

$\lambda\mu$

RELATING CONSTRUCTIVE, CLASSICAL  
AND SUBSTRUCTURAL LOGICS

Greg Restall

Arché · Philosophy Department  
University of St Andrews

**HANDOUT** ▷ <https://consequently.org/p/2024/lm1>

Second Workshop on the Objects  
and Grounds of Structural Rules

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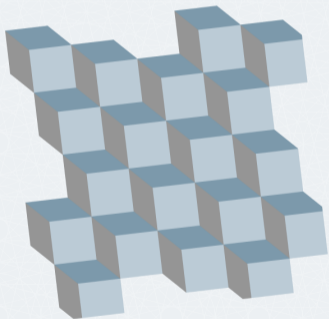


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St Andrews



AN INTRODUCTION TO  
SUBSTRUCTURAL  
LOGICS

GREG RESTALL

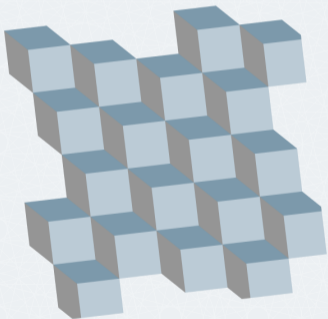


*Substructural Logics*



AN INTRODUCTION TO  
SUBSTRUCTURAL  
LOGICS

GREG RESTALL



*Substructural Logics*

LogicalPluralism

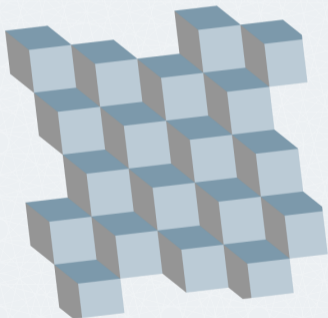
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*Logical Pluralism*



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*Substructural Logics*

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*Logical Pluralism*

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*Philosophy of Proof Theory*

1 • TWO RULES / FOUR LOGICS

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

3 · TRANSLATION / NORMALISATION

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

3 · TRANSLATION / NORMALISATION

4 · MEANINGS



1 • TWO RULES / FOUR LOGICS

$$\frac{\begin{array}{c} [A]^i \\ \Pi \\ B \end{array}}{A \rightarrow B} \rightarrow I^i$$

$$\frac{\begin{array}{c} \Pi \\ A \rightarrow B \end{array} \quad \begin{array}{c} \Pi' \\ A \end{array}}{B} \rightarrow E$$

$$\begin{array}{ccc}
 & & [A]^i \\
 & & \Pi \\
 & & B \\
 & \frac{}{A \rightarrow B} & \rightarrow I^i \\
 A & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \Pi & \Pi' \\
 & A \rightarrow B & A \\
 \frac{}{B} & & \rightarrow E
 \end{array}$$

$$\begin{array}{c}
 \frac{[p \rightarrow (q \rightarrow r)]^3 \quad [p]^1}{q \rightarrow r} \rightarrow E \quad \frac{[p \rightarrow q]^2 \quad [p]^1}{q} \rightarrow E \\
 \frac{\frac{r}{p \rightarrow r} \rightarrow I^1}{(p \rightarrow q) \rightarrow (p \rightarrow r)} \rightarrow I^2 \\
 \frac{(p \rightarrow q) \rightarrow (p \rightarrow r)}{(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))} \rightarrow I^3
 \end{array}$$

$$\begin{array}{c}
\frac{[x : p \rightarrow (q \rightarrow r)] \quad [z : p]}{\quad} \rightarrow_E \quad \frac{[y : p \rightarrow q] \quad [z : p]}{\quad} \rightarrow_E \\
\frac{(xz) : q \rightarrow r \quad (xy) : q}{\quad} \rightarrow_E \\
\frac{\quad}{\lambda z((xz)(xy)) : r} \rightarrow_{I^z} \\
\frac{\lambda z((xz)(xy)) : p \rightarrow r}{\quad} \rightarrow_{I^y} \\
\frac{\lambda y \lambda z((xz)(xy)) : (p \rightarrow q) \rightarrow (p \rightarrow r)}{\quad} \rightarrow_{I^x} \\
\lambda x \lambda y \lambda z((xz)(xy)) : (p \rightarrow (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))
\end{array}$$

$$\begin{array}{c}
 [x : A] \\
 \vdots \\
 M : B \\
 \hline
 \lambda x M : A \rightarrow B \quad \rightarrow I^x
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 M : A \rightarrow B \quad N : A \\
 \hline
 (M N) : B \quad \rightarrow E
 \end{array}$$

$x : A$

$$\begin{array}{c}
 [x : A] \\
 \vdots \\
 M : B \\
 \hline
 x : A \quad \lambda x M : A \rightarrow B \quad \rightarrow I^x
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 M : A \rightarrow B \quad N : A \\
 \hline
 (M N) : B \quad \rightarrow E
 \end{array}$$

You can think of these *terms* as representing processes of *justification* or of *construction*.

(Justify  $A \rightarrow B$  by taking  $A$  as *given*, and using this to justify  $B$ .)

You can *use* a such a justification of  $A \rightarrow B$  by *applying* it to a justification of  $A$  to produce a justification of  $B$ .)

$$\frac{
 \begin{array}{c}
 [x : A] \\
 \vdots \\
 M : B
 \end{array}
 \xrightarrow{\rightarrow I^x}
 \begin{array}{c}
 \vdots \\
 N : A
 \end{array}
 }{
 \begin{array}{c}
 \lambda x M : A \rightarrow B \\
 \hline
 (\lambda x M N) : B
 \end{array}
 \xrightarrow{\rightarrow E}
 \begin{array}{c}
 \vdots \\
 N : A \\
 \vdots \\
 M \{N/x\} : B
 \end{array}
 }
 \triangleright$$



What about this ‘proof’?

$$\frac{\frac{[x : p]}{\lambda y x : q \rightarrow p} \rightarrow I^y}{\lambda x \lambda y x : p \rightarrow (q \rightarrow p)} \rightarrow I^x$$

We never *used* the supposition of  $q$  in the justification of  $p$ , and this is reflected in the term structure: the  $\lambda y$  is *vacuous*.

What about this ‘proof’?

$$\frac{\frac{[x : p]}{\lambda y x : q \rightarrow p} \rightarrow I^y}{\lambda x \lambda y x : p \rightarrow (q \rightarrow p)} \rightarrow I^x$$

We never *used* the supposition of  $q$  in the justification of  $p$ , and this is reflected in the term structure: the  $\lambda y$  is *vacuous*.

We have a choice: *allow* vacuous binding, or *forbid* it.

$$\begin{array}{c}
\frac{[x : p] \quad [y : q]}{\langle x, y \rangle : p \wedge q} \wedge I \\
\frac{\langle x, y \rangle : p \wedge q}{\text{fst} \langle x, y \rangle : p} \wedge E \\
\frac{\text{fst} \langle x, y \rangle : p}{\lambda y \text{fst} \langle x, y \rangle : q \rightarrow p} \rightarrow I^y \\
\frac{\lambda y \text{fst} \langle x, y \rangle : q \rightarrow p}{\lambda x \lambda y \text{fst} \langle x, y \rangle : p \rightarrow (q \rightarrow p)} \rightarrow I^x
\end{array}$$

BEWARE: You must restrict or modify the usual rules for conjunction if you want to forbid vacuous binding, since with  $\text{fst} \langle M, y \rangle$  you can mimic the use of an assumption  $y$  in the otherwise  $y$ -free  $M$ .

$$\begin{array}{c}
\frac{[x : p \rightarrow (q \rightarrow r)] \quad [z : p]}{\quad} \rightarrow_E \quad \frac{[y : p \rightarrow q] \quad [z : p]}{\quad} \rightarrow_E \\
\frac{((xz)(yz)) : r}{\quad} \rightarrow_E \quad \frac{(yz) : q}{\quad} \rightarrow_E \\
\frac{\lambda z((xz)(yz)) : p \rightarrow r}{\quad} \rightarrow_{I^z} \\
\frac{\lambda y \lambda z((xz)(yz)) : (p \rightarrow q) \rightarrow (p \rightarrow r)}{\quad} \rightarrow_{I^y} \\
\frac{\lambda x \lambda y \lambda z((xz)(yz)) : (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))}{\quad} \rightarrow_{I^x}
\end{array}$$

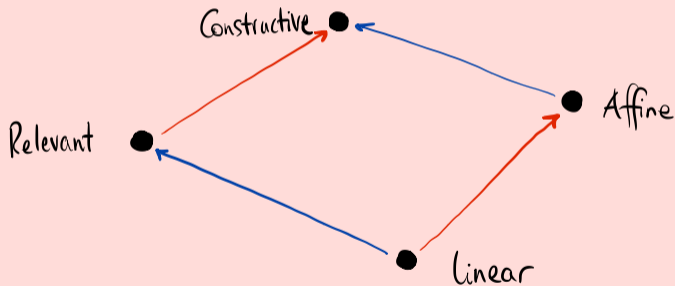
Here we bound *two* instances of  $z$  in one go.

$$\begin{array}{c}
\frac{[x : p \rightarrow (q \rightarrow r)] \quad [z : p]}{\quad} \rightarrow_E \quad \frac{[y : p \rightarrow q] \quad [z : p]}{\quad} \rightarrow_E \\
\frac{((xz)(yz)) : r}{\lambda z((xz)(yz)) : p \rightarrow r} \rightarrow_{I^z} \\
\frac{\lambda y \lambda z((xz)(yz)) : (p \rightarrow q) \rightarrow (p \rightarrow r)}{\lambda x \lambda y \lambda z((xz)(yz)) : (p \rightarrow (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))} \rightarrow_{I^y} \rightarrow_{I^x}
\end{array}$$

Here we bound *two* instances of  $z$  in one go.

There are two options for *duplicate* binding: *allow* and *forbid*.

## The Four Logics



→ Add duplicate discharge:  
**CONTRACTION**

→ Add vacuous discharge:  
**WEAKENING**

We *keep* the  $\rightarrow I$  and  $\rightarrow E$  rules fixed  
and change the *context* in which they apply.

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and change the *context* in which they apply.

Can we extend this analysis to *classical* logic?



In the *sequent* calculus, sort of!

$$\frac{X \succ A \quad B, X' \succ C}{X, A \rightarrow B, X' \succ C} \rightarrow^L \qquad \frac{X, A \succ B}{X \succ A \rightarrow B} \rightarrow^R$$

In the *sequent* calculus, sort of!

$$\frac{X \succ A, Y \quad B, X' \succ Y'}{X, A \rightarrow B, X' \succ Y, Y'} \rightarrow L$$

$$\frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y} \rightarrow R$$

In the *sequent* calculus, sort of!

$$\frac{\succ A \quad B \succ}{A \rightarrow B \succ} \rightarrow L \qquad \frac{A \succ B}{\succ A \rightarrow B} \rightarrow R$$

$\rightarrow L/R$  operate in different *contexts*.

In the *sequent* calculus, sort of!

$$\frac{X \succ A, Y \quad B, X' \succ Y'}{X, A \rightarrow B, X' \succ Y, Y'} \rightarrow_L \qquad \frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y} \rightarrow_R$$

$\rightarrow L/R$  operate in different *contexts*.

The *classical* context allows for more than one *positive* formula occurrence.

In the *sequent* calculus, sort of!

$$\frac{X \succ A \quad B, X' \succ C}{X, A \rightarrow B, X' \succ C} \rightarrow_L \qquad \frac{X, A \succ B}{X \succ A \rightarrow B} \rightarrow_R$$

$\rightarrow L/R$  operate in different *contexts*.

The *classical* context allows for more than one *positive* formula occurrence.

The *constructive* context imposes a tighter restriction.

In the *sequent* calculus, sort of!

$$\frac{X \succ A \quad B, X' \succ C}{X, A \rightarrow B, X' \succ C} \rightarrow_L \qquad \frac{X, A \succ B}{X \succ A \rightarrow B} \rightarrow_R$$

$\rightarrow L/R$  operate in different *contexts*.

The *classical* context allows for more than one *positive* formula occurrence.

The *constructive* context imposes a tighter restriction.

(This is totally independent of contraction and weakening.)

Let's try to do this in a natural deduction setting.

Find a *structural* extension to natural deduction that renders our four *constructive* logics, **classical**.

Let's try to do this in a natural deduction setting.

Find a *structural* extension to natural deduction that renders our four *constructive* logics, **classical**.

(For bonus points, extend the simply typed  $\lambda$  calculus and our understanding of processes of justification or construction.)



1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

***Bilateralism***: assertion and denial are on equal footing.

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*One option*: proofs involve positively tagged formulas ( $+A$ )  
and negatively tagged formulas ( $-A$ ).

$+ \rightarrow \text{-I:}$ 

$$\frac{\text{_____ (i)}}{+ A}$$

$$\vdots$$

$$\frac{+ B \text{ (i)}}{+ (A \rightarrow B)}$$

 $+ \rightarrow \text{-E:}$ 

$$+ (A \rightarrow B)$$

$$\frac{+ A}{+ B}$$

 $- \rightarrow \text{-I:}$ 

$$\frac{+ A \quad - B}{- (A \rightarrow B)}$$

 $- \rightarrow \text{-E:}$ 

$$\frac{- (A \rightarrow B)}{+ A} \quad \frac{- (A \rightarrow B)}{- B}$$

 $+ \neg \text{-I:}$ 

$$\frac{- A}{+ (\neg A)}$$

 $+ \neg \text{-E:}$ 

$$\frac{+ (\neg A)}{- A}$$

 $- \neg \text{-I:}$ 

$$\frac{+ A}{- (\neg A)}$$

 $- \neg \text{-E:}$ 

$$\frac{- (\neg A)}{+ A}$$

$+ \rightarrow \text{-I:}$ 

$$\frac{\begin{array}{c} \text{_____ (i)} \\ + A \\ \vdots \\ + B \end{array}}{\text{_____ (i)}} \\ + (A \rightarrow B)$$

 $+ \rightarrow \text{-E:}$ 

$$\frac{\begin{array}{c} + (A \rightarrow B) \\ \text{_____} \\ + A \end{array}}{+ B}$$

 $- \rightarrow \text{-I:}$ 

$$\frac{\begin{array}{c} + A \quad - B \\ \text{_____} \\ - (A \rightarrow B) \end{array}}$$

 $- \rightarrow \text{-E:}$ 

$$\frac{\begin{array}{c} - (A \rightarrow B) \\ \text{_____} \\ + A \end{array}}$$

$$\frac{\begin{array}{c} - (A \rightarrow B) \\ \text{_____} \\ - B \end{array}}$$

 $+ \neg \text{-I:}$ 

$$\frac{\begin{array}{c} - A \\ \text{_____} \\ + (\neg A) \end{array}}$$

 $+ \neg \text{-E:}$ 

$$\frac{\begin{array}{c} + (\neg A) \\ \text{_____} \\ - A \end{array}}$$

 $- \neg \text{-I:}$ 

$$\frac{\begin{array}{c} + A \\ \text{_____} \\ - (\neg A) \end{array}}$$

 $- \neg \text{-E:}$ 

$$\frac{\begin{array}{c} - (\neg A) \\ \text{_____} \\ + A \end{array}}$$

This is **more** than just a change in the structural context.

The key bilateralist idea is that for  $A, B \succ C, D$

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if  $C$  is the *conclusion* of your proof, then  $D$  is  
part of the context, with *opposite polarity* to  $A$  and  $B$ .

The key bilateralist idea is that for  $A, B \succ \boxed{C}$ ,  $D$   
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*Given  $A$  and  $B$ ,  $C$  follows, unless  $D$ .*



The key bilateralist idea is that for  $A, B \succ \boxed{C}$ , D  
if C is the *conclusion* of your proof, then D is  
part of the context, with *opposite polarity* to A and B.

*Given A and B, C follows, unless D.*

*Granting A and B, and setting D aside, we have C.*

The key bilateralist idea is that for  $A, B \succ \boxed{C}$ , D  
if C is the *conclusion* of your proof, then D is  
part of the context, with *opposite polarity* to A and B.

*Given A and B, C follows, unless D.*

*Granting A and B, and setting D aside, we have C.*

We can use this idea to make a *purely structural* addition  
to natural deduction, keeping our existing rules unchanged.

$$\frac{\begin{matrix} \Pi \\ A \end{matrix}}{\#} \quad \overset{\cancel{A}}{\uparrow}$$

$$\begin{matrix} [\cancel{A}]^i \\ \Pi \\ \# \\ \hline A \end{matrix} \quad \downarrow^i$$

$$\frac{\begin{array}{c} \Pi \\ A \end{array} \quad \cancel{A}}{\#} \uparrow$$

$$\frac{[\cancel{A}]^i}{\#} \downarrow^i$$

$$\frac{X \succ \boxed{A}, Y}{X \succ \square A, Y}$$

$$\frac{\Pi \quad \cancel{A}}{\cancel{A} \quad \uparrow} \#$$

$$\frac{[\cancel{A}]^i \quad \Pi \quad \#}{\cancel{A} \quad \downarrow^i}$$

$$\frac{X \succ \boxed{A}, Y}{X \succ \boxed{\phantom{A}}, Y}$$

$$\frac{\begin{array}{c} \Pi \\ A \end{array} \quad \cancel{A}}{\#} \uparrow$$

$$\begin{array}{c} [\cancel{A}]^i \\ \Pi \\ \# \\ \frac{\quad}{A} \downarrow^i \end{array}$$

$$\frac{X \succ \boxed{A}, Y}{X \succ \square A, Y}$$

$$\frac{X \succ \square A, Y}{X \succ \boxed{A}, Y}$$

$$\frac{\begin{array}{c} \Pi \\ A \end{array} \quad \cancel{A}}{\#} \uparrow$$

$$\begin{array}{c} [\cancel{A}]^i \\ \Pi \\ \# \\ \cancel{A} \end{array} \downarrow^i$$

$$\frac{X \succ \boxed{A}, Y}{X \succ \square A, Y}$$

$$\frac{X \succ \square A, Y}{X \succ \boxed{A}, Y}$$

Add these rules for alternatives.

Keep the connective rules fixed.

*(Employing whichever discharge/binding discipline you prefer.)*

Now you have a classical version of your logic.



$$\begin{array}{c}
 \frac{[p]^1 \quad [p]^2}{\# \downarrow} \uparrow \\
 \frac{[(p \rightarrow q) \rightarrow p]^3 \quad \frac{p \rightarrow q}{\# \downarrow} \rightarrow I^1}{p \rightarrow E} \rightarrow E \quad [p]^2 \\
 \frac{p \rightarrow E \quad [p]^2}{\# \downarrow^2} \uparrow \\
 \frac{p \rightarrow E \quad [p]^2}{((p \rightarrow q) \rightarrow p) \rightarrow p} \rightarrow I^3
 \end{array}$$

$$\begin{array}{c}
 \frac{[p]^1 \quad [p]^2}{\#} \uparrow \\
 \frac{\#}{q} \downarrow \\
 \frac{[(p \rightarrow q) \rightarrow p]^3 \quad \frac{p \rightarrow q}{\rightarrow I^1}}{p \rightarrow q} \rightarrow E}{p} \rightarrow E \quad [p]^2 \\
 \frac{\#}{p} \downarrow^2}{((p \rightarrow q) \rightarrow p) \rightarrow p} \rightarrow I^3
 \end{array}$$

$p \succ \boxed{p}$

$$\begin{array}{c}
 \frac{[p]^1 \quad [p]^2}{\quad} \uparrow \\
 \quad \quad \quad \# \\
 \quad \quad \quad \frac{\overline{q}}{\quad} \downarrow \\
 \frac{[(p \rightarrow q) \rightarrow p]^3 \quad \frac{\overline{p \rightarrow q}}{\quad} \rightarrow I^1}{\quad} \rightarrow E \\
 \frac{\quad}{p} \quad \quad \quad \frac{[p]^2}{\quad} \uparrow \\
 \quad \quad \quad \# \\
 \quad \quad \quad \frac{\overline{p}}{\quad} \downarrow^2 \\
 \frac{\quad}{((p \rightarrow q) \rightarrow p) \rightarrow p} \rightarrow I^3
 \end{array}$$

$p \succ \Box p$

$$\begin{array}{c}
 \frac{[p]^1 \quad [p]^2}{\#} \uparrow \\
 \frac{[(p \rightarrow q) \rightarrow p]^3 \quad \frac{p \rightarrow q}{\#} \downarrow}{p \rightarrow q} \rightarrow I^1 \\
 \frac{p \quad \frac{p \rightarrow q}{\#} \downarrow}{p} \rightarrow E \quad [p]^2 \uparrow \\
 \frac{((p \rightarrow q) \rightarrow p) \rightarrow p}{\#} \downarrow^2 \rightarrow I^3
 \end{array}$$

$p \succ \boxed{q}, p$

$$\begin{array}{c}
 \frac{[(p \rightarrow q) \rightarrow p]^3}{p} \\
 \frac{\frac{\frac{[p]^1 \quad [p]^2}{\#} \uparrow}{q} \downarrow}{p \rightarrow q} \rightarrow I^1}{\rightarrow E} \\
 \frac{p \quad [p]^2}{\#} \uparrow \\
 \frac{\#}{p} \downarrow^2}{((p \rightarrow q) \rightarrow p) \rightarrow p} \rightarrow I^3
 \end{array}$$

$\succ \boxed{p \rightarrow q}, p$

$$\begin{array}{c}
 \frac{[p]^1 \quad [p]^2}{\# \downarrow} \uparrow \\
 \frac{[(p \rightarrow q) \rightarrow p]^3 \quad \frac{p \rightarrow q}{\rightarrow I^1}}{p \rightarrow q} \rightarrow E}{p} \\
 \frac{\# \downarrow^2}{((p \rightarrow q) \rightarrow p) \rightarrow p} \rightarrow I^3
 \end{array}$$

$$(p \rightarrow q) \rightarrow p \succ \boxed{p}, p$$

$$\begin{array}{c}
 \frac{[p]^1 \quad [\cancel{p}]^2}{\# \downarrow} \uparrow \\
 \frac{[(p \rightarrow q) \rightarrow p]^3 \quad \frac{p \rightarrow q}{\rightarrow I^1}}{p \rightarrow q} \rightarrow E \quad [\cancel{p}]^2 \\
 \hline
 p \quad \uparrow \\
 \frac{\# \downarrow^2}{p} \\
 \hline
 ((p \rightarrow q) \rightarrow p) \rightarrow p \rightarrow I^3
 \end{array}$$

$$(p \rightarrow q) \rightarrow p \succ \Box p, p$$

$$\begin{array}{c}
 \frac{[p]^1 \quad [p]^2}{\#} \uparrow \\
 \frac{\#}{q} \downarrow \\
 \frac{[(p \rightarrow q) \rightarrow p]^3 \quad \frac{p \rightarrow q}{\rightarrow I^1}}{p \rightarrow q} \rightarrow E \\
 \frac{p \quad [p]^2}{\#} \uparrow \\
 \frac{\#}{p} \downarrow^2 \\
 \frac{((p \rightarrow q) \rightarrow p) \rightarrow p}{\rightarrow I^3}
 \end{array}$$

$$(p \rightarrow q) \rightarrow p \succ \boxed{p}$$



$$\begin{array}{c}
 \frac{[p]^1 \quad [p]^2}{\#} \uparrow \\
 \frac{\#}{q} \downarrow \\
 \frac{[(p \rightarrow q) \rightarrow p]^3 \quad \frac{p \rightarrow q}{\rightarrow I^1}}{p \rightarrow q} \rightarrow E \quad [p]^2}{p} \uparrow \\
 \frac{\#}{p} \downarrow^2 \\
 \frac{((p \rightarrow q) \rightarrow p) \rightarrow p}{\rightarrow I^3}
 \end{array}$$

$$\succ \boxed{((p \rightarrow q) \rightarrow p) \rightarrow p}$$

$$\begin{array}{c}
 [x : A] \\
 \vdots \\
 M : B \\
 \hline
 \lambda x M : A \rightarrow B \rightarrow I^x
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 M : A \rightarrow B \qquad N : A \\
 \hline
 (MN) : B \rightarrow E
 \end{array}$$

Terms

$$\begin{array}{c}
 [x : A] \\
 \vdots \\
 M : B \\
 \hline
 \lambda x M : A \rightarrow B \quad \rightarrow I^x
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \qquad \qquad \vdots \\
 M : A \rightarrow B \quad N : A \\
 \hline
 (MN) : B \quad \rightarrow E
 \end{array}$$

$$\begin{array}{c}
 \vdots \\
 M : A \quad \alpha : \cancel{A} \\
 \hline
 \langle M | \alpha \rangle : \# \quad \uparrow
 \end{array}$$

Terms

$$\begin{array}{c}
 [x : A] \\
 \vdots \\
 M : B \\
 \hline
 \lambda x M : A \rightarrow B \rightarrow I^x
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 M : A \rightarrow B \qquad N : A \\
 \hline
 (MN) : B \rightarrow E
 \end{array}$$

$$\begin{array}{c}
 \vdots \\
 M : A \quad \alpha : \cancel{A} \\
 \hline
 \langle M | \alpha \rangle : \# \uparrow
 \end{array}$$

Terms      Labels

$$\begin{array}{c}
 [x : A] \\
 \vdots \\
 M : B \\
 \hline
 \lambda x M : A \rightarrow B \rightarrow I^x
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 M : A \rightarrow B \qquad N : A \\
 \hline
 (MN) : B \rightarrow E
 \end{array}$$

$$\begin{array}{c}
 \vdots \\
 M : A \quad \alpha : \cancel{A} \\
 \hline
 \langle M | \alpha \rangle : \# \uparrow
 \end{array}$$

Terms

Labels

Packages

$$\begin{array}{c}
 [x : A] \\
 \vdots \\
 M : B \\
 \hline
 \lambda x M : A \rightarrow B \quad \rightarrow I^x
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 M : A \rightarrow B \quad N : A \\
 \hline
 (MN) : B \quad \rightarrow E
 \end{array}$$

$$\begin{array}{c}
 \vdots \\
 M : A \quad \alpha : \cancel{A} \\
 \hline
 \langle M | \alpha \rangle : \# \quad \uparrow
 \end{array}
 \qquad
 \begin{array}{c}
 [\alpha : \cancel{A}] \\
 \vdots \\
 P : \# \\
 \hline
 \mu \alpha P : A \quad \downarrow^\alpha
 \end{array}$$

Terms

Labels

Packages

$$\begin{array}{c}
\frac{[y : p] \quad [\alpha : p]}{\langle y | \alpha \rangle : \#} \uparrow \\
\frac{\langle y | \alpha \rangle : \#}{\mu\beta \langle y | \alpha \rangle : q} \downarrow \beta \\
\frac{[x : (p \rightarrow q) \rightarrow p] \quad \lambda y \mu\beta \langle y | \alpha \rangle : p \rightarrow q}{\lambda x \mu\beta \langle y | \alpha \rangle : p \rightarrow q} \rightarrow I^y \\
\frac{\lambda x \mu\beta \langle y | \alpha \rangle : p \rightarrow q}{(\lambda x \lambda y \mu\beta \langle y | \alpha \rangle) : p} \rightarrow E \\
\frac{(\lambda x \lambda y \mu\beta \langle y | \alpha \rangle) : p \quad [\alpha : p]}{\langle (\lambda x \lambda y \mu\beta \langle y | \alpha \rangle) | \alpha \rangle : \#} \uparrow \\
\frac{\langle (\lambda x \lambda y \mu\beta \langle y | \alpha \rangle) | \alpha \rangle : \#}{\mu\alpha \langle (\lambda x \lambda y \mu\beta \langle y | \alpha \rangle) | \alpha \rangle : p} \downarrow \alpha \\
\frac{\mu\alpha \langle (\lambda x \lambda y \mu\beta \langle y | \alpha \rangle) | \alpha \rangle : p}{\lambda x \mu\alpha \langle (\lambda x \lambda y \mu\beta \langle y | \alpha \rangle) | \alpha \rangle : ((p \rightarrow q) \rightarrow p) \rightarrow p} \rightarrow I^x
\end{array}$$

Given  $\sharp$ , it makes sense to *use* it to define *negation*:

$\neg A$  is  $A \rightarrow f$ , where  $f$  is the formula representative of the empty conclusion.

$$\frac{P : \sharp}{\mu P : f} \text{ } fI \qquad \frac{M : f}{\langle M \rangle : \sharp} \text{ } fE$$



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$$\frac{\frac{\frac{[x : A] \quad [\alpha : \cancel{A}]}{\langle x | \alpha \rangle : \sharp} \uparrow}{\mu \langle x | \alpha \rangle : f} \text{ fI}}{\lambda x \mu \langle x | \alpha \rangle : A \rightarrow f} \rightarrow I^x}{\frac{\frac{\frac{y : (A \rightarrow f) \rightarrow f}{(y \lambda x \mu \langle x | \alpha \rangle) : f}}{\langle (y \lambda x \mu \langle x | \alpha \rangle) \rangle : \sharp} \text{ fE}}{\mu \alpha \langle (y \lambda x \mu \langle x | \alpha \rangle) \rangle : A} \downarrow^\alpha} \rightarrow E$$

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$$\frac{\frac{\frac{\frac{[x : A] \quad [\alpha : \cancel{A}]}{\langle x | \alpha \rangle : \sharp} \uparrow}{\mu \langle x | \alpha \rangle : f} \text{ fI}}{\lambda x \mu \langle x | \alpha \rangle : A \rightarrow f} \rightarrow I^x}{y : (A \rightarrow f) \rightarrow f \quad \lambda x \mu \langle x | \alpha \rangle : A \rightarrow f} \rightarrow E}{\frac{\frac{\frac{(y \lambda x \mu \langle x | \alpha \rangle) : f}{\langle (y \lambda x \mu \langle x | \alpha \rangle) \rangle : \sharp} \text{ fE}}{\mu \alpha \langle (y \lambda x \mu \langle x | \alpha \rangle) \rangle : A} \downarrow \alpha} \rightarrow E$$

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$$\frac{\frac{\frac{[x : A] \quad [\alpha : \cancel{A}]}{\langle x | \alpha \rangle : \sharp} \uparrow}{\mu \langle x | \alpha \rangle : f} \text{ fI}}{\lambda x \mu \langle x | \alpha \rangle : A \rightarrow f} \rightarrow I^x}{\frac{y : (A \rightarrow f) \rightarrow f \quad \lambda x \mu \langle x | \alpha \rangle : A \rightarrow f}{(y \lambda x \mu \langle x | \alpha \rangle) : f} \rightarrow E} \text{ fE} \quad \frac{\langle (y \lambda x \mu \langle x | \alpha \rangle) \rangle : \sharp}{\mu \alpha \langle (y \lambda x \mu \langle x | \alpha \rangle) \rangle : A} \downarrow^\alpha$$

$$\frac{\frac{\frac{[x : A] \quad [\alpha : \cancel{A}]}{\langle x | \alpha \rangle : \sharp} \uparrow}{\lambda x \mu \langle x | \alpha \rangle : \neg A} \rightarrow I^x}{y : \neg \neg A \quad \lambda x \mu \langle x | \alpha \rangle : \neg A} \rightarrow E} \langle (y \lambda x \mu \langle x | \alpha \rangle) \rangle : \sharp \downarrow^\alpha \quad \mu \alpha \langle (y \lambda x \mu \langle x | \alpha \rangle) \rangle : A$$

$(\lambda x M N) \triangleright M\{N/x\}$

$$\frac{\frac{\frac{[x : A]}{\vdots} M : B}{\lambda x M : A \rightarrow B} \rightarrow I^x \quad \frac{\vdots}{N : A}}{\frac{(\lambda x M N) : B}{M\{N/x\} : B} \rightarrow E} \triangleright \frac{\vdots}{N : A} \quad \frac{\vdots}{M\{N/x\} : B}$$

$(\lambda x M N) \triangleright M\{N/x\}$

$\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$

$$\frac{\begin{array}{c} [\alpha : \mathbf{A}] \\ \vdots \\ P : \# \end{array}}{\mu \alpha P : \mathbf{A}} \downarrow^{\alpha} \quad \beta : \mathbf{A} \quad \triangleright \quad \begin{array}{c} [\beta : \mathbf{A}] \\ \vdots \\ P\{\beta/\alpha\} : \# \end{array}$$
$$\frac{\mu \alpha P : \mathbf{A} \quad \beta : \mathbf{A}}{\langle \mu \alpha P | \beta \rangle : \#} \uparrow$$

$$(\lambda x M N) \triangleright M\{N/x\}$$

$$\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$$

$$\langle \mu P \rangle \triangleright P$$

$$\frac{\frac{\vdots}{P : \#} fI}{\langle \mu P \rangle : \#} fE \triangleright \frac{\vdots}{P : \#}$$

$(\lambda x M N) \triangleright M\{N/x\}$

$\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$

$\langle \mu P \rangle \triangleright P$

$(\mu \alpha P N) \triangleright ???$

$(\mu\alpha P N) \triangleright ???$

$$\frac{\begin{array}{c} [\alpha : \cancel{A} \rightarrow B] \\ \vdots \\ P : \# \\ \hline \mu\alpha P : A \rightarrow B \end{array} \downarrow^{\alpha} \quad \begin{array}{c} \vdots \\ N : A \end{array}}{\mu\alpha P N : B} \rightarrow E$$



$(\mu\alpha P N) \triangleright ???$

$$\frac{\begin{array}{c} \vdots \\ * : A \rightarrow B \quad [\alpha : \cancel{A} \rightarrow B] \end{array}}{\langle * | \alpha \rangle : \#} \uparrow$$
$$\frac{\begin{array}{c} \vdots \\ P : \# \end{array}}{\mu\alpha P : A \rightarrow B} \downarrow^{\alpha} \quad \begin{array}{c} \vdots \\ N : A \end{array}$$
$$\frac{\mu\alpha P : A \rightarrow B \quad N : A}{(\mu\alpha P N) : B} \rightarrow_E$$

$(\mu\alpha P N) \triangleright ???$

$$\begin{array}{c}
 \vdots \\
 * : A \rightarrow B \quad [\alpha : \cancel{A \rightarrow B}] \\
 \hline
 \langle * | \alpha \rangle : \# \\
 \vdots \\
 \frac{P : \#}{\mu\alpha P : A \rightarrow B} \downarrow^{\alpha} \\
 \hline
 \frac{\mu\alpha P : A \rightarrow B \quad N : A}{(\mu\alpha P N) : B} \rightarrow E
 \end{array}$$

$\triangleright$

$$\begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 * : A \rightarrow B \quad N : A \\
 \hline
 (* N) : B \quad \rightarrow E \quad [\beta : \cancel{B}] \\
 \hline
 \langle (* N) | \beta \rangle : \# \\
 \vdots \\
 \frac{P \{ \langle (* N) | \beta \rangle / \langle * | \alpha \rangle \} : \#}{\mu\beta P \{ \langle (* N) | \beta \rangle / \langle * | \alpha \rangle \} : B} \downarrow^{\beta}
 \end{array}$$

$$(\mu\alpha P N) \triangleright \mu\beta P \{ \langle (* N) | \beta \rangle / \langle * | \alpha \rangle \}$$

$$\begin{array}{c}
 \vdots \\
 * : A \rightarrow B \quad [\alpha : \cancel{A \rightarrow B}] \\
 \hline
 \langle * | \alpha \rangle : \# \quad \uparrow \\
 \vdots \\
 P : \# \\
 \hline
 \mu\alpha P : A \rightarrow B \quad \downarrow^\alpha \\
 \hline
 (\mu\alpha P N) : B \quad \rightarrow^E \quad N : A
 \end{array}
 \triangleright
 \begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 * : A \rightarrow B \quad N : A \\
 \hline
 (* N) : B \quad \rightarrow^E \\
 \hline
 \langle (* N) | \beta \rangle : \# \quad \uparrow \\
 [\beta : \cancel{B}] \\
 \vdots \\
 P \{ \langle (* N) | \beta \rangle / \langle * | \alpha \rangle \} : \# \\
 \hline
 \mu\beta P \{ \langle (* N) | \beta \rangle / \langle * | \alpha \rangle \} : B \quad \downarrow^\beta
 \end{array}$$

$$(\mu\alpha P N) \triangleright \mu P \{ \langle (* N) \rangle / \langle * | \alpha \rangle \}$$

$$\begin{array}{c}
 \vdots \\
 * : A \rightarrow f \quad [\alpha : \cancel{A} \rightarrow f] \\
 \hline
 \langle * | \alpha \rangle : \# \\
 \vdots \\
 P : \# \\
 \hline
 \mu\alpha P : A \rightarrow f \quad \downarrow^{\alpha} \\
 \hline
 (\mu\alpha P N) : f \quad \rightarrow E
 \end{array}
 \triangleright
 \begin{array}{c}
 \vdots \qquad \qquad \qquad \vdots \\
 * : A \rightarrow f \quad N : A \\
 \hline
 (* N) : f \\
 \hline
 \langle (* N) \rangle : \# \quad fE \\
 \vdots \\
 P \{ \langle (* N) \rangle / \langle * | \alpha \rangle \} : \# \\
 \hline
 \mu P \{ \langle (* N) \rangle / \langle * | \alpha \rangle \} : f \quad fI
 \end{array}$$

$(\lambda x M N) \triangleright M\{N/x\}$

$\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$

$\langle \mu P \rangle \triangleright P$

$(\mu \alpha P N) \triangleright \mu \beta P\{\langle (* N) | \beta \rangle / \langle * | \alpha \rangle\}$  (if not type f)

$\triangleright \mu P\{\langle (* N) \rangle / \langle * | \alpha \rangle\}$  (if type f)

$$\begin{array}{c}
\frac{x : p \rightarrow (p \rightarrow q) \quad [y : p]}{(xy) : p \rightarrow q} \rightarrow^E \quad [\alpha : \cancel{p \rightarrow q}] \\
\frac{\langle (xy) | \alpha \rangle : \#}{\lambda y \mu \langle (xy) | \alpha \rangle : \neg p} \neg^I y \quad [\beta : \cancel{p}]! \\
\frac{\langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle : \#}{\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle : p \rightarrow q} \downarrow^\alpha \quad [z : p] \\
\frac{(\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) : q}{(\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) : q} \rightarrow^E \quad [\gamma : \cancel{q}] \\
\frac{\langle (\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) | \gamma \rangle : \#}{\lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) | \gamma \rangle : \neg p} \neg^I z \quad [\beta : \cancel{p}]! \\
\frac{\langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle : \#}{\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle : \neg p} \downarrow^{\beta!!} \quad [w : p] \\
\frac{\langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle w) \rangle : \#}{\mu \gamma \langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle w) \rangle : q} \downarrow^\gamma \\
\frac{\lambda w \mu \gamma \langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle w) \rangle : q}{\lambda w \mu \gamma \langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (xy) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle w) \rangle : p \rightarrow q} \rightarrow^{I^w}
\end{array}$$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda \omega \mu \gamma \langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (x y) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle \omega) \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (x y) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle w) \rangle$



$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $\langle \mu \alpha P N \rangle \triangleright \dots$

$\lambda \omega \mu \gamma \langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (x y) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle \omega) \rangle \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $\langle \mu \alpha P N \rangle \triangleright \dots$

$\lambda w \mu \gamma \langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (x y) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle w) \rangle \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $\langle \mu \alpha P N \rangle \triangleright \dots$

$\lambda w \mu \gamma \langle (\mu \beta \langle \lambda z \mu \langle (\mu \alpha \langle \lambda y \mu \langle (x y) | \alpha \rangle | \beta \rangle z) | \gamma \rangle | \beta \rangle w) \rangle \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \boxed{\dots}$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle (\mu \alpha \langle (\lambda y \mu \langle (x y) | \alpha \rangle w) \rangle z) | \gamma \rangle w) \rangle \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle (\mu \alpha \langle (\lambda y \mu \langle (x y) | \alpha \rangle w) \rangle z) | \gamma \rangle w) \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle (\mu \alpha \langle (\lambda y \mu \langle (x y) | \alpha \rangle w) \rangle z) | \gamma \rangle w) \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda \omega \mu \gamma \langle (\lambda z \mu \langle \mu \alpha \langle \mu \langle (x \omega) | \alpha \rangle \rangle z) | \gamma \rangle \omega \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle \mu \alpha \langle \mu \langle (x w) | \alpha \rangle \rangle z) | \gamma \rangle w \rangle$



$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright \boxed{P}$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle (\mu \alpha \langle (x w) | \alpha \rangle z) | \gamma \rangle w) \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $\langle \mu \alpha P N \rangle \triangleright \dots$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle \mu \alpha \langle (x w) | \alpha \rangle z \rangle | \gamma \rangle w) \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \boxed{\dots}$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle \mu \delta \langle ((xw)z) | \delta \rangle | \gamma \rangle w) \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle \mu \delta \langle ((xw)z) | \delta \rangle | \gamma \rangle w) \rangle$

$(\lambda x M N) \triangleright M \{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P \{ \beta / \alpha \}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle (\lambda z \mu \langle ((xw)z) | \gamma \rangle w) \rangle$

$$(\lambda x M N) \triangleright M\{N/x\} \quad \langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\} \quad \langle \mu P \rangle \triangleright P \quad (\mu \alpha P N) \triangleright \dots$$

$$\lambda w \mu \gamma \langle (\lambda z \mu \langle ((xw)z) | \gamma \rangle w) \rangle$$

$$(\lambda x M N) \triangleright M\{N/x\} \quad \langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\} \quad \langle \mu P \rangle \triangleright P \quad (\mu \alpha P N) \triangleright \dots$$

$$\lambda w \mu \gamma \langle \mu \langle ((xw)w) | \gamma \rangle \rangle$$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle \mu \langle ((xw)w) | \gamma \rangle \rangle$



$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright \boxed{P}$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle ((xw)w) | \gamma \rangle$

$(\lambda x M N) \triangleright M\{N/x\}$      $\langle \mu \alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$      $\langle \mu P \rangle \triangleright P$      $(\mu \alpha P N) \triangleright \dots$

$\lambda w \mu \gamma \langle ((xw)w) | \gamma \rangle$

$$\begin{array}{c}
x : p \rightarrow (p \rightarrow q) \quad [w : p] \\
\hline
(xw) : p \rightarrow q \quad [w : p] \xrightarrow{E} \\
\hline
((xw)w) : q \quad [\gamma : \mathbf{q}] \xrightarrow{E} \uparrow \\
\hline
\langle ((xw)w) | \gamma \rangle : \# \\
\hline
\mu\gamma \langle ((xw)w) | \gamma \rangle : q \quad \downarrow \gamma \\
\hline
\lambda w \mu \gamma \langle ((xw)w) | \gamma \rangle : p \rightarrow q \quad \rightarrow I^w
\end{array}$$

$$\begin{array}{c}
x : p \rightarrow (p \rightarrow q) \quad [w : p] \\
\hline
(xw) : p \rightarrow q \quad [w : p] \xrightarrow{E} \\
\hline
((xw)w) : q \quad [\gamma : \mathbf{q}] \xrightarrow{E} \uparrow \\
\hline
\langle ((xw)w) | \gamma \rangle : \# \\
\hline
\mu\gamma \langle ((xw)w) | \gamma \rangle : q \quad \downarrow \gamma \\
\hline
\lambda w \mu\gamma \langle ((xw)w) | \gamma \rangle : p \rightarrow q \xrightarrow{I^w}
\end{array}$$

$$\begin{array}{c}
M : A \rightarrow B \quad x : A \\
\hline
(Mx) : B \xrightarrow{E} \\
\hline
\lambda x (Mx) : A \rightarrow B \xrightarrow{I^x}
\end{array}$$

$$\begin{array}{c}
x : p \rightarrow (p \rightarrow q) \quad [w : p] \\
\hline
(xw) : p \rightarrow q \quad [w : p] \xrightarrow{E} \\
\hline
((xw)w) : q \quad [\gamma : \mathbf{q}] \xrightarrow{E} \uparrow \\
\hline
\langle ((xw)w) | \gamma \rangle : \# \\
\hline
\mu\gamma \langle ((xw)w) | \gamma \rangle : q \quad \downarrow \gamma \\
\hline
\lambda w \mu \gamma \langle ((xw)w) | \gamma \rangle : p \rightarrow q \xrightarrow{I^w}
\end{array}$$

$$\begin{array}{c}
M : A \rightarrow B \quad x : A \\
\hline
(Mx) : B \xrightarrow{E} \\
\hline
\lambda x (Mx) : A \rightarrow B \xrightarrow{I^x}
\end{array}$$

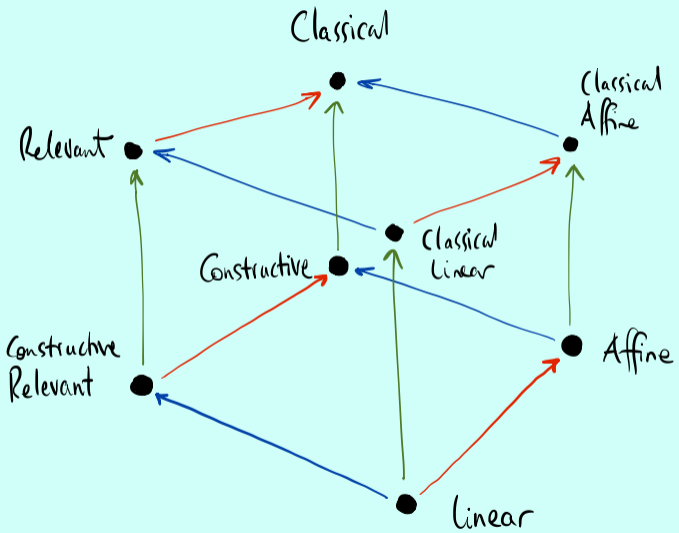
 $\triangleright_{\eta}$ 
 $M : A \rightarrow B$

$$\begin{array}{c}
\frac{x : p \rightarrow (p \rightarrow q) \quad [w : p]}{(xw) : p \rightarrow q} \rightarrow E \\
\frac{\frac{(xw) : p \rightarrow q \quad [w : p]}{((xw)w) : q} \rightarrow E \quad [\gamma : \cancel{q}]}{\langle ((xw)w) | \gamma \rangle : \#} \uparrow \triangleright_{\eta} \\
\frac{\langle ((xw)w) | \gamma \rangle : \#}{\mu \gamma \langle ((xw)w) | \gamma \rangle : q} \downarrow^{\gamma} \\
\frac{\mu \gamma \langle ((xw)w) | \gamma \rangle : q}{\lambda w \mu \gamma \langle ((xw)w) | \gamma \rangle : p \rightarrow q} \rightarrow I^w
\end{array}$$

$$\begin{array}{c}
\frac{x : p \rightarrow (p \rightarrow q) \quad [w : p]}{(xw) : p \rightarrow q} \rightarrow E \\
\frac{(xw) : p \rightarrow q \quad [w : p]}{((xw)w) : q} \rightarrow E \\
\frac{((xw)w) : q}{\lambda w ((xw)w) : p \rightarrow q} \rightarrow I^w
\end{array}$$

$$\begin{array}{c}
\frac{M : A \rightarrow B \quad x : A}{(Mx) : B} \rightarrow E \\
\frac{(Mx) : B}{\lambda x (Mx) : A \rightarrow B} \rightarrow I^x \\
\triangleright_{\eta}
\end{array}$$

$M : A \rightarrow B$



→ Add  $M$   
ALTERNATIVES

→ Add duplicate discharge:  
CONTRACTION

→ Add vacuous discharge:  
WEAKENING

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

3 · TRANSLATION / NORMALISATION



It's well known that classical logic  
can be found inside intuitionistic logic  
using *double negation* translations.

It's well known that classical logic  
can be found inside intuitionistic logic  
using *double negation* translations.

Classical logic is found inside intuitionistic logic  
not only at the level of *provability*, but also  
at the level of *proofs*, and even at the level  
of proof dynamics—*normalisation*.

It's well known that classical logic can be found inside intuitionistic logic using *double negation* translations.

Classical logic is found inside intuitionistic logic not only at the level of *provability*, but also at the level of *proofs*, and even at the level of proof dynamics—*normalisation*.

These results are *robust*. They extend to all four structural settings, linear, relevant, affine, and full.

We can translate a *classical* logic  
(either linear, relevant, affine or full)  
inside its constructive counterpart.

FIRST: For formulas,  $\sharp$  and ~~slashed~~ formulas, add  
to the *constructive* language a fresh atom  $q$ , and set:

$$\begin{array}{lll} \bar{\sharp} = q & \bar{p} = p \rightarrow q & \overline{\cancel{A \rightarrow B}} = (\bar{A} \rightarrow \bar{B}) \rightarrow q \\ \bar{f} = q & \bar{p} = \cancel{p} \rightarrow q & \overline{A \rightarrow B} = (\overline{\cancel{A \rightarrow B}}) \rightarrow q \end{array}$$

We can translate a *classical* logic  
(either linear, relevant, affine or full)  
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FIRST: For formulas,  $\sharp$  and ~~slashed~~ formulas, add  
to the *constructive* language a fresh atom  $q$ , and set:

$$\begin{array}{lll} \bar{\sharp} = q & \bar{\cancel{p}} = \neg_q p & \overline{\cancel{A \rightarrow B}} = \neg_q (\bar{A} \rightarrow \bar{B}) \\ \bar{f} = q & \bar{p} = \neg_q \cancel{\bar{p}} & \overline{A \rightarrow B} = \neg_q (\overline{\cancel{A \rightarrow B}}) \end{array}$$

SECOND: for terms variables and labels,  
whenever  $x : A$ , we choose a unique variable  $\bar{x} : \bar{A}$ ,  
and whenever  $\alpha : \cancel{A}$ , we choose a unique variable  $\bar{\alpha} : \bar{\cancel{A}}$ .

We extend this translation to all terms and packages as follows . . .

$$\overline{\lambda x N} = \lambda y (y \lambda \bar{x} \bar{N})$$

$$\overline{(M N)} = \lambda z (\bar{M} \lambda y ((y \bar{N}) z))$$

$$\overline{\langle M | \alpha \rangle} = (\bar{M} \bar{\alpha})$$

$$\overline{\mu \alpha P} = \lambda \bar{\alpha} \bar{P}$$

$$\overline{\mu P} = \bar{P}$$

$$\overline{\langle N \rangle} = \bar{N}$$

$$\begin{aligned}
\overline{\lambda x N} &= \lambda y (y \lambda \bar{x} \bar{N}) \\
\overline{(M N)} &= \lambda z (\bar{M} \lambda y ((y \bar{N}) z)) \\
\overline{\langle M | \alpha \rangle} &= (\bar{M} \bar{\alpha}) \\
\overline{\mu \alpha P} &= \lambda \bar{\alpha} \bar{P} \\
\overline{\mu P} &= \bar{P} \\
\overline{\langle N \rangle} &= \bar{N}
\end{aligned}$$

$$\frac{[y : \neg_q(\bar{A} \rightarrow \bar{B})] \quad \frac{[\bar{x} : \bar{A}] \quad \vdots \quad \bar{N} : \bar{B}}{\lambda \bar{x} \bar{N} : \bar{A} \rightarrow \bar{B}}}{(y \lambda \bar{x} \bar{N}) : q} \quad \frac{[y : \neg_q(\bar{A} \rightarrow \bar{B})] \quad \lambda \bar{x} \bar{N} : \bar{A} \rightarrow \bar{B}}{\lambda y (y \lambda \bar{x} \bar{N}) : \neg_q \neg_q(\bar{A} \rightarrow \bar{B})}}$$



$$\begin{array}{l}
 \overline{\lambda x N} = \lambda y (y \lambda \bar{x} \bar{N}) \\
 \overline{(M N)} = \lambda z (\overline{M} \lambda y ((y \bar{N}) z)) \\
 \overline{\langle M | \alpha \rangle} = (\overline{M} \bar{\alpha}) \\
 \overline{\mu \alpha P} = \lambda \bar{\alpha} \bar{P} \\
 \overline{\mu P} = \bar{P} \\
 \overline{\langle N \rangle} = \bar{N}
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 \frac{[y : \bar{A} \rightarrow \bar{B}] \quad \bar{N} : \bar{A}}{(y \bar{N}) : \bar{B}} \quad [z : \bar{B}] \\
 \frac{((y \bar{N}) z) : q}{\lambda y ((y \bar{N}) z) : \neg_q (\bar{A} \rightarrow \bar{B})} \\
 \frac{\overline{M} : \neg_q \neg_q (\bar{A} \rightarrow \bar{B}) \quad \lambda y ((y \bar{N}) z) : q}{\lambda z (\overline{M} \lambda y ((y \bar{N}) z)) : q} \\
 \frac{\lambda z (\overline{M} \lambda y ((y \bar{N}) z)) : q}{\lambda z (\overline{M} \lambda y ((y \bar{N}) z)) : \bar{B}}
 \end{array}$$

$$\begin{aligned}
\overline{\lambda x N} &= \lambda y (y \lambda \bar{x} \bar{N}) \\
\overline{(M N)} &= \lambda z (\bar{M} \lambda y ((y \bar{N}) z)) \\
\overline{\langle M | \alpha \rangle} &= (\bar{M} \bar{\alpha}) \\
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\overline{\mu P} &= \bar{P} \\
\overline{\langle N \rangle} &= \bar{N}
\end{aligned}$$

$$\begin{array}{c} \vdots \\ \frac{\bar{M} : \bar{A} \quad \bar{\alpha} : \bar{A}}{(\bar{M} \bar{\alpha}) : \mathfrak{q}} \end{array}$$

$$\begin{aligned}
\overline{\lambda x N} &= \lambda y (y \lambda \bar{x} \bar{N}) \\
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\overline{\langle N \rangle} &= \bar{N}
\end{aligned}$$

$$\frac{\begin{array}{c} \vdots \\ \bar{M} : \bar{A} \end{array} \quad \bar{\alpha} : \bar{A}}{(\bar{M} \bar{\alpha}) : q} \quad \frac{\begin{array}{c} [\bar{\alpha} : \bar{A}] \\ \vdots \\ \bar{P} : q \end{array}}{\lambda \bar{\alpha} \bar{P} : \neg_q \bar{A}}$$

$$\begin{aligned}
\overline{\lambda x N} &= \lambda y (y \lambda \bar{x} \bar{N}) \\
\overline{(M N)} &= \lambda z (\bar{M} \lambda y ((y \bar{N}) z)) \\
\overline{\langle M | \alpha \rangle} &= (\bar{M} \bar{\alpha}) \\
\overline{\mu \alpha P} &= \lambda \bar{\alpha} \bar{P} \\
\overline{\mu P} &= \bar{P} \\
\overline{\langle N \rangle} &= \bar{N}
\end{aligned}$$

$$\begin{array}{c}
\vdots \\
\frac{\bar{M} : \bar{A} \quad \bar{\alpha} : \bar{A}}{(\bar{M} \bar{\alpha}) : q} \\
\vdots \\
\frac{\bar{P} : q}{\lambda \bar{\alpha} \bar{P} : \neg_q \bar{A}} \\
\vdots \\
\frac{\bar{P} : q}{\overline{\mu P} : q} \quad \frac{\bar{N} : q}{\overline{\langle N \rangle} : q}
\end{array}$$

$$\begin{array}{l}
\overline{\lambda x N} = \lambda y (y \lambda \bar{x} \bar{N}) \\
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\overline{\mu P} = \bar{P} \\
\overline{\langle N \rangle} = \bar{N}
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\frac{\bar{M} : \bar{A} \quad \bar{\alpha} : \bar{A}}{(\bar{M} \bar{\alpha}) : q} \\
\vdots \\
\frac{\bar{P} : q}{\lambda \bar{\alpha} \bar{P} : \neg_q \bar{A}} \\
\vdots \\
\frac{\bar{P} : q}{\overline{\mu P} : q} \qquad \frac{\bar{N} : q}{\overline{\langle N \rangle} : q}
\end{array}
\qquad
\begin{array}{c}
[\bar{\alpha} : \bar{A}] \\
\vdots \\
\frac{\bar{P} : q}{\lambda \bar{\alpha} \bar{P} : \neg_q \bar{A}}
\end{array}$$

This translation sends a classical proof of  $A$  to a constructive proof of  $\bar{A}$ .

NOTICE: If source terms are linear (or relevant, or affine), so are their translations.

$$(\lambda x M N) \triangleright M \{N/x\}$$

$$\begin{aligned} \overline{(\lambda x M N)} &= \lambda z (\overline{\lambda x M} \lambda w ((w \overline{N}) z)) \\ &= \lambda z (\lambda v (v \lambda \overline{x M}) \lambda w ((w \overline{N}) z)) \\ &\triangleright \lambda z (\lambda w ((w \overline{N}) z) \lambda \overline{x M}) \\ &\triangleright \lambda z ((\lambda \overline{x M} \overline{N}) z) \\ &\triangleright_{\eta} (\lambda \overline{x M} \overline{N}) \\ &\triangleright \overline{M} \{N/\overline{x}\} \\ &= \overline{M \{N/x\}} \end{aligned}$$

$$\langle \mu\alpha P | \beta \rangle \triangleright P\{\beta/\alpha\}$$

$$\begin{aligned} \overline{\langle \mu\alpha P | \beta \rangle} &= (\overline{\mu\alpha P} \overline{\beta}) \\ &= (\lambda \overline{\alpha} \overline{P} \overline{\beta}) \\ &\triangleright \overline{P}\{\overline{\beta}/\overline{\alpha}\} \\ &= \overline{P\{\beta/\alpha\}} \end{aligned}$$

$$\langle \mu P \rangle \triangleright P$$

$$\overline{\langle \mu P \rangle} = \overline{P}$$



$$(\mu\alpha P N) \triangleright \mu\beta P \{ \langle (*N) | \beta \rangle / \langle * | \alpha \rangle \}$$

$$\overline{(\mu\alpha P N)} = \lambda y (\overline{\mu\alpha P} \lambda x ((x \bar{N}) y))$$

$$= \lambda y (\lambda \bar{\alpha} \bar{P} \lambda x ((x \bar{N}) y))$$

$$\triangleright \lambda y \bar{P} \{ \lambda x ((x \bar{N}) y) / \bar{\alpha} \}$$

$$\triangleright \lambda y \bar{P} \{ \langle (*N) y \rangle / \langle * | \bar{\alpha} \rangle \}$$

$$= \lambda \bar{\beta} \bar{P} \{ \langle \langle (*N) | \beta \rangle / \langle * | \alpha \rangle \rangle \}$$

$$= \overline{\mu\beta P \{ \langle \langle (*N) | \beta \rangle / \langle * | \alpha \rangle \rangle \}}$$

All the behaviour of *classical* proof  
lives inside *constructive* proof,  
for formulas of the form  $\overline{A}$ .

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

3 · TRANSLATION / NORMALISATION

4 · MEANINGS

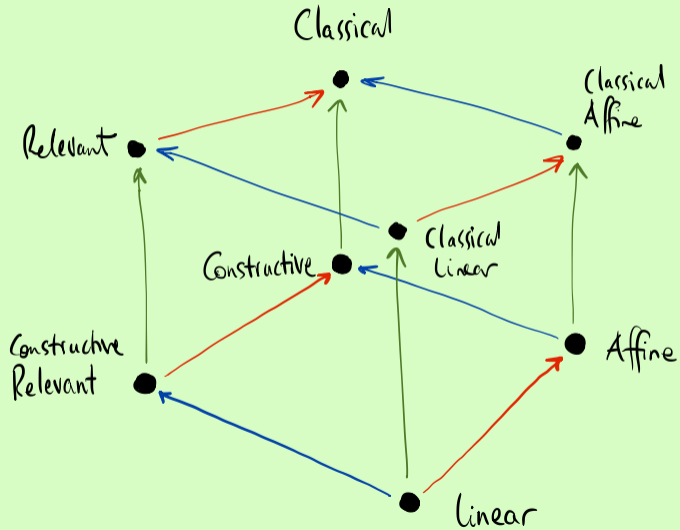
What does this mean for the relationship  
between classical and constructive reasoning?

To see how some of the most basic results of classical analysis lack computational meaning, take the assertion that every bounded non-void set  $A$  of real numbers has a least upper bound. (The real number  $b$  is the *least upper bound* of  $A$  if  $a \leq b$  for all  $a$  in  $A$ , and if there exist elements of  $A$  that are arbitrarily close to  $b$ .) To avoid unnecessary complications, we actually consider the somewhat less general assertion that every bounded sequence  $(x_k)$  of rational numbers has a least upper bound  $b$  (in the set of real numbers). If this assertion were constructively valid, we could compute  $b$ , in the sense of computing a rational number approximating  $b$  to within any desired accuracy; in fact, we could program a digital computer to compute the approximations for us. For instance, the computer could be programmed to produce, one by one, a sequence  $((b_k, m_k))$  of ordered pairs, where each  $b_k$  is a rational number and each  $m_k$  is a positive integer, such that  $x_j \leq b_k + k^{-1}$  for all positive integers  $j$  and  $k$ , and  $x_{m_k} \geq b_k - k^{-1}$  for all positive integers  $k$ . Unless there exists a general method  $M$  that produces such a computer program corresponding to each bounded, constructively given sequence  $(x_k)$  of rational numbers, we are not justified, by constructive standards, in asserting that each of the se-

### PERSPECTIVE #1:

Classical reasoning *extends* constructive reasoning.

There are statements which can be *proved* classically  
that *cannot* be proved constructively.



The law of the excluded middle provides a prime example. Constructively, this principle is not universally valid, as we have seen in Exercise 12.1. Classically, however, it is valid, because every proposition is either false or not false, and being not false is the same as being true. Nevertheless, classical logic is consistent with constructive logic in that constructive logic does not refute classical logic. As we have seen, constructive logic proves that the law of the excluded middle is positively not refuted (its double negation is constructively true). Consequently, **constructive logic is stronger (more expressive) than classical logic, because it can express more distinctions (namely, between affirmation and irrefutability), and because it is consistent with classical logic.**

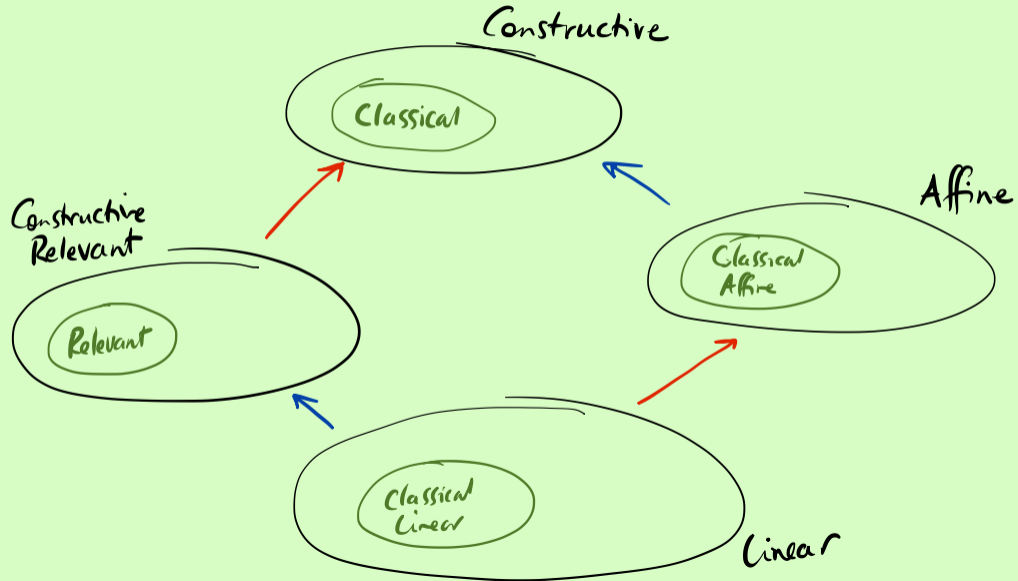
Proofs in constructive logic have computational content: they can be executed as programs, and their behavior is described by their type. Proofs in classical logic also have computational content, but in a weaker sense than in constructive logic. Rather than positively affirm a proposition, a proof in classical logic is a computation that cannot be refuted. Computationally, a refutation consists of a continuation, or control stack, that takes a proof of a proposition and derives a contradiction from it. So a proof of a proposition in classical logic is a computation that, when given a refutation of that proposition derives a contradiction, witnessing the impossibility of refuting it. In this sense, the law of the excluded middle has a proof, precisely because it is irrefutable.



## PERSPECTIVE #2:

Constructive language *extends* classical language.

There are things we can *state* constructively  
that *cannot* be stated classically.



Which of these pictures is *correct*?

Which of these pictures is *correct*?

It depends on what you *mean*.

Which of these pictures is *correct*?

It depends on what you *mean*.

That is, it depends on how you individuate the claims we make in our reasoning—the things that have meaning.

We usually take this as *given*: we have one field of statements, and classical and constructive mathematicians argue about which statements in that field are correct.

“Take ***the assertion*** *that every bounded non-void set  $A$  of real numbers has a least upper bound . . .*”

This fits the picture of classical logic as an extension of constructive logic, allowing for more proofs.

If you take it that propositional content is determined by what *norms* govern it, then the usual picture is not the *only* one.

Constructive justification is *stricter* than classical justification.

Since there are fewer ways to give constructive justification, you can do more with such a justification when you have one.

CLASSICALLY: to state something is to rule something out, in that if you and I *rule out* the same things, we have *said* the same thing.

CONSTRUCTIVELY:  $p$  and  $\neg\neg p$  *rule out* the same things, but they might (constructively) entail *different* things.



PERSPECTIVE #2A: The constructive distinction between  $p$  and  $\neg\neg p$  is a meaningful difference in what is *said*.

The classical logician erases or ignores differences that are present in propositional content.

PERSPECTIVE #2B: The constructive distinction between  $p$  and  $\neg\neg p$  is not a difference in propositional content.

If we allow only constructive justification, we are in a wider field of *pre*-propositions, only some of which are governed by all the norms that determine propositional content.

Our *formal* results are consistent with  
PERSPECTIVES #1, #2A and #2B.

Take time to *recognise* these different perspectives,  
and *learn* what is involved in taking up each stance.

Thank You!