More Matchings

\[ p \land (q \lor r) \approx_{Hz} (p \land q) \lor (p \land r) \]
More Matchings

\[ p \land (q \lor r) \cong_{Hz} (p \land q) \lor (p \land r) \]

\[ (p \land q) \lor (p \land r) \]

\[ p \land (q \lor r) \]

\[ (p \land q) \lor (p \land r) \]
FACT: If an atom $p$ occurs positively in $A$ but not in $B$, then $A$ and $B$ do not $Mx$-match.
**FACT:** If an atom \( p \) occurs positively in \( A \) but not in \( B \), then \( A \) and \( B \) do not \( Mx \)-match.

**PROOF:** \( Mx(A) : A \rightarrow A \) contains the link from that occurrence of \( p \) in the premise \( A \) to its corresponding occurrence in the conclusion \( A \).
**FACT:** If an atom $p$ occurs positively in $A$ but not in $B$, then $A$ and $B$ do not $Mx$-match.

**PROOF:** $Mx(A) : A \Rightarrow A$ contains the link from that occurrence of $p$ in the premise $A$ to its corresponding occurrence in the conclusion $A$. No proof from $A$ to $B$ contains a link from that occurrence to anything in $B$ (since there is no positive occurrence in $B$ at all).
**FACT:** If an atom $p$ occurs positively in $A$ but not in $B$, then $A$ and $B$ do not $Mx$-match.

**PROOF:** $Mx(A)$ : $A : A$ contains the link from that occurrence of $p$ in the premise $A$ to its corresponding occurrence in the conclusion $A$.

No proof from $A$ to $B$ contains a link from that occurrence to anything in $B$ (since there is no positive occurrence in $B$ at all).

So, in the composition proof from $A$ to $A$, there is no link from the premise occurrence to the conclusion occurrence. No proof from $A$ to $B$ and back can recreate $Mx(A)$.  

**FACT:** If an atom $p$ occurs positively [negatively] in $A$ but not in $B$, then $A$ and $B$ do not $Mx$-match.

**PROOF:** $Mx(A) : A \to A$ contains the link from [to] that occurrence of $p$ in the premise $A$ to [from] its corresponding occurrence in the conclusion $A$.

No proof from $A$ to $B$ contains a link from [to] that occurrence to anything in $B$ (since there is no positive [negative] occurrence in $B$ at all).

So, in the composition proof from $A$ to $A$, there is no link from [to] the premise occurrence to [from] the conclusion occurrence. No proof from $A$ to $B$ and back can recreate $Mx(A)$. 
**FACT:** If an atom occurs positively [negatively] in A but not in B then A and B do not $Mx$-match.
**FACT:** If an atom occurs positively [negatively] in $A$ but not in $B$ then $A$ and $B$ do not $Mx$-match.

**COROLLARY:** $p \not\equiv_{Mx} p \land (q \lor \neg q)$.
**FACT:** If an atom occurs positively [negatively] in A but not in B then A and B do not $Mx$-match.

**COROLLARY:** $p \not\equiv_{Mx} p \land (q \lor \neg q)$.

$p \land \neg p \not\equiv_{Mx} q \land \neg q$. 
Hz-matching $\subseteq Mx$-matching

$(p \land \neg p) \land (q \lor \neg q) \approx_{Mx} (p \lor \neg p) \land (q \land \neg q)$
Hz-matching $\subseteq Mx$-matching

$$(p \land \neg p) \land (q \lor \neg q) \cong_{Mx} (p \lor \neg p) \land (q \land \neg q)$$
Isomorphism $\subset$ Hz-Matching $\subset$ Mx-Matching $\subset$ Logical Equivalence
So what *are* the *matching* relations?
MATCHING & LOGICS OF ANALYTIC CONTAINMENT
Angell's Logic of Analytic Containment

**AC1** \( A \leftrightarrow \neg\neg A \)

**AC2** \( A \leftrightarrow (A \land A) \)

**AC3** \( (A \land B) \leftrightarrow (B \land A) \)

**AC4** \( A \land (B \land C) \leftrightarrow (A \land B) \land C \)

**AC5** \( A \lor (B \land C) \leftrightarrow (A \lor B) \land (A \lor C) \)

**R1** \( A \leftrightarrow B, C(A) \Rightarrow C(B) \)
Angell's Logic of Analytic Containment

AC1  $A \leftrightarrow \neg \neg A$

AC2  $A \leftrightarrow (A \land A)$

AC3  $(A \land B) \leftrightarrow (B \land A)$

AC4  $A \land (B \land C) \leftrightarrow (A \land B) \land C$

AC5  $A \lor (B \land C) \leftrightarrow (A \lor B) \land (A \lor C)$

R1   $A \leftrightarrow B, C(A) \Rightarrow C(B)$

Here, $A \lor B$ is shorthand for $\neg (\neg A \land \neg B)$. 
### Angell's Logic of Analytic Containment

<table>
<thead>
<tr>
<th>Rule</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AC1</strong></td>
<td>$A \leftrightarrow \neg \neg A$</td>
</tr>
<tr>
<td><strong>AC2</strong></td>
<td>$A \leftrightarrow (A \land A)$</td>
</tr>
<tr>
<td><strong>AC3</strong></td>
<td>$(A \land B) \leftrightarrow (B \land A)$</td>
</tr>
<tr>
<td><strong>AC4</strong></td>
<td>$A \land (B \land C) \leftrightarrow (A \land B) \land C$</td>
</tr>
<tr>
<td><strong>AC5</strong></td>
<td>$A \lor (B \land C) \leftrightarrow (A \lor B) \land (A \lor C)$</td>
</tr>
<tr>
<td><strong>R1</strong></td>
<td>$A \leftrightarrow B, C(A) \Rightarrow C(B)$</td>
</tr>
</tbody>
</table>

Here, $A \lor B$ is shorthand for $\neg(\neg A \land \neg B)$.

You can define $A \to B$ as $A \leftrightarrow (A \land B)$.
Angell's Logic of Analytic Containment

\textbf{AC1} \quad A \leftrightarrow \neg \neg A \\
\textbf{AC2} \quad A \leftrightarrow (A \land A) \\
\textbf{AC3} \quad (A \land B) \leftrightarrow (B \land A) \\
\textbf{AC4} \quad A \land (B \land C) \leftrightarrow (A \land B) \land C \\
\textbf{AC5} \quad A \lor (B \land C) \leftrightarrow (A \lor B) \land (A \lor C) \\
\textbf{R1} \quad A \leftrightarrow B, C(A) \Rightarrow C(B)

Here, $A \lor B$ is shorthand for $\neg(\neg A \land \neg B)$.

You can define $A \to B$ as $A \leftrightarrow (A \land B)$.

Famously, $A \to (A \lor B)$ is not derivable in Angell’s logic.

We cannot prove $A \leftrightarrow (A \land (A \lor B))$. 

\textcopyright{2023} Greg Restall
Extensions of Angell's Logic

- The first degree fragment of Parry's Logic of Analytic Containment is found by adding \((A \lor (B \land \neg B)) \to A\) to Angell's Logic.
  - Parry's logic still satisfies this relevance constraint: \(A \to B\) is provable only when the atoms in \(B\) are present in \(A\).

- First Degree Entailment (FDE) is found by adding \(A \to (A \lor B)\) to Angell's Logic.
  - FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that \(p \lor \neg p\), and \(q \land \neg q\) are both non-trivial, and ineliminable.
  - A simple translation encodes FDE inside classical logic. Choose, for each atom \(p\), a fresh atom \(p'\), its shadow. For each FDE formula \(A\), its translation is the formula \(A'\) found by replacing the negative occurrences of atoms \(p\) in \(A\) by their shadows. An argument is FDE valid iff its translation is classically valid.
**DEFINITION:** $Mx(A, B)$ is the set of all possible linkings which could occur in any proof from $A$ to $B$. 
**DEFINITION:** $Mx(A, B)$ is the set of all possible linkings which could occur in any proof from $A$ to $B$.

That is, it contains a link from 

\{positive atoms in $A$, negative atoms in $B$\}

to matching \{positive atoms in $B$, negative atoms in $A$\}.
**DEFINITION:** $Mx(A, B)$ is the set of all possible linkings which could occur in any proof from $A$ to $B$.

That is, it contains a link from

\{positive atoms in $A$, negative atoms in $B$\}

to matching \{positive atoms in $B$, negative atoms in $A$\}.

**FACT:** $Mx(A, B)$ is a proof iff there is some proof from $A$ to $B$. 
$Mx(A, B)$ examples

$Mx(p, q)$

$p$

$q$
$Mx(A, B)$ examples

$Mx(p, q)$

$p$

$q$

No links.
\( Mx(A, B) \) examples

\[ Mx(p, q) \]

\[ Mx(p \lor \neg p, p \land \neg q) \]

No links.
$Mx(A, B)$ examples

$Mx(p, q)$

$Mx(p \lor \neg p, p \land \neg q)$

$p$

$q$

$p \lor \neg p$

$p \land \neg q$

No links.

Not a proof.
**Lemma:** If $A \approx_{Mx} B$, then $Mx(A, B)$ and $Mx(B, A)$ are proofs, and $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$
**Lemma**: If \( A \cong_{Mx} B \), then \( Mx(A, B) \) and \( Mx(B, A) \) are proofs, and \( Mx(A, B), Mx(B, A) : A \cong_{Mx} B \).

**Proof**: If \( \pi, \pi' : A \cong_{Mx} B \), then \( \pi \subseteq Mx(A, B) \) and \( \pi' \subseteq Mx(B, A) \), so \( Mx(A, B) \) and \( Mx(B, A) \) are both proofs.
LEMMA: If \( A \cong_{Mx} B \), then \( Mx(A, B) \) and \( Mx(B, A) \) are proofs, and \( Mx(A, B), Mx(B, A) : A \cong_{Mx} B \).

PROOF: If \( \pi, \pi' : A \cong_{Mx} B \), then \( \pi \subseteq Mx(A, B) \) and \( \pi' \subseteq Mx(B, A) \), so \( Mx(A, B) \) and \( Mx(B, A) \) are both proofs.

Since \( \pi' \circ \pi = Mx(A) \), we have
\[
Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A),
\]
and similarly, \( Mx(B) = Mx(A, B) \circ Mx(A) \), so \( Mx(A, B), Mx(B, A) : A \cong_{Mx} B \).
FACT: If A is classically logically equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and vice versa, then A and B Mx-match — and conversely.
Proof

If \( A \) is logically equivalent to \( B \), then \( Mx(A, B) \) and \( Mx(B, A) \) are both proofs. It suffices to show that \( Mx(B, A) \circ Mx(A, B) = Mx(A) \) (and similarly for \( B \)). To show this, we need to show that each positive [negative] occurrence of an atom in \( A \) is linked to any positive [negative] occurrence of that atom in \( A \) by way of some link in \( Mx(A, B) \) composed with a link in \( Mx(B, A) \). But since that atom occurs positively [negatively] also in \( B \) at least once, the links to accomplish this occur in \( Mx(A, B) \) and \( Mx(B, A) \).

Conversely, if \( A \approx_{Mx} B \), we have already seen that \( A \) and \( B \) must be equivalent, and no atom occurs positively [negatively] in \( A \) but not \( B \).
This is *not* Equivalence in Parry's Logic

**FACT:** A is equivalent to B in Parry's logic of analytic containment iff A is classically equivalent to B and the atoms present in A are present in B and *vice versa.*
FACT: A is equivalent to B in Parry’s logic of analytic containment iff
A is classically equivalent to B and the atoms present in A are present in B and vice versa.

\((p \land \neg p) \land q \cong_{Mx} (p \land \neg p) \land \neg q\)

But this pair satisfies Parry’s variable sharing criteron.
Does the equivalence relation of $Mx$-matching occur elsewhere in the literature?
**DEFINITION:** $Hz(A, B)$ is the set of all possible linkings which could occur in any proof from $A$ to $B$, excluding caps and cups.
**DEFINITION:** $Hz(A, B)$ is the set of all possible linkings which could occur in any proof from $A$ to $B$, excluding caps and cups.

That is, it contains a link from positive atoms in $A$ to corresponding positive atoms in $B$ and negative atoms in $A$ to corresponding negative atoms in $B$. 
$Hz(A, B)$ examples

$Hz(p \land \neg p, q \lor \neg q)$
**Hz(A, B) examples**

\[ \text{Hz}(p \land \neg p, q \lor \neg q) \]

\[ p \land \neg p \]

\[ q \lor \neg q \]
Hz(A, B) examples

\[ \neg p \land p \]

\[ Hz(p \land \neg p, q \lor \neg q) \]

\[ q \lor \neg q \]

No links!
$Hz(A, B)$ examples

$p \land \neg p$

$Hz(p \land \neg p, q \lor \neg q)$

$q \lor \neg q$

$Hz(p \land \neg p, p \lor \neg p)$
$Hz(A, B)$ examples

$p \land \neg p$

$Hz(p \land \neg p, q \lor \neg q)$

$q \lor \neg q$

$Hz(p \land \neg p, p \lor \neg p)$

$p \lor \neg p$
$Hz(A, B)$ examples

$p \land \neg p$

$Hz(p \land \neg p, q \lor \neg q)$

$q \lor \neg q$

$Hz(p \land \neg p, p \lor \neg p)$

$p \lor \neg p$

\begin{align*}
\text{No links!} \\
\text{A proof, but not the maximal one.}
\end{align*}
**FACT:** \( Hz(A, B) \) is a proof iff the argument from \( A \) to \( B \) is FDE valid.
FACT: \( Hz(A, B) \) is a proof iff the argument from \( A \) to \( B \) is FDE valid.

- From FDE-validity to \( Hz \)-proof: straightforward induction on an FDE-axiomatisation.
FACT: \( \text{Hz}(A, B) \) is a proof iff the argument from \( A \) to \( B \) is \text{FDE} valid.

- From \text{FDE-validity} to \text{Hz-proof}: straightforward induction on an \text{FDE-axiomatisation}.
- From the \text{Hz-proof} \( \text{Hz}(A, B) \) to \text{FDE-validity}. Notice that no negative occurrences of atoms in \( A \) or \( B \) are linked to any positive occurrences of atoms in \( A \) or \( B \). So, there is another \text{Hz-proof} \( \text{Hz}(A', B') \) for the \text{FDE translations} for \( A \) and \( B \).
LEMMA: If \( A \approx_{Hz} B \), then \( Hz(A, B) \) and \( Hz(B, A) \) are proofs, and \( Hz(A, B), Hz(B, A) : A \approx_{Hz} B \).
Hz(A, B) and Hz-matching

**Lemma:** If \( A \approx_{Hz} B \), then \( Hz(A, B) \) and \( Hz(B, A) \) are proofs, and \( Hz(A, B), Hz(B, A) : A \approx_{Hz} B \)

**Proof:** If \( \pi, \pi' : A \approx_{Hz} B \), then since \( \pi' \circ \pi = Hz(A) \) and \( \pi \circ \pi' = Hz(B) \), \( \pi \) and \( \pi' \) are cap- and cup-free, so \( \pi \subseteq Hz(A, B) \) and \( \pi' \subseteq Hz(B, A) \), so \( Hz(A, B) \) and \( Hz(B, A) \) are both proofs.
LEMMA: If $A \approx_{Hz} B$, then $Hz(A, B)$ and $Hz(B, A)$ are proofs, and $Hz(A, B), Hz(B, A) : A \approx_{Hz} B$

PROOF: If $\pi, \pi' : A \approx_{Hz} B$, then since $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$, $\pi$ and $\pi'$ are cap- and cup-free, so $\pi \subseteq Hz(A, B)$ and $\pi' \subseteq Hz(B, A)$, so $Hz(A, B)$ and $Hz(B, A)$ are both proofs.

Since $\pi' \circ \pi = Hz(A)$, we have
$Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$,
and similarly, $Hz(B) = Hz(A, B) \circ Hz(A)$,
so $Hz(A, B), Hz(B, A) : A \approx_{Mx} B$. 
FACT: If A is fde-equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and vice versa, then A and B Hz-match — and conversely.
Proof

If $A$ is FDE-equivalent to $B$, then $Hz(A, B)$ and $Hz(B, A)$ are both proofs.

It suffices to show that $Hz(B, A) \circ Hz(A, B) = Hz(A)$ (and similarly for $B$). To show this, we need to show that each positive [negative] occurrence of an atom in $A$ is linked to any positive [negative] occurrence of that atom in $A$ by way of some link in $Hz(A, B)$ composed with a link in $Hz(B, A)$. But since that atom occurs positively [negatively] also in $B$ at least once, the links to accomplish this occur in $Hz(A, B)$ and $Hz(B, A)$.

Conversely, if $A \approx_{Hz} B$, we have already seen that $A$ and $B$ must be FDE-equivalent, and no atom occurs positively [negatively] in $A$ but not $B$. 

Greg Restall Isomorphisms in a Category of Proofs ɇɆ of ɈɃ
FACT: (Fine, Ferguson) $A$ is equivalent to $B$ in Angell’s logic of analytic containment iff $A$ is fde equivalent to $B$, and any atom occurs positively [negatively] in $A$ iff it occurs positively [negatively] in $B$. 

$Hz$-matching $\equiv$ Angellic Equivalence
FACT: (Fine, Ferguson) $A$ is equivalent to $B$ in Angell’s logic of analytic containment iff $A$ is FDE equivalent to $B$, and any atom occurs positively [negatively] in $A$ iff it occurs positively [negatively] in $B$.

So, $Hz$-matching is equivalence in Angell’s Logic of Analytic Containment.
MATCHING AS ISOMORPHISM
Hz(A) and Mx(A) are Idempotents

- \( Hz(A) \circ Hz(A) = Hz(A) \), \( Mx(A) \circ Mx(A) = Mx(A) \).
\[Hz(A) \circ Hz(A) = Hz(A), \ Mx(A) \circ Mx(A) = Mx(A).\]

For any category \(C\), if \(i_A\) is an idempotent for each object \(A\), we can form a new category \(C_i\) with the same objects as \(C\), and with arrows \(i_B \circ f \circ i_A : A \to B\).
Hz(A) and Mx(A) are Idempotents

- \( Hz(A) \circ Hz(A) = Hz(A) \), \( Mx(A) \circ Mx(A) = Mx(A) \).

- For any category \( C \), if \( i_A \) is an idempotent for each object \( A \), we can form a new category \( C_i \) with the same objects as \( C \), and with arrows \( i_B \circ f \circ i_A : A \to B \).

- In this new category, the idempotents \( i_A \) are the new identity arrows.
Hz(A) and Mx(A) are Idempotents

- Hz(A) \circ Hz(A) = Hz(A), Mx(A) \circ Mx(A) = Mx(A).

- For any category \( C \), if \( i_A \) is an idempotent for each object \( A \), we can form a new category \( C_i \) with the same objects as \( C \), and with arrows \( i_B \circ f \circ i_A : A \to B \).

- In this new category, the idempotents \( i_A \) are the new identity arrows.

- So, \( C_{Hz} \) and \( C_{Mx} \) are both categories — like \( C \), but less discriminating, with fewer arrows.
\( Hz(A) \) and \( Mx(A) \) are Idempotents

- \( Hz(A) \circ Hz(A) = Hz(A) \), \( Mx(A) \circ Mx(A) = Mx(A) \).

- For any category \( C \), if \( i_A \) is an idempotent for each object \( A \), we can form a new category \( C_i \) with the same objects as \( C \), and with arrows \( i_B \circ f \circ i_A : A \to B \).

- In this new category, the idempotents \( i_A \) are the new identity arrows.

- So, \( C_{Hz} \) and \( C_{Mx} \) are both categories — like \( C \), but less discriminating, with fewer arrows.

- \( Hz \)-matching is isomorphism in \( C_{Hz} \).
\(\text{Hz}(A)\) and \(\text{Mx}(A)\) are Idempotents

- \(\text{Hz}(A) \circ \text{Hz}(A) = \text{Hz}(A), \ \text{Mx}(A) \circ \text{Mx}(A) = \text{Mx}(A)\).

- For any category \(C\), if \(i_A\) is an idempotent for each object \(A\), we can form a new category \(C_i\) with the same objects as \(C\), and with arrows \(i_B \circ f \circ i_A : A \to B\).

- In this new category, the idempotents \(i_A\) are the new identity arrows.

- So, \(\mathcal{C}_\text{Hz}\) and \(\mathcal{C}_\text{Mx}\) are both categories — like \(\mathcal{C}\), but less discriminating, with fewer arrows.

- \(\text{Hz}\)-matching is \textit{isomorphism} in \(\mathcal{C}_\text{Hz}\).

- \(\text{Mx}\)-matching is \textit{isomorphism} in \(\mathcal{C}_\text{Mx}\).
$\mathcal{C}_{Mx}$ and $\mathcal{C}_{Hz}$ are nontrivial, nonetheless

These are each different proofs in $\mathcal{C}_{Mx}$ and $\mathcal{C}_{Hz}$. 
$\mathcal{C}_{Mx}$ and $\mathcal{C}_{Hz}$ are nontrivial, nonetheless

These are each different proofs in $\mathcal{C}_{Mx}$ and $\mathcal{C}_{Hz}$. 
To Conclude

- Proof theoretical resources *for classical logic* provide tools for fine-grained hyperintensional distinctions.
To Conclude

- Proof theoretical resources for classical logic provide tools for fine-grained hyperintensional distinctions.
- Extending these results to include the units $\top$ and $\bot$ are not difficult. (They were left out only to ease the presentation).
To Conclude

- Proof theoretical resources for classical logic provide tools for fine-grained hyperintensional distinctions.
- Extending these results to include the units $\top$ and $\bot$ are not difficult. (They were left out only to ease the presentation).
- Relate these results to models of logics of content.
To Conclude

- Proof theoretical resources for classical logic provide tools for fine-grained hyperintensional distinctions.
- Extending these results to include the units $\top$ and $\bot$ are not difficult. (They were left out only to ease the presentation).
- Relate these results to models of logics of content.
- Extend these results to first order logic, and beyond!
THANK YOU!
Thank you!

SLIDES: http://consequently.org/presentation/

FEEDBACK: @consequently on Twitter, or email at restall@unimelb.edu.au