Our Aim

To introduce *proof theory*, with a focus in its applications in philosophy, linguistics and computer science.
Examine the connections between proof theory and semantics, both formal \textit{model theory}, and more general philosophical considerations concerning meaning.
Today's Plan

Speech Acts and Norms

Proofs and Models

Beyond
SPEECH ACTS AND NORMS
An idea found in Brandom’s *Making It Explicit* is that the *meaning* of linguistic items should first be understood in terms of their *use*.

The linguistic (conceptual) practices of communities set up *norms* governing their behavior.

These practices have features that we can make explicit through the introduction of new vocabulary.
Rules as Definitions

The rules that govern a connective are taken to define the new connective. This appears to make it really easy to introduce new logical terms. Specify a set of rules governing a connective, and you’ve got a new connective.
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Specify a set of rules governing a connective, and you’ve got a new connective.

But, there’s a problem.
Arthur Prior pointed out that if a set of rules is enough to define a connective, then *tonk* is legitimate.

\[ X, A \vdash C \quad [\text{tonkL}] \]
\[ X, A \otimes B \vdash C \]

\[ X \vdash B \quad [\text{tonkR}] \]
\[ X \vdash A \otimes B \]
Arthur Prior pointed out that if a set of rules is enough to define a connective, then *tonk* is legitimate.

\[
\frac{X, A \vdash C}{\frac{X, A \otimes B \vdash C}{X, A \vdash B}} \quad \text{[tonkL]} \quad \frac{X \vdash B}{X \vdash A \otimes B} \quad \text{[tonkR]}
\]

\[
\frac{B \vdash B}{B \vdash A \otimes B} \quad \frac{A \vdash A}{A \otimes B \vdash A} \quad \frac{B \vdash A \otimes B}{B \vdash A} \quad \text{[Cut]}
\]
Responding to tonk

Nuel Belnap responded to Prior’s article, saying that additional conditions need to be satisfied in order to define a connective.

Connectives aren’t introduced out of thin air, there is a context of deducibility, e.g. the full set of Gentzen’s structural rules.

In order to be a definition, an extension has to be conservative, while tonk manifestly is not.

In order to be a definition, an addition has to be uniquely specified.

These ideas have been taken up and developed by Dummett and others in discussions of harmony.
Many philosophers and logicians take \textit{assertion} to be the primary speech act, which is used to define others.

Others argue that \textit{denial} should be understood as a primitive act on its own.

We take logic, in particular valid sequents, as presenting normative relations between assertions and denials.

\( X \vdash Y \) tells us that one should not assert everything in \( X \) while denying everything in \( Y \).
Positions

\[ X \vdash Y \]
\( X \supset Y \)
Positions

Invalid sequents can be viewed as *positions* in a discourse

\[ X : Y \]
What do the structural rules say in terms of assertion and denial?
A \vdash A

Asserting $A$ clashes with denying $A$
Structural Rules

\[
\begin{align*}
& X, Y \vdash Z \quad [\text{KL}] \\
\Rightarrow & \quad X, A, Y \vdash Z \\
& X \vdash Y, Z \quad [\text{KR}] \\
\Rightarrow & \quad X \vdash Y, A, Z
\end{align*}
\]

If asserting $X$, $Y$ clashes with denying $Z$, then asserting more stuff still clashes
Structural Rules

\[
\frac{X, A, AY \vdash Z}{X, A, Y \vdash Z} \quad [\text{WL}]
\]

\[
\frac{X \vdash Y, A, A, Z}{X \vdash Y, A, Z} \quad [\text{WR}]
\]

If asserting or denying \( A \) twice results in a clash, then asserting or denying \( A \) just once results in a clash.
Structural Rules

\[
X, A, B, Y \vdash Z \quad [\text{CL}]
\]

\[
X, B, A, Y \vdash Z
\]

\[
X \vdash Y, A, B, Z \quad [\text{CR}]
\]

\[
X \vdash Y, B, A, Z
\]

If some assertions and denials clash, then asserting and denying the same things in a different order still clashes.
Structural Rules

\[
\frac{X \vdash Y, A \quad A, X \vdash Y}{X \vdash Y} \text{[Cut]}
\]

If asserting \(X\) and denying \(A\) and \(Y\) clashes, and asserting \(X\) and \(A\) while denying \(Y\) clashes, then asserting \(X\) and denying \(Y\)

Contrapositively, if asserting \(X\) and denying \(Y\) does not clash, then either asserting \(X\) and \(A\) while denying \(Y\) does not clash or asserting \(X\) while denying \(Y\) and \(A\) does not clash
Belnap argued that a systematic logical treatment of language should give equal weight to imperatives and interrogatives.

1. THE DECLARATIVE FALLACY

My thesis is simple: systematic theorists should not only stop neglecting interrogatives and imperatives, but should begin to give them equal weight with declaratives. A study of the grammar, semantics, and pragmatics of all three types of sentence is needed for every single serious program in philosophy that involves giving important attention to language.¹

Attempting to understand all linguistic behavior in terms of assertions commits the Declarative Fallacy.

The hope is that the view of sequents and logic can be extended to other speech acts.
PROOFS AND MODELS
How might *truth* enter this picture?

Models are ways of systematically elaborating finite positions into ideal, infinite positions that settle every proposition.

In the propositional case, valuations are generated by ideal positions.
The members of $X$ are $true$ and the members of $Y$ are $false$
The members of $X$ are *true* and the members of $Y$ are *false* (relative to $[X : Y]$).
Example

\[ p \lor q, \ r : \neg p \]
Example

\[ [p \lor q, r : \neg p] \]

\[
\begin{align*}
\text{p} \lor \text{q, r} \\
\text{p} \lor \text{q, r}
\end{align*}
\]
Example

\[
[p \lor q, \ r : \neg p]
\]

\[
\begin{align*}
\text{true} & \\
\hline
p \lor q, \ r & \text{true}
\end{align*}
\]
Example

\[ [p \lor q, r : \neg p] \]

\[
\begin{align*}
  & \quad p \lor q, r \quad \text{true} \\
  & \quad \neg p \\
\end{align*}
\]
Example

\[ [p \lor q, r : \neg p] \]

\[
\begin{array}{c}
p \lor q, r \quad true \\
\neg p \quad false \\
\end{array}
\]
Example

\[ [p \lor q, r : \neg p] \]

\[ \begin{array}{c}
\neg p & false \\
p & true \\
p & p \\
\end{array} \]
Example

\[[p \lor q, r : \neg p]\]

\[
\begin{array}{c}
p \lor q, r & \text{true} \\
\neg p & \text{false} \\
p & \text{???}
\end{array}
\]
Example

\[ [p \lor q, r : \neg p] \]

\[ \begin{array}{ll}
\text{true} & p \lor q, r \\
\text{false} & \neg p \\
\text{??} & p \\
\end{array} \]

\textbf{Definition:} \( A \) is \textit{true} at \([X : Y]\) iff \( X \vdash A, Y \).

\textbf{Definition:} \( A \) is \textit{false} at \([X : Y]\) iff \( X, A \vdash Y \).
Example

\[ [p \lor q, r : \neg p] \]

\[ \begin{align*}
\text{p } \lor \text{ q, r } \quad \text{true} \\
\neg \text{p } \quad \text{false} \\
\text{p } \quad \text{true}
\end{align*} \]

**Definition:** A is *true* at \([X : Y]\) iff \(X \vdash A, Y\).

**Definition:** A is *false* at \([X : Y]\) iff \(X, A \vdash Y\).
Example

\[ [p \lor q, r : \neg p] \]

\[
\begin{array}{c}
p \lor q, r \quad \text{true} \\
\neg p \quad \text{false} \\
p \quad \text{true} \\
p \land r \\
\end{array}
\]

**Definition:** \( A \) is *true* at \([X : Y]\) iff \( X \vdash A, Y \).

**Definition:** \( A \) is *false* at \([X : Y]\) iff \( X, A \vdash Y \).
Example

\[ [p \lor q, \ r : \neg p] \]

\[
\begin{array}{ccc}
p \lor q, \ r & true \\
\neg p & false \\
p & true \\
p \land r & true \\
\end{array}
\]

**Definition:** \( A \) is *true* at \([X : Y]\) iff \( X \vdash A, Y \).

**Definition:** \( A \) is *false* at \([X : Y]\) iff \( X, A \vdash Y \).
A \land B \text{ is true at } [X : Y] \text{ iff } A \text{ and } B \text{ are true at } [X : Y].

A \lor B \text{ is false at } [X : Y] \text{ iff } A \text{ and } B \text{ are false at } [X : Y].

\neg A \text{ is true at } [X : Y] \text{ iff } A \text{ is false at } [X : Y].

\neg A \text{ is false at } [X : Y] \text{ iff } A \text{ is true at } [X : Y].
A \land B \text{ is true at } [X : Y] \text{ iff } A \text{ and } B \text{ are true at } [X : Y].

A \lor B \text{ is false at } [X : Y] \text{ iff } A \text{ and } B \text{ are false at } [X : Y].

\neg A \text{ is true at } [X : Y] \text{ iff } A \text{ is false at } [X : Y].

\neg A \text{ is false at } [X : Y] \text{ iff } A \text{ is true at } [X : Y].

However, p \land q \text{ is false at } [\ : p \land q]
A \land B \text{ is true at } [X : Y] \text{ iff } A \text{ and } B \text{ are true at } [X : Y].

A \lor B \text{ is false at } [X : Y] \text{ iff } A \text{ and } B \text{ are false at } [X : Y].

\neg A \text{ is true at } [X : Y] \text{ iff } A \text{ is false at } [X : Y].

\neg A \text{ is false at } [X : Y] \text{ iff } A \text{ is true at } [X : Y].

However, p \land q \text{ is false at } [ : p \land q] \text{ but neither } p \text{ nor } q \text{ is false at } [ : p \land q]
\text{ since neither } p \vdash p \land q \text{ nor } q \vdash p \land q.
A \land B \text{ is true at } [X : Y] \iff A \text{ and } B \text{ are true at } [X : Y].

A \lor B \text{ is false at } [X : Y] \iff A \text{ and } B \text{ are false at } [X : Y].

\neg A \text{ is true at } [X : Y] \iff A \text{ is false at } [X : Y].

\neg A \text{ is false at } [X : Y] \iff A \text{ is true at } [X : Y].

However, p \land q \text{ is false at } [ : p \land q] \text{ but neither } p \text{ nor } q \text{ is false at } [ : p \land q] \text{ since neither } p \vdash p \land q \text{ nor } q \vdash p \land q.

Similarly, r is neither true nor false at [p : q].
FACT: If $A$ is neither true nor false in $[X : Y]$ then both $[X, A : Y]$ and $[X : A, Y]$ is invalid, and each sequent settles $A$ — one as true and the other as false.
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So, if $[X : Y]$ doesn’t settle the truth of a statement $A$, then we can throw $A$ in on either side, to get a more comprehensive sequent which *does* settle it.
Extensions

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In general, if $X \not\vdash Y$ then either $X, A \not\vdash Y$ or $X \not\vdash A, Y$. 

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So, if $[X : Y]$ doesn’t settle the truth of a statement $A$, then we can throw $A$ in on either side, to get a more comprehensive sequent which *does* settle it.

In general, if $X \nvdash Y$ then either $X, A \nvdash Y$ or $X \nvdash A, Y$.

\[
\frac{X \vdash Y, A \quad A, X \vdash Y}{X \vdash Y} \quad \text{[Cut]}
\]
Maximal Sequents

A maximal sequent is the limit of the process of throwing in each sentence in either the left or the right hand side. You can think of it as:

▶ A pair \([X:Y]\) of infinite sets, such that \(X \not\vdash Y\) and \(X \cup Y\) is the whole language.

Fact: Every maximal sequent makes each sentence either true or false.

Fact: If \(X \not\vdash Y\), there's a maximal \([X:Y]\) extending \([X:Y]\).
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[X : Y] is finitary, where X and Y are sets (or multisets or lists ...).

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Maximal Sequents

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A maximal sequent is the limit of the process of throwing in each sentence in either the left or the right hand side. You can think of it as:

- A pair \([\mathcal{X} : \mathcal{Y}]\) of infinite sets, such that \(\mathcal{X} \nvdash \mathcal{Y}\) and \(\mathcal{X} \cup \mathcal{Y}\) is the whole language.
Maximal Sequents

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**FACT:** If \( X \not\vDash Y \), there’s a maximal \( [X : Y] \) extending \( [X : Y] \).
Assign truth values relative to *maximal* positions.
Models

Assign truth values relative to *maximal* positions.

In a slogan, *truth value* = *location in a maximal position*,
Variations

The ideal position construction handles classical logic

With some small adjustments, it can be used to provide models for intuitionistic logic

The system LJ is single-conclusion, but there is an intuitionistic sequent system that has multiple conclusions

The construction with these two systems yield *Kripke models* and *Beth models*
The hypersequent system for S5 can be used to give a similar construction.

Each component of a hypersequent describes a possible world.
S5 hypersequents

\[ \frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']} \mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y'] }{\Box L} \]
\[ \frac{\mathcal{H}[X \vdash Y \mid \vdash A]} \mathcal{H}[X \vdash \Box A, Y]} \]

\[ \frac{\mathcal{H}[X \vdash Y \mid A \vdash]} \mathcal{H}[\Diamond A, X \vdash Y]} \]
\[ \frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']} \mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']} \]

\[ \frac{\mathcal{H}[\Diamond A, X \vdash Y]} \mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']} \]
\[ \frac{\mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']} {\Diamond R} \]
Extending positions

Invalid sequents \([X : Y]\)

Invalid hypersequents \([\left[ X : Y \right], \left[ X' : Y' \right], \ldots \)]
Extending positions

Invalid sequents \([X : Y]\)

Invalid hypersequents \(\llbracket X : Y \rrbracket, \llbracket X' : Y' \rrbracket, \ldots\)

Say one set of pairs \(\mathcal{H}\) extends another \(\mathcal{G}\), \(\mathcal{G} \subseteq \mathcal{H}\), just in case for each component \([X : Y]\) in \(\mathcal{G}\), there is a component \([U : V]\) in \(\mathcal{H}\) such that \(X \subseteq U\) and \(Y \subseteq V\)

Example: \(\{[p : q], [s : r]\}\) is extended by both \(\{[p, s : r, q, t]\}\) and by \(\{[p, t : q], [s : r, p]\}\)
Where are the truth values now?
Where are the truth values *now*?

Maximal positions \([X : Y]\)

Maximal modal positions \([[[X : Y], [X' : Y']], \ldots]\)
Where are the truth values *now*?

**Maximal positions** \([X : Y]\)

**Maximal modal positions** \([[[X : Y], [X' : Y']], \ldots]\)

A set of pairs \(\mathcal{H}\) is a *modal position* iff there is no valid hypersequent

\[
X_1 \vdash Y_1 \mid \cdots \mid X_n \vdash Y_n\ 	ext{extended by } \mathcal{H}
\]

A modal position \(\mathcal{H}\) is *maximal* iff there is no modal position \(\mathcal{I}\) such that \(\mathcal{H} \prec \mathcal{I}\).
Building maximal modal positions

The process of expanding a modal position can add formulas to a component as well as adding more components.

Some maximal modal positions, however, will contain finitely many components.

The construction builds connected chunks of the S5 canonical model, taking the accessibility relation to be an equivalence relation rather than the universal relation.
As in the classical case, the Cut rule adds new formulas to individual components.

The modal rules can extend a position with new components.
Building maximal modal positions

\[
\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']} \quad \text{[\Box L]}
\]

\[
\frac{\mathcal{H}[X \vdash Y \mid \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]} \quad \text{[\Box R]}
\]

If \([X : \Box A, Y], [X_i : Y_i]\) isn’t derivable, then \([X : \Box A, Y], [ : A], [X_i : Y_i]\) can’t be either

If the latter were derivable then the former would be by [\Box R]

Similarly but for [\Box L], if, e.g. \([X, \Box A : Y], [X' : Y'], [X_i : Y_i]\) isn’t derivable, then \([X, \Box A : Y], [X', A : Y'], [X_i : Y_i]\) can’t be
Necessity in maximal modal positions

For a maximal modal position \(\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\}\),
\(\Box A\) is true at \([\mathcal{X}_i : \mathcal{Y}_i]\) iff \(A\) is true at each \([\mathcal{X}_j : \mathcal{Y}_j], j \in I\).

(\Rightarrow) If \(\Box A\) is true at \([\mathcal{X}_i : \mathcal{Y}_i]\) and \(A\) were not true at some component \([\mathcal{X} : \mathcal{Y}]\),
then since \([\mathcal{X} : \mathcal{Y}]\) is a maximal position, we would have \(A \in \mathcal{Y}\) but
\(\Box A \vdash \vdash A\) is a valid sequent (by \([\Box L]\) from the axiom \(\vdash \vdash A\)), so
\([[\mathcal{X}_i : \mathcal{Y}_i], [\mathcal{X}_j : \mathcal{Y}_j]]\) would not be a position, as \(\Box A \in \mathcal{X}_i\) and \(A \in \mathcal{Y}_j\), so
\([[\mathcal{X}_i : \mathcal{Y}_i] : i \in I]\) isn’t a position. As it is, whenever \(\Box A \in \{\mathcal{X}_i : \mathcal{Y}_i\}\), \(A\) is true at every component.
Necessity in maximal modal positions

For a maximal modal position \{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\},
\[ \square A \text{ is true at } [\mathcal{X}_i : \mathcal{Y}_i] \text{ iff } A \text{ is true at each } [\mathcal{X}_j : \mathcal{Y}_j], j \in I \]

\((\Leftarrow)\) Suppose \(\square A\) isn’t true at \([\mathcal{X}_i : \mathcal{Y}_i]\). So we have \(\square A \in \mathcal{Y}_i\). Take \(\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\} \cup [ : A]\). This is a position. Suppose that it is not. Then there is a derivable hypersequent \(\vdash A \mid X \vdash Y \mid \mathcal{H}\), where \(X \subseteq \mathcal{X}_i\), \(Y \subseteq \mathcal{Y}_i\) and \(\mathcal{H}\) is extended by the other components of the modal position. If that were the case, then by \([\Box R]\), we could derive \(X \vdash \square A, Y \mid \mathcal{H}\), but that is extended by the original modal position. It is, then, not valid. So, \(\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\} \cup [ : A]\) is a position, so it is extended by a maximal modal position, which must be \(\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\}\), as that is not extended by any modal positions. Therefore, for some \(j \in I\), \(A \in \mathcal{Y}_j\).
A modal position extended by a finite, maximal modal position

\[\vdash \Box p \lor \Box \neg p \text{ is not valid}\]

So \[ [ : \Box p \lor \Box \neg p]\] is a position.

Using the rules, one obtains

\[[[ : ], [ : ], [ : \Box p \lor \Box \neg p]]\]

One can then choose extensions in such a way that no additional components are needed.
A modal position extended by a finite, maximal modal position

\[ \vdash \Box p \lor \Box \neg p \] is not valid

So \[ [ : \Box p \lor \Box \neg p] \] is a position

Using the rules, one obtains

\[ [[ : ], [ : ], [ : \Box p, \Box \neg p, \Box p \lor \Box \neg p]] \]

One can then choose extensions in such a way that no additional components are needed
A modal position extended by a finite, maximal modal position

\[ \vdash \Box p \lor \Box \neg p \text{ is not valid} \]

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Using the rules, one obtains

\[ [ [ : p], [ : \neg p], [ : \Box p, \Box \neg p, \Box p \lor \Box \neg p] ] \]

One can then choose extensions in such a way that no additional components are needed
A modal position extended by a finite, maximal modal position

\[ \vdash \Box p \lor \Box \neg p \text{ is not valid} \]

So \( [ : \Box p \lor \Box \neg p ] \) is a position

Using the rules, one obtains

\[ [ : p ], [ p : \neg p ], [ : \Box p, \Box \neg p, \Box p \lor \Box \neg p ] \]

One can then choose extensions in such a way that no additional components are needed
Maximality Facts

Each modal position can be extended to a maximal modal position

Each component of a maximal modal position is a maximal position

Each maximal modal position corresponds to a simple Kripke model: $\Box A$ is true at $\{[x_i; y_i] : i \in I\}$ iff $A$ is true in every position in the modal position
BEYOND
Further directions

There are many directions one could go from here

One could add quantifiers and predicates

One could add axioms to obtain theories
Quantifiers

\[
\begin{align*}
A(\alpha), X & \vdash Y \quad [\forall L] \\
\forall x A(x), X & \vdash Y \\
X & \vdash Y, A(y) \quad [\forall R] \\
X & \vdash Y, \forall x A(x) \\
A(y), X & \vdash Y \quad [\exists L] \\
\exists x A(x), X & \vdash Y \\
X & \vdash Y, A(\alpha) \quad [\exists R] \\
X & \vdash Y, \exists x A(x)
\end{align*}
\]

In [\forall R] and [\exists L], y cannot occur freely in the lower sequent.

These rules permit an Elimination Theorem.
Identity

\[ \vdash a = a \]

\[ A(s), X \vdash Y \]

\[ s = t, A(t), X \vdash Y \]

\[ X \vdash Y, A(s) \]

\[ s = t, X \vdash Y, A(t) \]
Identity, alternative rules

Alternative rules make it easier to eliminate Cut

\[
\frac{\alpha = \alpha, X \vdash Y}{X \vdash Y} \quad [= \text{R}]
\]

\[
\frac{s = t, A(t), A(s), X \vdash Y}{s = t, A(t), X \vdash Y} \quad [= \text{L}]
\]

One can use the Dragalin-style proof to show that Cut can be eliminated
These rules are inconsistent in classical logic, so one will need to go non-classical to hang onto them.

They take complex formulas to atomic formulas, which leads to complications for showing that Cut can be eliminated.
Arithmetic

Take a language with =, 0, ′, +, ×

\[ \vdash x + 0 = x \]
\[ \vdash x + y' = (x + y)' \]
\[ \vdash x \times 0 = 0 \]
\[ \vdash x \times y' = (x \times y) + x \]

\[ x' = y' \vdash x = y \]
\[ 0 = x' \vdash \]
\[ X \vdash A(0), Y \quad X, A(x) \vdash A(x'), Y \]
\[ \quad \vdash X, A(x), Y \]
\[ X, A(x') \vdash A(x), Y \quad A(0), X \vdash Y \]
\[ \quad \vdash A(x), X \vdash Y \]
Inferentialism

NUEL BELNAP
“Tonk, Plonk and Plink.”

NUEL BELNAP
“Declaratives Are Not Enough.”

JAROSLAV PEREGRIN
“An Inferentialist Approach to Semantics: Time for a New Kind of Structuralism?”

ARTHUR PRIOR
“The Runabout Inference Ticket.”
Positions and Models

GREG RESTALL
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