MODELS FOR IDENTITY in three-valued logics

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1. THREE VIEWS OF $\{0, i, 1\}$ – K3, LP, ST
2. K3, LP, ST & CLASSICAL SEQUENT CALCULUS
3. IDENTITY in K3 & LP
4. SEQUENT RULES for IDENTITY
5. THE VARIETY OF $ST_=$ MODELS

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5. THE VARIETY OF $ST_=$ MODELS

$[\![\neg A]\!]_\alpha = 1$ iff $[\![A]\!]_\alpha = 0$

$[\![\neg A]\!] = 0$ iff $[\![A]\!]_\alpha = 1$

$[\![A \wedge B]\!]_\alpha = 1$ iff $[\![A]\!]_\alpha = 1$ & $[\![B]\!]_\alpha = 1$

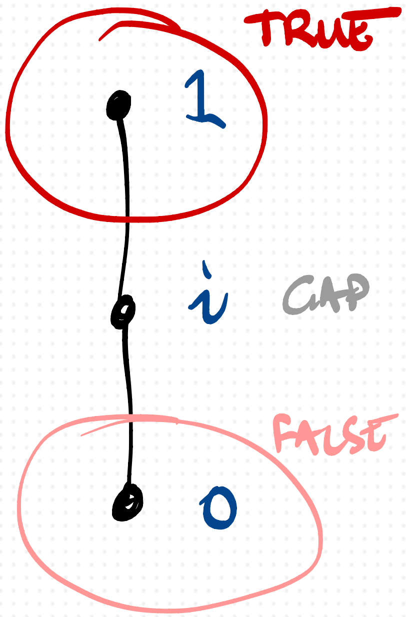
$[\![A \wedge B]\!]_\alpha = 0$ iff $[\![A]\!]_\alpha = 0$ or $[\![B]\!]_\alpha = 0$

$[\![A \vee B]\!]_\alpha = 1$ iff $[\![A]\!]_\alpha = 1$ or $[\![B]\!]_\alpha = 1$

$[\![A \vee B]\!]_\alpha = 0$ iff $[\![A]\!]_\alpha = 0$ & $[\![B]\!]_\alpha = 0$

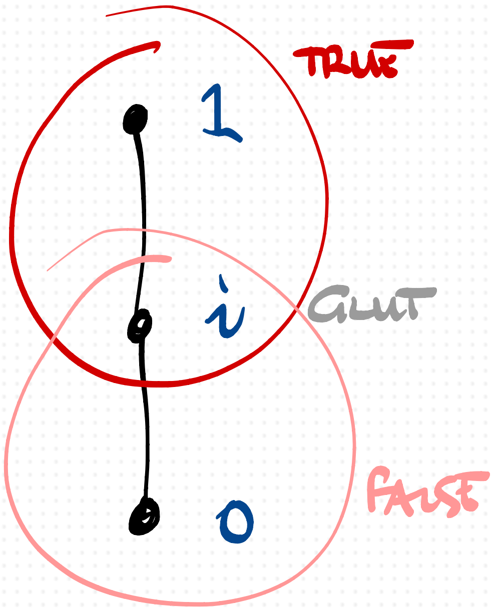
$[\![\forall x A]\!]_\alpha = 1$ iff $[\![A]\!]_{\alpha'}$ for every x-variant α' of α.

$[\![\forall x A]\!]_\alpha = 0$ iff $[\![A]\!]_{\alpha'}$ for some x-variant α' of α.



$A \models_{K3} B$ iff for every interpretation $[\![\cdot]\!]$,
if $[\![A]\!]_\alpha = 1$ then $[\![B]\!]_\alpha = 1$,

ie, there is no $[\![\cdot]\!]$ where $[\![A]\!]_\alpha = 1$ & $[\![B]\!]_\alpha \neq 1$.



$A \vDash_{K3} B$ iff for every interpretation $[\![\cdot]\!]$,

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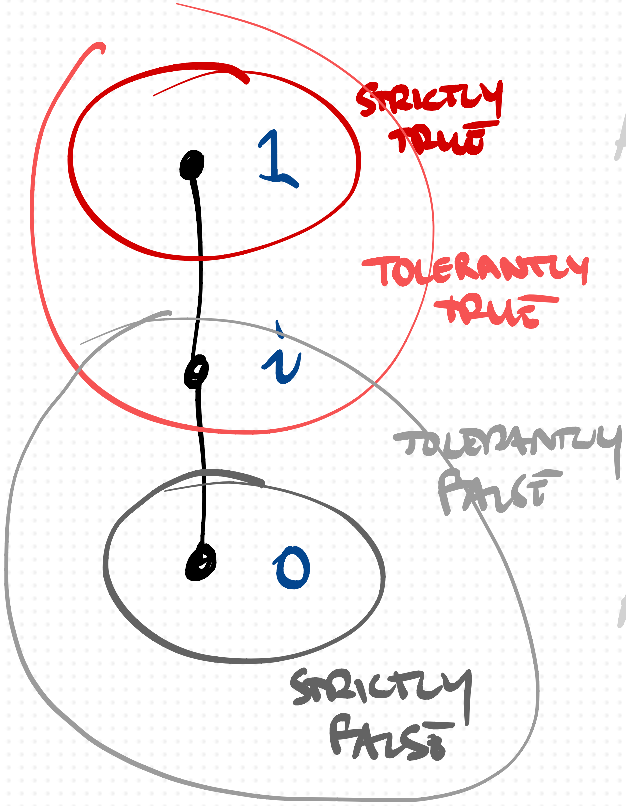
$A \vDash_{LP} B$ iff for every $[\![\cdot]\!]$,

if $[\![A]\!]_\alpha = 1$ or i then $[\![B]\!]_\alpha = 1$ or i.

ie, if $[\![B]\!]_\alpha = 0$ then $[\![A]\!]_\alpha = 0$ too.



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$A \vDash_{LP} B$ iff for every $[\![\cdot]\!]$,

if $[\![A]\!]_\alpha = 1$ or i then $[\![B]\!]_\alpha = 1$ or i.

ie, if $[\![B]\!]_\alpha = 0$ then $[\![A]\!]_\alpha = 0$ too.

$A \vDash_{ST} B$ iff for every $[\![\cdot]\!]$,

if $[\![A]\!]_\alpha = 1$ then $[\![B]\!]_\alpha = 1$ or i.

ie, there is no $[\![\cdot]\!]$ where $[\![A]\!]_\alpha = 1$ & $[\![B]\!]_\alpha = 0$.

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$A \vDash_{LP} B$

$\neg(\llbracket A \rrbracket = 1 \text{ or } i \ \& \ \llbracket B \rrbracket = 0)$

$A \vDash_{ST} B$

$\neg(\llbracket A \rrbracket = 1 \ \& \ \llbracket B \rrbracket = 0)$

$A \vDash_{K3} B$

$\neg(\llbracket A \rrbracket = 1 \ \& \ \llbracket B \rrbracket = 0 \text{ or } i)$

$A \vDash_{ST} B$ iff $A \vDash_{CL} B$

$A \vDash_{LP} B$

$\neg([\![A]\!] = 1 \text{ or } i \ \& \ [\![B]\!] = 0)$

$\vDash_{LP} B$ iff $\vDash_{CL} B$

$p \wedge \neg p \nvDash_{LP} q$

$A \vDash_{ST} B$

$\neg([\![A]\!] = 1 \ \& \ [\![B]\!] = 0)$

$A \vDash_{ST} B$ iff $A \vDash_{CL} B$

$p \wedge \neg p \vDash_{ST} q$

$p \vDash_{ST} q \vee \neg q$

$A \vDash_{K3} B$

$\neg([\![A]\!] = 1 \ \& \ [\![B]\!] = 0 \text{ or } i)$

$A \vDash_{K3}$ iff $A \vDash_{CL}$

$p \nvDash_{K3} q \vee \neg q$

$A \models_{LP} B$

$\neg([\![A]\!] = 1 \text{ or } i \;\&\; [\![B]\!] = 0)$

$\models_{LP} B$ iff $\models_{CL} B$

$A \models_{ST} B$

$\neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0)$

$A \models_{ST} B$ iff $A \models_{CL} B$

$A \models_{K3} B$

$\neg([\![A]\!] = 1 \;\&\; [\![B]\!] = 0 \text{ or } i)$

$A \models_{K3}$ iff $A \models_{CL}$

$$\frac{A \models_{ST} B \qquad B \models_{ST} C}{A \models_{ST} C}$$

Admissible for the logical vocabulary

$A \models_{LP} B$

$\neg(\llbracket A \rrbracket = 1 \text{ or } i \ \& \ \llbracket B \rrbracket = 0)$

$\models_{LP} B$ iff $\models_{CL} B$

$A \models_{ST} B$

$\neg(\llbracket A \rrbracket = 1 \ \& \ \llbracket B \rrbracket = 0)$

$A \models_{ST} B$ iff $A \models_{CL} B$

$A \models_{K3} B$

$\neg(\llbracket A \rrbracket = 1 \ \& \ \llbracket B \rrbracket = 0 \text{ or } i)$

$A \models_{K3}$ iff $A \models_{CL}$

$$\frac{A \models_{ST} B \qquad B \models_{ST} C}{A \models_{ST} C}$$

Admissible for the logical vocabulary

Extend the language with a formula λ whose $\llbracket \lambda \rrbracket = i$

$\top \models_{ST_\lambda} \lambda \qquad \lambda \models_{ST_\lambda} \bot$

$\top \not\models_{ST_\lambda} \bot$

But not a principle for all ST theories!

THESE ARE <u>ALL</u> ST-Valid INFERENCE PRINCIPLES

$$X, A \succ A, Y$$

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ \wedge L \qquad \frac{X \succ A, Y}{X, \neg A \succ Y}\ \neg L \qquad \frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \vee B, Y}\ \vee L$$

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \wedge B, Y}\ \wedge R \qquad \frac{X, A \succ Y}{X \succ \neg A, Y}\ \neg R \qquad \frac{X \succ A, B, Y}{X \succ A \vee B, Y}\ \vee R$$

$$\frac{X, A(t) \succ Y}{X, \forall x A(x) \succ Y}\ \forall L \qquad \frac{X \succ A(t), Y}{X \succ \exists x A(x), Y}\ \exists R$$

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\ \forall R^{*} \qquad \frac{X, A(n) \succ Y}{X, \exists x A(x) \succ Y}\ \exists L^{*}$$

* n must be fresh.

BUT CUT ISN'T

$$\frac{X \vdash A, Y \qquad X, A \vdash Y}{X \vdash Y}$$

(Take $[\![A]\!] = i$ to validate $X \models A, Y$ & $X, A \models Y$ without $X \models Y$.)

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The final part of first order machinery, identity, can be simply accommodated. We merely take '=' to be a particular two-place predicate such that

$$d^+(=) = \{<x,x>|x \in D\}.$$

$d^-(=)$ is arbitrary, except that $d^+(=) \cup d^-(=) = D^2$. (There may be philosophical arguments for placing other constraints on $d^-(=)$, but they need not concern us here.) We can now state the final Fact.

Graham Priest, *In Contradiction*, §5.4

$[\![s=t]\!]_\alpha = 0$ if $[\![s]\!]_\alpha \neq [\![t]\!]_\alpha$

$= 1$ or i if $[\![s]\!]_\alpha = [\![t]\!]_\alpha$

(here, $[\![s]\!]_\alpha$ & $[\![t]\!]_\alpha$ are always defined)

$\mathbb{I} = \mathbb{I}$	d_1	d_2	d_3	$\cdots$
d_1	i/1	0	0	$\cdots$
d_2	0	i/1	0	$\cdots$
d_3	0	0	i/1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

STEPHEN BLAMEY, Handbook of Philosophical Logic, Ed 2

$$M_s(\top) = \top,$$

$$M_s(\bot) = \bot,$$

$$M_s(t_1 = t_2) = \begin{cases} \top & \text{iff} \quad M_s(t_1), M_s(t_2) \in D_M, \text{ and } M_s(t_1) = M_s(t_2) \\ \bot & \text{iff} \quad M_s(t_1), M_s(t_2) \in D_M, \text{ and } M_s(t_1) \neq M_s(t_2), \end{cases}$$

$$M_s(Pt_1 \ldots t_{\lambda(P)}) = \begin{cases} \top & \text{iff} \quad P_M(M_s(t_1, \ldots, t_{\lambda(P)}) = \top \\ \bot & \text{iff} \quad P_M(M_s(t_1, \ldots, t_{\lambda(P)}) = \bot, \end{cases}$$

$$M_s(\neg\phi) = \begin{cases} \top & \text{iff} \quad M_s(\phi) = \bot \\ \bot & \text{iff} \quad M_s(\phi) = \top, \end{cases}$$

$$M_s(\phi \wedge \psi) = \begin{cases} \top & \text{iff} \quad M_s(\phi) = \top \quad \text{and} \quad M_s(\psi) = \top \\ \bot & \text{iff} \quad M_s(\phi) = \bot \quad \text{or} \quad M_s(\psi) = \bot, \end{cases}$$

$$[\![s = t]\!]_\alpha = \begin{cases} 1 & \text{if } [\![s]\!]_\alpha, [\![t]\!]_\alpha \text{ are defined } \& \ [\![s]\!]_\alpha = [\![t]\!]_\alpha. \\ 0 & \text{if } [\![s]\!]_\alpha, [\![t]\!]_\alpha \text{ are defined } \& \ [\![s]\!]_\alpha = [\![t]\!]_\alpha \end{cases}$$

a proxy "undefined" object

$[\![I = T]\!]$	$*$	d_1	d_2	d_3	$\cdots$
$*$	i	i	i	$\cdots$	
d_1	i	1	0	$\cdots$	
d_2	i	0	1	$\cdots$	
d_3	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$\vdots$					

LP

$[\![=]\!]$	d_1	d_2	d_3	...
d_1	$i/1$	0	0	...
d_2	0	$i/1$	0	...
d_3	0	0	$i/1$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

K3

$[\![=]\!]$	$*$	d_1	d_2	d_3 ...
$*$	i	i	i	...
d_1	i	1	0	...
d_2	i	0	1	...
d_3	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\vdots$				

What about ST?

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Which rules?

A defining rule:

IDENTITY is indistinguishability

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a=b, Y} =Df$$

Which rules?

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a=b, Y} = Df$$

difficult to work with.
Replace by equivalent LEFT/RIGHT rules.

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a=b, Y} = R$$

$$\frac{X \succ A(a), Y \qquad X, A(b) \succ Y}{X, a=b \succ Y} = L$$

$$\frac{X \succ A(b), Y \qquad X, A(a) \succ Y}{X, a=b \succ Y} = L$$

Which rules?

$$\dfrac{X, fa \succ fb, Y \qquad X, fb \succ fa, Y}{X \succ a=b, Y} = Df$$

$$\dfrac{X, fa \succ fb, Y \qquad X, fb \succ fa, Y}{X \succ a=b, Y} = R$$

MORE COMPLEX THAN NECESSARY

$$\dfrac{X \succ A(a), Y \qquad X, A(b) \succ Y}{X, a=b \succ Y} = L$$

$$\dfrac{X \succ A(b), Y \qquad X, A(a) \succ Y}{X, a=b \succ Y} = L$$

THESE ARE CUT + IDENTITY properties

$$\succ a=a \quad (Refl)$$

$$\dfrac{X \succ A(a), Y}{X, a=b \succ A(b), Y} = L \qquad \dfrac{X, A(a) \succ Y}{X, a=b, A(b) \succ Y} = L$$

Identity Axioms

$\vdash a = a$ $\qquad$ $a = b, Fa \vdash Fb$ $\qquad$ $a = b, Fb \vdash Fa$

Here, F is any predicate of any arity.

Identity Axioms

$\vdash a = a$ $\qquad$ $a = b, Fa \vdash Fb$ $\qquad$ $a = b, Fb \vdash Fa$

let Fx be $x = a$.

$$\dfrac{\vdash a = a \qquad a = b,\ a = a \vdash b = a}{a = b \vdash b = a}\ \text{cut}$$

Identity Axioms

$$\vdash a=a \qquad\qquad a=b,\, Fa \vdash Fb \qquad\qquad a=b,\, Fb \vdash Fa$$

$$\dfrac{\vdash a=a \qquad a=b,\, a=a \vdash b=a}{a=b \vdash b=a}\ \text{cut}$$

$$\dfrac{a=b,\, Fa \vdash Fb \qquad \dfrac{b=c,\, Fb \vdash Fc \qquad d=c,\, Fc \vdash Fd}{b=c,\, d=c,\, Fb \vdash Fd}\ \text{cut}}{a=b,\, b=c,\, d=c,\, Fa \vdash Fd}\ \text{cut}$$

IF WE CLOSE THOSE AXIOMS UNDER CUT, WE GET....

$$X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y$$

I^a_b is any set of identity statements linking a to b.

$$\boxed{X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y}$$

I^a_b is any set of identity statements linking a to b.

- $\emptyset$ links a to a for all a.
- If X links a to b, $\quad a=c, X$ & $c=a, X$ links b to c,

  $b=c, X$ & $c=b, X$ links a to c,

  (as well as linking all pairs linked by X)

$$\boxed{X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y}$$

I^a_b is any set of identity statements linking a to b.

- These axioms are classically valid.

- If you add them to the sequent rules for first order predicate logic, the resulting system is complete & cut is admissible.

$$\boxed{X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y}$$

I^a_b is any set of identity statements linking a to b.

What do ST-models for these axioms look like?

$$\boxed{X, I^a_b \vdash a=b, Y \qquad X, I^a_b, Fa \vdash Fb, Y}$$

I^a_b is any set of identity statements linking a to b.

What do ST-models for these axioms look like?

* $[\![a]\!]$ & $[\![b]\!]$ are strictly connected iff either $[\![a]\!] = [\![b]\!]$, or some sequence of identity statements linking a & b are strictly true.

Ax 1

$$X, I^a_b \vdash a=b, Y \qquad\qquad X, I^a_b, Fa \vdash Fb, Y$$

I^a_b is any set of identity statements linking a to b.

What do ST-models for these axioms look like?

★ $[\![a]\!]$ & $[\![b]\!]$ are strictly connected iff either $[\![a]\!] = [\![b]\!]$, or some sequence of identity statements linking a & b are strictly true.

Ax1 • If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.

Ax1

$$X, I^a_b \vdash a=b, Y$$

Ax2

$$X, I^a_b, Fa \vdash Fb, Y$$

I^a_b is any set of identity statements linking a to b.

What do ST-models for these axioms look like?

* $[\![a]\!]$ & $[\![b]\!]$ are strictly connected iff either $[\![a]\!] = [\![b]\!]$, or some sequence of identity statements linking a & b are strictly true.

Ax1 • If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.

Ax2 • If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

[$0 \not\approx 1$; $1 \not\approx 0$; otherwise $x \approx y$]

What is the logic of such models?

$$X \models_{ST^=} Y$$

iff

$$X \models_{CL^=} Y$$

What is the logic of such models?

$$\models_{LP^=} Y$$

iff

$$\models_{CL^=} Y$$

Since $\models_{LP^=} Y$ iff $\models_{ST^=} Y$

$$X \models_{ST^=} Y$$

iff

$$X \models_{CL^=} Y$$

cut elimination

$ST_= \subseteq CL_=$ by definition

$$X \models_{K3^=}$$

iff

$$X \models_{CL^=}$$

Since $X \models_{K3^=}$ iff $X \models_{ST^=}$

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- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.
- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

THE OLD LP & K_3 MODELS ARE SPECIAL CASES...

LP

$[\![=]\!]$	d_1	d_2	d_3	...
d_1	i/1	0	0	...
d_2	0	i/1	0	...
d_3	0	0	i/1	...
⋮	⋮	⋮	⋮	⋱

K_3

$[\![=]\!]$	*	d_1	d_2	d_3 ...
*	i	i	i	...
d_1	i	1	0	...
d_2	i	0	1	...
d_3	⋮	⋮	⋮	⋱
⋮				

(Objects are only ever strictly connected to themselves in these models)

- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.
- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

Lax Identity Models

$$\begin{array}{c|cccccc}
[\![=]\!] & d_1 & d_2 & \cdots & d_i & & \\
\hline
d_1 & i & i & \cdots & i & \cdots & \\
d_2 & i & i & & \vdots & & \\
\vdots & \vdots & & \ddots & \vdots & & \\
d_i & i & \cdots & \cdots & i & \cdots & \\
 & \vdots & & & \vdots & \ddots &
\end{array}$$

- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![a=b]\!] \neq 0$.
- If $[\![a]\!]$ & $[\![b]\!]$ are strictly connected, then $[\![Fa]\!] \approx [\![Fb]\!]$.

In General...

If $\mathcal{M}$ is an $ST^=$ model & $\mathcal{M}'$ is a blurring of $\mathcal{M}$, it is an $ST^=$ model too.

$\mathcal{M}'$ is a blurring of $\mathcal{M}$ iff $[\![F]\!]^{\mathcal{M}'}_{\alpha} \sqsubseteq [\![F]\!]^{\mathcal{M}}_{\alpha}$ for all F.

$\sqsubseteq$: 

(the specification ordering)

(this is <u>much</u> more general than LP or K_3 models)

Symmetry "failures"

$[\![=]\!]$	a	b
a	1	i
b	0	1

$$a=b \models_{ST} b=a$$

$$a=b \not\models_{LP} b=a$$

Compatible with any predicates on $D=\{a,b\}$

Symmetry "failures"

$[\![=]\!]$	a	b
a	1	i
b	0	1

$[\![=]\!]$	a	b
a	1	1
b	i	1

Compatible with any predicates on $D = \{a, b\}$

Requires $[\![Fa]\!] \approx [\![Fb]\!]$ for every predicate F.

– cannot have $[\![Fa]\!] = 1$, $[\![Fb]\!] = 0$, for example.

Stronger Indiscernibility Rules

$$\frac{X,\ Fa \succ Y}{X, a=b, Fb \succ Y} =L$$

If $[\![a=b]\!]=1$ & $[\![Fb]\!]=1$ then $[\![Fa]\!]=1$

Stronger Indiscernibility Rules

$$\frac{X, Fa \succ Y}{X, a=b, Fb \succ Y}{=}L$$

If $[\![a=b]\!]=1$ & $[\![Fb]\!]=1$ then $[\![Fa]\!]=1$

$$\frac{X, Fb \succ Y}{X, a=b, Fa \succ Y}{=}L$$

If $[\![a=b]\!]=1$ & $[\![Fa]\!]=1$ then $[\![Fb]\!]=1$

$$\frac{X \succ Fa, Y}{X, a=b \succ Fb, Y}{=}L$$

If $[\![a=b]\!]=1$ & $[\![Fb]\!]=0$, then $[\![Fa]\!]=0$

$$\frac{X \succ Fb, Y}{X, a=b \succ Fa, Y}{=}L$$

If $[\![a=b]\!]=1$ & $[\![Fa]\!]=0$, then $[\![Fb]\!]=0$

Symmetry

$$\frac{X, a=b \vdash Y}{X, b=a \vdash Y} = \mathrm{SwapL}$$

If $[\![b=a]\!] = 1$ then $[\![a=b]\!] = 1$

$$\frac{X \vdash a=b, Y}{X \vdash b=a, Y} = \mathrm{SwapR}$$

If $[\![b=a]\!] = 0$ then $[\![a=b]\!] = 0$

LP-style Indiscernibility

$$\frac{X \vdash a=b, Y \qquad X \vdash Fa, Y}{X \vdash Fb, Y} = LP_I$$

If $[\![Fb]\!]=0$ then either $[\![a=b]\!]=0$ or $[\![Fa]\!]=0$

LP-style Indiscernibility

$$\frac{X \vdash a=b, Y \qquad X \vdash Fa, Y}{X \vdash Fb, Y} = LPI$$

If $[\![Fb]\!]=0$ then either $[\![a=b]\!]=0$ or $[\![Fa]\!]=0$

If $[\![a=b]\!]=1$ or i & $[\![Fa]\!]=1$ or i, then $[\![Fb]\!]=1$ or i.

A "Drop" Rule

$$\frac{X, a=a \vdash Y}{X \vdash Y} = \text{Drop}$$

$[\![a=a]\!] = 1$

There is plenty more here for you to explore. The logic-agnostic (or pluralist) perspective on models gives us a number of new tools for developing distinctive three-valued models for identity.

Thanks!