Natural Deduction with Alternatives

on *structural rules*, and *identifying assumptions*

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https://consequently.org/presentation/2022/natural-deduction-with-alternatives-bochum

My Aim

To introduce *natural deduction with alternatives*,
a well-behaved, *mildly* bilateralist, single-conclusion natural deduction framework for a range of logical systems, including *classical*, *linear*, *relevant* logic and *affine* logic, by varying the policy for managing discharging of assumptions and retrieval of alternatives.

Natural Deduction with Alternatives

Weakening and Explosion

Varieties of Conjunction

Contraction, Composition, and Assumptions

NATURAL DEDUCTION WITH ALTERNATIVES

Gentzen–Prawitz Natural Deduction

$$\dfrac{\dfrac{\dfrac{[A \land \neg B]^1}{\neg B}{\scriptstyle \land E} \qquad \dfrac{A \to B \qquad \dfrac{[A \land \neg B]^1}{A}{\scriptstyle \land E}}{B}{\scriptstyle \to E}}{\sharp}{\scriptstyle \neg E}}{\neg(A \land \neg B)}{\scriptstyle \neg I^1}$$

Gentzen–Prawitz Natural Deduction

$$
\dfrac{\dfrac{\dfrac{[A \wedge \neg B]^1}{\neg B}\,{\wedge E} \qquad \dfrac{A \to B \qquad \dfrac{[A \wedge \neg B]^1}{A}\,{\wedge E}}{B}\,{\to E}}{\sharp}\,{\neg E}}{\neg(A \wedge \neg B)}\,{\neg I^1}
$$

Gentzen–Prawitz Natural Deduction

$$\dfrac{\dfrac{[A \wedge \neg B]^1}{\neg B}\wedge E \qquad \dfrac{A \to B \qquad \dfrac{[A \wedge \neg B]^1}{A}\wedge E}{B}\to E}{\dfrac{\sharp}{\neg(A \wedge \neg B)}\neg I^1}\neg E$$

Gentzen–Prawitz Natural Deduction

$$\dfrac{\dfrac{\dfrac{[A \land \neg B]^1}{\neg B}\,{\scriptstyle \land E} \qquad \dfrac{A \to B \qquad \dfrac{[A \land \neg B]^1}{A}\,{\scriptstyle \land E}}{B}\,{\scriptstyle \to E}}{\sharp}\,{\scriptstyle \neg E}}{\neg(A \land \neg B)}\,{\scriptstyle \neg I^1}$$

Gentzen–Prawitz Natural Deduction

$$
\dfrac{
  \dfrac{\dfrac{[A \wedge \neg B]^1}{\neg B}\,{\scriptstyle \wedge E}
  \qquad
  \dfrac{A \to B \qquad \dfrac{[A \wedge \neg B]^1}{A}\,{\scriptstyle \wedge E}}{B}\,{\scriptstyle \to E}}
  {\sharp}\,{\scriptstyle \neg E}
}{\neg(A \wedge \neg B)}\,{\scriptstyle \neg I^1}
$$

Gentzen–Prawitz Natural Deduction

$$
\begin{prooftree}
\AxiomC{$[A \wedge \neg B]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\neg B$}
\AxiomC{$A \to B$}
\AxiomC{$[A \wedge \neg B]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{A}
\RightLabel{$\to E$}
\BinaryInfC{B}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg(A \wedge \neg B)$}
\end{prooftree}
$$

Gentzen–Prawitz Natural Deduction

$$
\dfrac{\dfrac{[A \land \neg B]^1}{\neg B}\land E \qquad \dfrac{A \to B \qquad \dfrac{[A \land \neg B]^1}{A}\land E}{B}\to E}{\dfrac{\sharp}{\neg(A \land \neg B)}\neg I^1}\neg E
$$

Natural Deduction Rules

A

Natural Deduction Rules

$$A$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B} \to\! I^i$$

$$\dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{B} \to\! E$$

Natural Deduction Rules

$$A \qquad \frac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \rightarrow B} \rightarrow I^i \qquad \frac{\begin{matrix}\Pi \\ A \rightarrow B\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{B} \rightarrow E$$

$$\frac{\begin{matrix}\Pi \\ A\end{matrix} \qquad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B} \wedge I \qquad \frac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{A} \wedge E \qquad \frac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{B} \wedge E$$

Natural Deduction Rules

$$A$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}{\to}I^i$$

$$\dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{B}{\to}E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B}{\wedge}I$$

$$\dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{A}{\wedge}E \qquad \dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{B}{\wedge}E$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ \sharp\end{matrix}}{\neg A}{\neg}I^i$$

Natural Deduction Rules

$$A$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\;{\to}I^i$$

$$\dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\;{\to}E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad \begin{matrix}\Pi' \\ B\end{matrix}}{A \land B}\;{\land}I$$

$$\dfrac{\begin{matrix}\Pi \\ A \land B\end{matrix}}{A}\;{\land}E \qquad \dfrac{\begin{matrix}\Pi \\ A \land B\end{matrix}}{B}\;{\land}E$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ \sharp\end{matrix}}{\neg A}\;\neg I^i$$

$$\dfrac{\begin{matrix}\Pi \\ \neg A\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{\sharp}\;\neg E$$

Natural Deduction Rules

$$A$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \rightarrow B}\,{\rightarrow}I^i$$

$$\dfrac{\begin{matrix}\Pi \\ A \rightarrow B\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\,{\rightarrow}E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B}\,{\wedge}I$$

$$\dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{A}\,{\wedge}E \qquad \dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{B}\,{\wedge}E$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ \sharp\end{matrix}}{\neg A}\,{\neg}I^i$$

$$\dfrac{\begin{matrix}\Pi \\ \neg A\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{\sharp}\,{\neg}E$$

$$\dfrac{\begin{matrix}\Pi \\ \sharp\end{matrix}}{A}\,{\sharp}E$$

$$
\begin{prooftree}
\AxiomC{$[A \land \neg B]^1$}
\RightLabel{$\land E$}
\UnaryInfC{$\neg B$}
\AxiomC{$A \to B$}
\AxiomC{$[A \land \neg B]^1$}
\RightLabel{$\land E$}
\UnaryInfC{A}
\RightLabel{$\to E$}
\BinaryInfC{B}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg(A \land \neg B)$}
\end{prooftree}
$$

$$A \to B \succ \neg(A \land \neg B)$$

Natural Deduction and Sequents

$$
\dfrac{\dfrac{[A \wedge \neg B]^1}{\neg B}\wedge E \qquad \dfrac{A \to B \qquad \dfrac{[A \wedge \neg B]^1}{A}\wedge E}{B}\to E}{\dfrac{\sharp}{\neg(A \wedge \neg B)}\neg I^1}\neg E
$$

$$A \to B, A \wedge \neg B \succ$$

Natural Deduction and Sequents

$$\dfrac{\dfrac{\dfrac{[A \wedge \neg B]^1}{\neg B}\,{\wedge E} \qquad \dfrac{A \to B \qquad \dfrac{[A \wedge \neg B]^1}{A}\,{\wedge E}}{B}\,{\to E}}{\sharp}\,{\neg E}}{\neg(A \wedge \neg B)}\,{\neg I^1}$$

$$A \wedge \neg B \succ \neg B$$

$$
\dfrac{\dfrac{[A \land \neg B]^1}{\neg B}\land E \qquad \dfrac{A \to B \qquad \dfrac{[A \land \neg B]^1}{A}\land E}{B}\to E}{\dfrac{\sharp}{\neg(A \land \neg B)}\neg I^1}\neg E
$$

$$A \to B, A \land \neg B \succ B$$

Natural Deduction and Sequents

$$
\dfrac{\dfrac{\dfrac{[A \wedge \neg B]^1}{\neg B}{\scriptstyle \wedge E} \qquad \dfrac{A \rightarrow B \qquad \dfrac{[A \wedge \neg B]^1}{A}{\scriptstyle \wedge E}}{B}{\scriptstyle \rightarrow E}}{\sharp}{\scriptstyle \neg E}}{\neg(A \wedge \neg B)}{\scriptstyle \neg I^1}
$$

$$A \wedge \neg B \succ A$$

There's no proof from $\neg(A \wedge \neg B)$ back to $A \to B$.

Classical Logic?

There's no proof from $\neg(A \land \neg B)$ back to $A \to B$.

One option: More sequents — not just $X \succ C$, but $X \succ Y$.

There's no proof from $\neg(A \wedge \neg B)$ back to $A \rightarrow B$.

One option: More sequents — not just $X \succ C$, but $X \succ Y$.

What does that mean for proofs?

Folding Multiple Conclusion Sequents

$$P_1, P_2 \succ C_1, C_2 \quad \textit{can become} \quad P_1, P_2, \cancel{C_1} \succ C_2$$

Folding Multiple Conclusion Sequents

$$P_1, P_2 \succ C_1, C_2 \quad \textit{can become} \quad P_1, P_2, \cancel{C_1} \succ C_2$$

$$\textit{or} \quad P_1, P_2, \cancel{C_2} \succ C_1$$

Folding Multiple Conclusion Sequents

$$P_1, P_2 \succ C_1, C_2 \quad \textit{can become} \quad P_1, P_2, \cancel{C_1} \succ C_2$$

$$\textit{or} \quad P_1, P_2, \cancel{C_2} \succ C_1$$

$$\textit{or} \quad P_1, P_2, \cancel{C_1}, \cancel{C_2} \succ$$

$$P_1, P_2 \succ C_1, C_2 \quad \textit{can become} \quad P_1, P_2, \cancel{C_1} \succ C_2$$

$$\textit{or} \quad P_1, P_2, \cancel{C_2} \succ C_1$$

$$\textit{or} \quad P_1, P_2, \cancel{C_1}, \cancel{C_2} \succ$$

Proofs *with alternatives* have *formulas* or *slashed formulas* at the leaves, and either one formula, or $\sharp$ as a conclusion.

Rules for Alternatives

$$\frac{A \qquad \cancel{A}}{\sharp}\uparrow$$

Rules for Alternatives

$$\begin{array}{c} X, \cancel{Y} \\ \Pi \\ A \qquad \cancel{A} \\ \hline \sharp \end{array}\uparrow$$

$$\frac{X, \cancel{Y} \succ A}{X, \cancel{Y}, \cancel{A} \succ}\uparrow$$

$$\cfrac{\begin{array}{c} X, \cancel{Y} \\ \Pi \\ A \end{array} \qquad \cancel{A}}{\sharp}\,\uparrow$$

$$\cfrac{X, \cancel{Y} \succ A}{X, \cancel{Y}, \cancel{A} \succ}\,\uparrow \qquad \cfrac{X \succ A; Y}{X \succ\ ; A, Y}\,\uparrow$$

$$\dfrac{\begin{array}{c} X, \cancel{Y} \\ \Pi \\ A \end{array} \qquad \cancel{A}}{\sharp}\uparrow \qquad\qquad \dfrac{\begin{array}{c} X, [\cancel{A}]^{i}, \cancel{Y} \\ \Pi \\ \sharp \end{array}}{A}\downarrow^{i}$$

$$\dfrac{X, \cancel{Y} \succ A}{X, \cancel{Y}, \cancel{A} \succ}\uparrow \qquad \dfrac{X \succ A; Y}{X \succ \ ; A, Y}\uparrow$$

$$\dfrac{\begin{array}{c} X, \cancel{Y} \\ \Pi \\ A \end{array} \qquad \cancel{A}}{\sharp}\uparrow \qquad\qquad \dfrac{\begin{array}{c} X, [\cancel{A}]^{i}, \cancel{Y} \\ \Pi \\ \sharp \end{array}}{A}\downarrow^{i}$$

$$\dfrac{X, \cancel{Y} \succ A}{X, \cancel{Y}, \cancel{A} \succ}\uparrow \quad \dfrac{X \succ A; Y}{X \succ \ ; A, Y}\uparrow \qquad \dfrac{X, \cancel{Y}, \cancel{A} \succ}{X, \cancel{Y} \succ A}\downarrow$$

Rules for Alternatives

$$
\dfrac{\begin{matrix} X, \cancel{Y} \\ \Pi \\ A \end{matrix} \qquad \cancel{A}}{\sharp}\uparrow
\qquad\qquad
\dfrac{\begin{matrix} X, [\cancel{A}]^i, \cancel{Y} \\ \Pi \\ \sharp \end{matrix}}{A}\downarrow^i
$$

$$
\dfrac{X, \cancel{Y} \succ A}{X, \cancel{Y}, \cancel{A} \succ}\uparrow
\quad
\dfrac{X \succ A; Y}{X \succ\ ; A, Y}\uparrow
\qquad
\dfrac{X, \cancel{Y}, \cancel{A} \succ}{X, \cancel{Y} \succ A}\downarrow
\quad
\dfrac{X \succ\ ; A, Y}{X \succ A, Y}\downarrow
$$

Rules for Alternatives

$$\dfrac{\begin{array}{c} X, \cancel{Y} \\ \Pi \\ A \end{array} \qquad \cancel{A}}{\sharp}\uparrow \qquad\qquad \dfrac{\begin{array}{c} X, [\cancel{A}]^i, \cancel{Y} \\ \Pi \\ \sharp \end{array}}{A}\downarrow^i$$

$$\dfrac{X, \cancel{Y} \succ A}{X, \cancel{Y}, \cancel{A} \succ}\uparrow \quad \dfrac{X \succ A; Y}{X \succ \ ; A, Y}\uparrow \qquad \dfrac{X, \cancel{Y}, \cancel{A} \succ}{X, \cancel{Y} \succ A}\downarrow \quad \dfrac{X \succ \ ; A, Y}{X \succ A, Y}\downarrow$$

We add the *store* and *retrieve* rules and keep the other rules *fixed*.

The store and retrieve rules are the only rules that manipulate alternatives.

An Example Proof

$$
\begin{prooftree}
\AxiomC{$\neg(A \wedge \neg B)$}
\AxiomC{$[A]^3$}
\AxiomC{$[B]^1$}
\AxiomC{$[\cancel{B}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg B$}
\RightLabel{$\wedge I$}
\BinaryInfC{$A \wedge \neg B$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{B}
\RightLabel{$\to I^3$}
\UnaryInfC{$A \to B$}
\end{prooftree}
$$

An Example Proof

$$
\begin{prooftree}
\AxiomC{$\neg(A \land \neg B)$}
\AxiomC{$[A]^3$}
\AxiomC{$[B]^1$}
\AxiomC{$[\cancel{B}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg B$}
\RightLabel{$\land I$}
\BinaryInfC{$A \land \neg B$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{B}
\RightLabel{$\to I^3$}
\UnaryInfC{$A \to B$}
\end{prooftree}
$$

$B \succ \ ; B$

An Example Proof

\begin{prooftree}
\AxiomC{$\neg(A \wedge \neg B)$}
\AxiomC{$[A]^3$}
\AxiomC{$[B]^1$}
\AxiomC{$[\cancel{B}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg B$}
\RightLabel{$\wedge I$}
\BinaryInfC{$A \wedge \neg B$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{B}
\RightLabel{$\to I^3$}
\UnaryInfC{$A \to B$}
\end{prooftree}

$\succ \neg B; B$

An Example Proof

$$\dfrac{\dfrac{\dfrac{\neg(A \land \neg B) \qquad \dfrac{[A]^3 \qquad \dfrac{\dfrac{[B]^1 \qquad [\cancel{B}]^2}{\sharp}\,\uparrow}{\neg B}\,\neg I^1}{A \land \neg B}\,\land I}{\sharp}\,\neg E}{B}\,\downarrow^2}{A \to B}\,\to I^3$$

$A \succ A \land \neg B; B$

An Example Proof

$$\cfrac{\cfrac{\neg(A \land \neg B) \qquad \cfrac{[A]^3 \qquad \cfrac{\cfrac{[B]^1 \qquad [\cancel{B}]^2}{\sharp}\,{\uparrow}}{\neg B}\,{\neg I^1}}{A \land \neg B}\,{\land I}}{\cfrac{\sharp}{B}\,{\downarrow^2}}\,{\neg E}}{A \to B}\,{\to I^3}$$

$$\neg(A \land \neg B), A \succ \ ; B$$

An Example Proof

$$
\begin{prooftree}
\AxiomC{$\neg(A \wedge \neg B)$}
\AxiomC{$[A]^3$}
\AxiomC{$[B]^1$}
\AxiomC{$[\cancel{B}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg B$}
\RightLabel{$\wedge I$}
\BinaryInfC{$A \wedge \neg B$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{B}
\RightLabel{$\to I^3$}
\UnaryInfC{$A \to B$}
\end{prooftree}
$$

$$\neg(A \wedge \neg B), A \succ B;$$

An Example Proof

$$\dfrac{\dfrac{\dfrac{\neg(A \land \neg B) \qquad \dfrac{[A]^3 \qquad \dfrac{\dfrac{[B]^1 \qquad [\cancel{B}]^2}{\sharp}\,{\uparrow}}{\neg B}\,{\neg I^1}}{A \land \neg B}\,{\land I}}{\sharp}\,{\neg E}}{B}\,{\downarrow^2}}{A \to B}\,{\to I^3}$$

$\neg(A \land \neg B) \succ A \to B;$

WEAKENING AND EXPLOSION

Paradoxes of Relevance

$p \succ q \to p$ $\qquad\qquad$ $p, \neg p \succ q$

Paradoxes of Relevance

$$p \succ q \to p$$

$$p, \neg p \succ q$$

$$\frac{p}{q \to p}\,{\to}I$$

$$p \succ q \to p$$

$$\dfrac{p}{q \to p}\,{\to}I$$

$$p, \neg p \succ q$$

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\,\sharp E$$

Given alternatives, $\sharp E$ is not a separate rule!

$$\frac{\sharp}{A}\,\sharp E$$

Given alternatives, $\sharp E$ is not a separate rule!

$$\frac{\sharp}{A}\,\sharp E$$

$$\begin{array}{c}[\cancel{A}]^i \\ \Pi \\ \sharp \\ \hline A\end{array}\,{\downarrow}^i$$

Discharge Policies

	DUPLICATES	NO DUPLICATES
VACUOUS	*Standard*	*Affine*
NO VACUOUS	*Relevant*	*Linear*

VARIETIES OF CONJUNCTION

Conjunction and Weakening

$$\dfrac{\dfrac{\dfrac{p \qquad [q]^1}{p \land q}\land I}{p}\land E}{q \to p}\to I^1$$

Conjunction and Weakening

$$\dfrac{\dfrac{\dfrac{p \qquad [q]^1}{p \wedge q}\wedge I}{p}\wedge E}{q \rightarrow p}\rightarrow I^1$$

Don't use $\wedge I$ with $\wedge E$ if you want to avoid weakening!

Start with $\wedge I$: *Multiplicative* Conjunction

$$\frac{A \qquad B}{A \otimes B}\otimes I$$

Start with $\wedge I$: *Multiplicative* Conjunction

$$\frac{A \qquad B}{A \otimes B}\,{\otimes}I \qquad\qquad \frac{A \otimes B \qquad \begin{array}{c}[A]^i, [B]^i \\ \Pi \\ C\end{array}}{C}\,{\otimes}E^i$$

Start with $\wedge I$: *Multiplicative* Conjunction

$$\frac{A \qquad B}{A \otimes B}\otimes I$$

$$\frac{A \otimes B \qquad \begin{array}{c}[A]^i, [B]^i \\ \Pi \\ C\end{array}}{C}\otimes E^i$$

$$\frac{X \succ A; Y \qquad X' \succ B; Y'}{X, X' \succ A \otimes B; Y, Y'}\otimes R$$

Start with $\wedge I$: *Multiplicative* Conjunction

$$\frac{A \qquad B}{A \otimes B}\,{\otimes}I \qquad\qquad \frac{A \otimes B \qquad \begin{array}{c}[A]^i, [B]^i \\ \Pi \\ C\end{array}}{C}\,{\otimes}E^i$$

$$\frac{X \succ A; Y \qquad X' \succ B; Y'}{X, X' \succ A \otimes B; Y, Y'}\,{\otimes}R \qquad\qquad \frac{X, A, B \succ C; Y}{X, A \otimes B \succ C; Y}\,{\otimes}L$$

Start with $\wedge$E: *Additive* Conjunction

$$\frac{A \sqcap B}{A}\,{\scriptstyle \sqcap E} \qquad \frac{A \sqcap B}{B}\,{\scriptstyle \sqcap E}$$

Start with $\wedge$E: *Additive* Conjunction

$$\frac{A \sqcap B}{A}\sqcap E \qquad \frac{A \sqcap B}{B}\sqcap E$$

$$\frac{X, A \succ C; Y}{X, A \sqcap B \succ C; Y}\sqcap L \qquad \frac{X, B \succ C; Y}{X, A \sqcap B \succ C; Y}\sqcap L$$

Start with $\wedge E$: *Additive* Conjunction

$$\frac{A \sqcap B}{A}\,{\sqcap E} \qquad \frac{A \sqcap B}{B}\,{\sqcap E}$$

$$\frac{X, A \succ C; Y}{X, A \sqcap B \succ C; Y}\,{\sqcap L} \qquad \frac{X, B \succ C; Y}{X, A \sqcap B \succ C; Y}\,{\sqcap L} \qquad \frac{X \succ A; Y \qquad X \succ B; Y}{X \succ A \sqcap B; Y}\,{\sqcap R}$$

Start with $\wedge$E: *Additive* Conjunction

$$\frac{A \sqcap B}{A}\,{\sqcap}E \qquad \frac{A \sqcap B}{B}\,{\sqcap}E \qquad \frac{\begin{array}{c} X, \cancel{Y} \\ \Pi_1 \\ A \end{array} \qquad \begin{array}{c} X, \cancel{Y} \\ \Pi_2 \\ B \end{array}}{A \sqcap B}\,{\sqcap}I$$

$$\frac{X, A \succ C; Y}{X, A \sqcap B \succ C; Y}\,{\sqcap}L \qquad \frac{X, B \succ C; Y}{X, A \sqcap B \succ C; Y}\,{\sqcap}L \qquad \frac{X \succ A; Y \qquad X \succ B; Y}{X \succ A \sqcap B; Y}\,{\sqcap}R$$

Combining Assumptions

$$
\dfrac{\begin{array}{c} X, \cancel{Y} \\ \Pi_1 \\ A \end{array} \qquad \begin{array}{c} X, \cancel{Y} \\ \Pi_2 \\ B \end{array}}{A \sqcap B} \sqcap I
$$

$$
\dfrac{X \succ A; Y \qquad X \succ B; Y}{X \succ A \sqcap B; Y} \sqcap R
$$

$$\dfrac{\begin{matrix} X, \cancel{Y} \\ \Pi_1 \\ A \end{matrix} \qquad \begin{matrix} [X, \cancel{Y}]^i \\ \Pi_2 \\ B \end{matrix}}{A \sqcap B}\,{\sqcap}I^i$$

$$\dfrac{X \succ A; Y \qquad X \succ B; Y}{X \succ A \sqcap B; Y}\,{\sqcap}R$$

You can't compose proofs using $\sqcap I$

$$\frac{p \sqcap q}{p}\,{\sqcap E} \qquad \frac{p \qquad [p]^1}{p \sqcap p}\,{\sqcap I^1}$$

$$\frac{p \sqcap q}{p}\,{\sqcap E} \qquad\qquad \frac{p \qquad [p]^1}{p \sqcap p}\,{\sqcap I^1}$$

$$\dfrac{\dfrac{p \sqcap q}{p}\,{\sqcap E} \qquad [p]^1}{p \sqcap p}\,\cancel{\sqcap I^1}$$

CONTRACTION, COMPOSITION, AND ASSUMPTIONS

Let's take a closer look at contraction

$$\dfrac{\dfrac{[p]^1 \qquad [p]^1}{p \otimes p}\,{\scriptstyle \otimes I}}{p \to (p \otimes p)}\,{\scriptstyle \to I}$$

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \otimes p}\,{\scriptstyle \otimes R}}{p \succ p \otimes p}\,{\scriptstyle W}}{\succ p \to (p \otimes p)}\,{\scriptstyle \to I}$$

Let's take a closer look at contraction

$$\dfrac{\dfrac{[p]^1 \qquad [p]^1}{p \otimes p}\,{\otimes}I}{p \to (p \otimes p)}\,{\to}I$$

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \otimes p}\,{\otimes}R}{p \succ p \otimes p}\,W}{\succ p \to (p \otimes p)}\,{\to}I$$

Let's take a closer look at contraction

$$\dfrac{\dfrac{[p]^1 \qquad [p]^1}{p \otimes p}\otimes I}{p \to (p \otimes p)}\to I$$

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \otimes p}\otimes R}{p \succ p \otimes p}W}{\succ p \to (p \otimes p)}\to I$$

Let's take a closer look at contraction

$$\dfrac{\dfrac{[p]^1 \quad [p]^1}{p \otimes p}\otimes I}{p \to (p \otimes p)}\to I$$

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \otimes p}\otimes R}{\color{red}{p \succ p \otimes p}}W}{\succ p \to (p \otimes p)}\to I$$

Let's take a closer look at contraction

$$\dfrac{\dfrac{[p]^1 \quad [p]^1}{p \otimes p}\otimes I}{p \to (p \otimes p)}\to I$$

$$\dfrac{\dfrac{\dfrac{p \succ p \quad p \succ p}{p, p \succ p \otimes p}\otimes R}{p \succ p \otimes p}W}{\succ p \to (p \otimes p)}\to I$$

We can *identify* assumptions before discharging them.

In a type theory, this is managed by *assumption variables*

$$\frac{p \qquad p}{p \otimes p}\otimes I$$

In a type theory, this is managed by *assumption variables*

$$\frac{p \qquad p}{p \otimes p}\otimes I \qquad\qquad \frac{x : p \qquad y : p}{\langle x, y\rangle : p \otimes p}\otimes I$$

In a type theory, this is managed by *assumption variables*

$$\frac{p \qquad p}{p \otimes p}\otimes I \qquad\qquad \frac{x : p \qquad y : p}{\langle x, y\rangle : p \otimes p}\otimes I$$

$$\frac{p^1 \qquad p^1}{p \otimes p}\otimes I$$

In a type theory, this is managed by *assumption variables*

$$\frac{p \qquad p}{p \otimes p}\otimes I \qquad\qquad \frac{x : p \qquad y : p}{\langle x, y\rangle : p \otimes p}\otimes I$$

$$\frac{p^1 \qquad p^1}{p \otimes p}\otimes I \qquad\qquad \frac{x : p \qquad x : p}{\langle x, x\rangle : p \otimes p}\otimes I$$

In a type theory, this is managed by *assumption variables*

$$\dfrac{p \qquad p}{p \otimes p}\otimes I \qquad\qquad \dfrac{x : p \qquad y : p}{\langle x, y\rangle : p \otimes p}\otimes I$$

$$\dfrac{p^1 \qquad p^1}{p \otimes p}\otimes I \qquad\qquad \dfrac{x : p \qquad x : p}{\langle x, x\rangle : p \otimes p}\otimes I$$

Here, proofs come with *equivalence classes*
on formula occurrences in the leaves, indicated by labelling.

Distinguishing two senses of *assumption*

- The *act* of assuming p.
- The *content* p assumed.
  - If the acts are the same, the contents are too.
  - But different acts can share the same content.

$$
\dfrac{\begin{array}{c} X^{\alpha}, \cancel{Y}^{\beta} \\ \Pi_1 \\ A \end{array} \qquad \begin{array}{c} X^{\alpha}, \cancel{Y}^{\beta} \\ \Pi_2 \\ B \end{array}}{A \sqcap B}\,{\sqcap I}
$$

Here, α and β *identify* the labellings in X and $\cancel{Y}$ respectively.
The equivalence relation links one class in Π_1 with one class in Π_2.

$$\dfrac{\begin{array}{c} X^{\alpha}, \cancel{Y}^{\beta} \\ \Pi_1 \\ A \end{array} \qquad \begin{array}{c} X'^{\alpha'}, \cancel{Y'}^{\beta'} \\ \Pi_2 \\ B \end{array}}{A \otimes B}\ \otimes I$$

Here, the labellings α, β and α', β' are *disjoint* if we do not allow *contraction* as a structural rule.

The equivalence classes in the two proofs are kept disjoint.

Compare

$$
\dfrac{\dfrac{\dfrac{p \sqcap (q \sqcap r)^1}{p}{\sqcap E} \qquad \dfrac{\dfrac{p \sqcap (q \sqcap r)^1}{q \sqcap r}{\sqcap E}}{q}{\sqcap E}}{p \sqcap q}{\sqcap I} \qquad \dfrac{\dfrac{p \sqcap (q \sqcap r)^1}{q \sqcap r}{\sqcap E}}{r}{\sqcap E}}{(p \sqcap q) \sqcap r}{\sqcap I}
$$

$$p \sqcap (q \sqcap r) \succ (p \sqcap q) \sqcap r$$

Compare

$$\dfrac{\dfrac{\dfrac{p \sqcap (q \sqcap r)^1}{p}\,{\sqcap}E \qquad \dfrac{\dfrac{p \sqcap (q \sqcap r)^1}{q \sqcap r}\,{\sqcap}E}{q}\,{\sqcap}E}{p \sqcap q}\,{\sqcap}I \qquad \dfrac{\dfrac{p \sqcap (q \sqcap r)^1}{q \sqcap r}\,{\sqcap}E}{r}\,{\sqcap}E}{(p \sqcap q) \sqcap r}\,{\sqcap}I$$

$$p \sqcap (q \sqcap r) \succ (p \sqcap q) \sqcap r$$

$$\dfrac{\dfrac{\dfrac{p \sqcap (q \sqcap r)^1}{p}\,{\sqcap}E \qquad \dfrac{\dfrac{p \sqcap (q \sqcap r)^1}{q \sqcap r}\,{\sqcap}E}{q}\,{\sqcap}E}{p \sqcap q}\,{\sqcap}I \qquad \dfrac{\dfrac{p \sqcap (q \sqcap r)^2}{q \sqcap r}\,{\sqcap}E}{r}\,{\sqcap}E}{(p \sqcap q) \otimes r}\,{\otimes}I$$

$$p \sqcap (q \sqcap r), p \sqcap (q \sqcap r) \succ (p \sqcap q) \otimes r$$

You *can* compose proofs — substitute on *the assumption*

$$\dfrac{p \sqcap q}{p}\,{\scriptstyle \sqcap E} \qquad \dfrac{p^1 \qquad p^1}{p \sqcap p}\,{\scriptstyle \sqcap I}$$

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Composition, in General

$$\frac{X \succ A; Y \qquad A, X' \succ B; Y'}{X, X' \succ B; Y, Y'}\,\mathit{Cut}$$

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$$\begin{array}{c} X, \cancel{Y} \\ \Pi \\ A \end{array} \qquad \begin{array}{c} A^i, X', \cancel{Y'} \\ \Pi' \\ B \end{array}$$

Composition, in General

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$$\begin{array}{c} X, \cancel{Y} \\ \Pi \\ A \end{array} \qquad\qquad \begin{array}{c} A^{i}, X', \cancel{Y'} \\ \Pi' \\ B \end{array} \qquad\qquad \begin{array}{c} \begin{array}{c} X^{\alpha}, \cancel{Y}^{\beta} \\ \Pi \\ A \end{array} \; X', \cancel{Y'} \\ \Pi' \\ B \end{array}$$

α and β are sets of new labels used to identify each distinct occurrence of the assumptions in X and $\cancel{Y}$.

Some Upshots

- *Alternatives* are a well-behaved addition to Gentzen–Prawitz natural deduction.

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- *Alternatives* are a well-behaved addition to Gentzen–Prawitz natural deduction.
- Alternatives help us *unify* the natural deduction account of *relevance/weakening*.
- The *act/content* distinction applies to assumptions, and this is important when it comes to different forms of *contraction*, and the composition of proofs.

Thank you!

SLIDES: `https://consequently.org/presentation/2022/natural-deduction-with-alternatives-bochum`

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