Natural Deduction Proof for Substructural, Constructive and Classical Logics

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Leeds *Logic Seminar*

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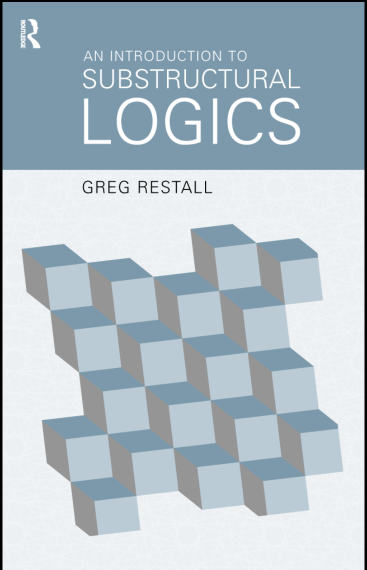
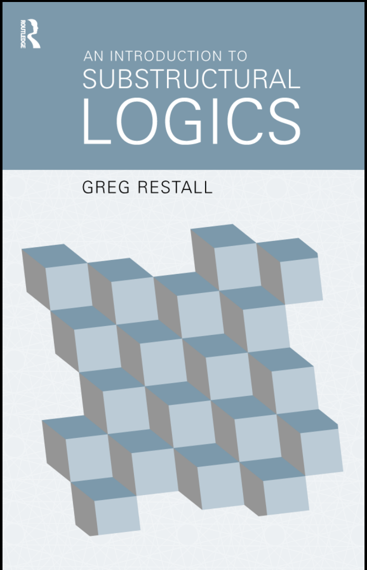


Substructural Logics



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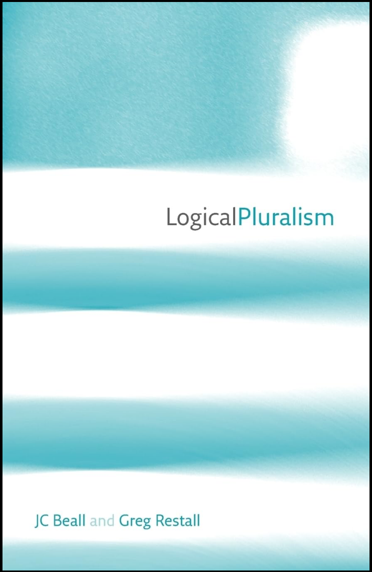
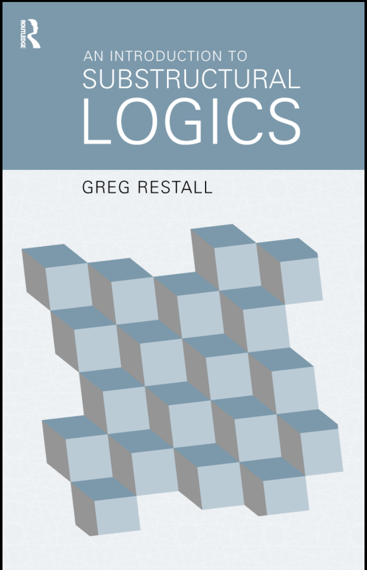
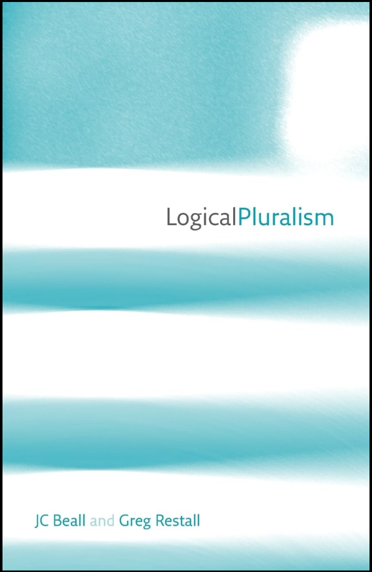


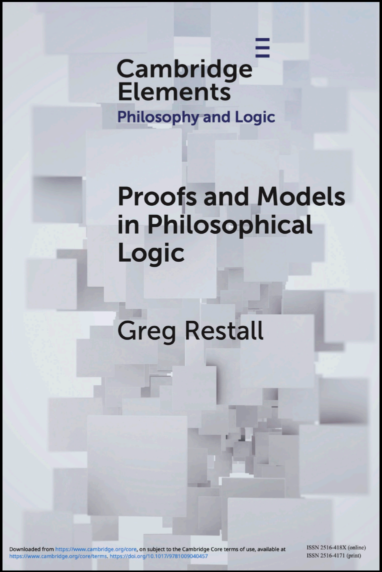
Logical Pluralism



Substructural Logics



Logical Pluralism



Philosophy of Proof Theory

1 · TWO RULES / FOUR LOGICS

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

3 · TRANSLATION / NORMALISATION

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

3 · TRANSLATION / NORMALISATION

4 · MEANINGS

1 · TWO RULES / FOUR LOGICS

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B} \to I^i$$

$$\dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{B} \to E$$

$$A \qquad \dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B} \to I^i \qquad \dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{B} \to E$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p \to (q \to r)]^3 \qquad [p]^1}{q \to r}\,{\to}E \qquad \dfrac{[p \to q]^2 \qquad [p]^1}{q}\,{\to}E}{r}\,{\to}E}{p \to r}\,{\to}I^1}{(p \to q) \to (p \to r)}\,{\to}I^2}{(p \to (q \to r) \to ((p \to q) \to (p \to r))}\,{\to}I^3
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x : p \to (q \to r)] \qquad [z : p]}{(xz) : q \to r}\,{\to}E \qquad \dfrac{[y : p \to q] \qquad [z : p]}{(xy) : q}\,{\to}E}{((xz)(xy)) : r}\,{\to}E}{\lambda z((xz)(xy)) : p \to r}\,{\to}I^{z}}{\lambda y \lambda z((xz)(xy)) : (p \to q) \to (p \to r)}\,{\to}I^{y}}{\lambda x \lambda y \lambda z((xz)(xy)) : (p \to (q \to r)) \to ((p \to q) \to (p \to r))}\,{\to}I^{x}$$

$$x : A$$

$$\dfrac{\begin{matrix}[x : A] \\ \vdots \\ M : B\end{matrix}}{\lambda x M : A \to B} \to I^{x}$$

$$\dfrac{\begin{matrix}\vdots \\ M : A \to B\end{matrix} \qquad \begin{matrix}\vdots \\ N : A\end{matrix}}{(M\,N) : B} \to E$$

$$x : A \qquad \dfrac{\begin{array}{c}[x : A] \\ \vdots \\ M : B\end{array}}{\lambda x M : A \to B}\,{\to}I^x \qquad \dfrac{\begin{array}{cc}\vdots & \vdots \\ M : A \to B & N : A\end{array}}{(M\,N) : B}\,{\to}E$$

You can think of these *terms* as representing processes of *justification* or of *construction*.

(Justify $A \to B$ by taking A as *given*, and using this to justify B.

You can *use* a such a justification of $A \to B$ by *applying* it to a justification of A to produce a justification of B.)

$$
\dfrac{\dfrac{\begin{matrix}[x : A] \\ \vdots \\ M : B\end{matrix}}{\lambda x M : A \to B}\;{\to}I^{x} \qquad \begin{matrix}\vdots \\ N : A\end{matrix}}{(\lambda x M\; N) : B}\;{\to}E
\quad \triangleright \quad
\begin{matrix}\vdots \\ N : A \\ \vdots \\ M\{^{N}/_{x}\} : B\end{matrix}
$$

What about this ‘proof’?

$$\dfrac{\dfrac{[x : p]}{\lambda y x : q \to p}\,{\to}I^{y}}{\lambda x \lambda y x : p \to (q \to p)}\,{\to}I^{x}$$

We never *used* the supposition of q in the justification of p, and this is reflected in the term structure: the λy is *vacuous*.

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We never *used* the supposition of q in the justification of p, and this is reflected in the term structure: the λy is *vacuous*.

We have a choice: *allow* vacuous binding, or *forbid* it.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[x : p] \quad [y : q]}{\langle x, y\rangle : p \wedge q}\,\wedge I}{\mathsf{fst}\langle x, y\rangle : p}\,\wedge E}{\lambda y\, \mathsf{fst}\langle x, y\rangle : q \to p}\,{\to}I^{y}}{\lambda x \lambda y\, \mathsf{fst}\langle x, y\rangle : p \to (q \to p)}\,{\to}I^{x}
$$

BEWARE: You must restrict or modify the usual rules for conjunction if you want to forbid vacuous binding, since with $\mathsf{fst}\langle M, y\rangle$ you can mimic the use of an assumption y in the otherwise y-free M.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x : p \to (q \to r)] \quad [z : p]}{(xz) : q \to r}{\to}E \qquad \dfrac{[y : p \to q] \quad [z : p]}{(yz) : q}{\to}E}{((xz)(yz)) : r}{\to}E}{\lambda z((xz)(yz)) : p \to r}{\to}I^{z}}{\lambda y \lambda z((xz)(yz)) : (p \to q) \to (p \to r)}{\to}I^{y}}{\lambda x \lambda y \lambda z((xz)(yz)) : (p \to (q \to r)) \to ((p \to q) \to (p \to r))}{\to}I^{x}
$$

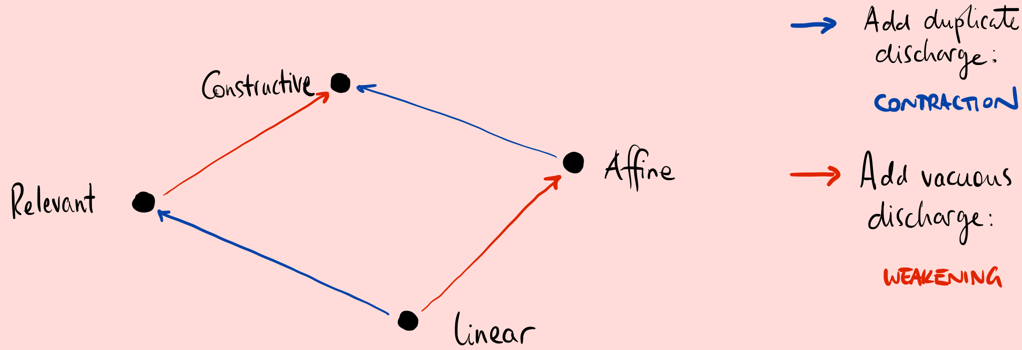
Here we bound *two* instances of z in one go.

$$
\dfrac{
  \dfrac{
    \dfrac{[x : p \to (q \to r)] \quad [z : p]}{(xz) : q \to r}{\to}E
    \qquad
    \dfrac{[y : p \to q] \quad [z : p]}{(yz) : q}{\to}E
  }{
    \dfrac{((xz)(yz)) : r}{
      \dfrac{\lambda z((xz)(yz)) : p \to r}{\lambda y \lambda z((xz)(yz)) : (p \to q) \to (p \to r)}{\to}I^{y}
    }{\to}I^{z}
  }{\to}E
}{\lambda x \lambda y \lambda z((xz)(yz)) : (p \to (q \to r)) \to ((p \to q) \to (p \to r))}{\to}I^{x}
$$

Here we bound *two* instances of z in one go.

There are two options for *duplicate* binding: *allow* and *forbid*.

The Four Logics



We *keep* the $\rightarrow I$ and $\rightarrow E$ rules fixed
and change the *context* in which they apply.

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and change the *context* in which they apply.

Can we extend this analysis to *classical* logic?

In the *sequent* calculus, sort of!

$$\frac{X \succ A \quad B, X' \succ C}{X, A \to B, X' \succ C}\,{\to}L \qquad \frac{X, A \succ B}{X \succ A \to B}\,{\to}R$$

In the *sequent* calculus, sort of!

$$\frac{X \succ A, Y \quad B, X' \succ Y'}{X, A \to B, X' \succ Y, Y'} {\to}L \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y} {\to}R$$

In the *sequent* calculus, sort of!

$$\frac{\succ A \qquad B \succ}{A \to B \succ} \to L \qquad\qquad \frac{A \succ B}{\succ A \to B} \to R$$

$\to L/R$ operate in different *contexts*.

In the *sequent* calculus, sort of!

$$\frac{X \succ A, Y \quad B, X' \succ Y'}{X, A \to B, X' \succ Y, Y'}{\to}L \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}{\to}R$$

$\to L/R$ operate in different *contexts*.

The *classical* context allows for more than one *positive* formula occurrence.

In the *sequent* calculus, sort of!

$$\frac{X \succ A \quad B, X' \succ C}{X, A \to B, X' \succ C}{\to}L \qquad \frac{X, A \succ B}{X \succ A \to B}{\to}R$$

$\to L/R$ operate in different *contexts*.

The *classical* context allows for more than one *positive* formula occurrence.

The *constructive* context imposes a tighter restriction.

In the *sequent* calculus, sort of!

$$\frac{X \succ A \qquad B, X' \succ C}{X, A \to B, X' \succ C}{\to}L \qquad \frac{X, A \succ B}{X \succ A \to B}{\to}R$$

$\to L/R$ operate in different *contexts*.

The *classical* context allows for more than one *positive* formula occurrence.

The *constructive* context imposes a tighter restriction.

(This is totally independent of contraction and weakening.)

Let's try to do this in a natural deduction setting.

Find a *structural* extension to natural deduction
that renders our four *constructive* logics, ***classical***.

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Find a *structural* extension to natural deduction
that renders our four *constructive* logics, ***classical***.

(For bonus points, extend the simply typed λ calculus and
our understanding of processes of justification or construction.)

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

Bilateralism: assertion and denial are on equal footing.

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One option: proofs involve positively tagged formulas ($+A$) and negatively tagged formulas ($-A$).

+-→-I:

$$\begin{array}{c} \overline{+A}^{\,(i)} \\ \vdots \\ +B \\ \hline +(A \to B) \end{array}\ (i)$$

+-→-E:

$$\frac{+(A \to B) \qquad +A}{+B}$$

−-→-I:

$$\frac{+A \qquad -B}{-(A \to B)}$$

−-→-E:

$$\frac{-(A \to B)}{+A} \qquad \frac{-(A \to B)}{-B}$$

+-¬-I:

$$\frac{-A}{+(\neg A)}$$

+-¬-E:

$$\frac{+(\neg A)}{-A}$$

−-¬-I:

$$\frac{+A}{-(\neg A)}$$

−-¬-E:

$$\frac{-(\neg A)}{+A}$$

+-→-I:

$$\begin{array}{c} \overline{\qquad\qquad}\,(i) \\ +A \\ \vdots \\ +B \\ \overline{\qquad\qquad}\,(i) \\ +(A \rightarrow B) \end{array}$$

+-→-E:

$$\dfrac{+(A \rightarrow B) \qquad +A}{+B}$$

–-→-I:

$$\dfrac{+A \quad -B}{-(A \rightarrow B)}$$

–-→-E:

$$\dfrac{-(A \rightarrow B)}{+A} \qquad \dfrac{-(A \rightarrow B)}{-B}$$

+-¬-I:

$$\dfrac{-A}{+(\neg A)}$$

+-¬-E:

$$\dfrac{+(\neg A)}{-A}$$

–-¬-I:

$$\dfrac{+A}{-(\neg A)}$$

–-¬-E:

$$\dfrac{-(\neg A)}{+A}$$

*This is **more** than just a change in the structural context.*

The key bilateralist idea is that for $A, B \succ C, D$

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Granting A *and* B, *and setting* D *aside, we have* C.

We can use this idea to make a *purely structural* addition
to natural deduction, keeping our existing rules unchanged.

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array} \qquad \cancel{A}}{\sharp}\,\uparrow \qquad\qquad \dfrac{\begin{array}{c}[\cancel{A}]^{i}\\ \Pi\\ \sharp\end{array}}{A}\,\downarrow^{i}$$

$$\dfrac{\begin{array}{c}\Pi \\ A\end{array} \qquad \not{A}}{\sharp}\uparrow \qquad\qquad \dfrac{\begin{array}{c}[\not{A}]^{i} \\ \Pi \\ \sharp\end{array}}{A}\downarrow^{i}$$

$$\dfrac{X \succ \boxed{A}, Y}{X \succ \boxed{}\; A, Y}$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad \cancel{A}}{\sharp}\uparrow \qquad\qquad \dfrac{\begin{matrix}[\cancel{A}]^{i} \\ \Pi \\ \sharp\end{matrix}}{A}\downarrow^{i}$$

$$\dfrac{X \succ \boxed{A}, Y}{X \succ \boxed{}\ A, Y}$$

$$\frac{\begin{matrix}\Pi\\ A\end{matrix}\qquad \cancel{A}}{\sharp}\uparrow \qquad\qquad \frac{\begin{matrix}[\cancel{A}]^{i}\\ \Pi\\ \sharp\end{matrix}}{A}\downarrow^{i}$$

$$\frac{X \succ \boxed{A}, Y}{X \succ \boxed{}\ A, Y} \qquad\qquad \frac{X \succ \boxed{}\ A, Y}{X \succ \boxed{A}, Y}$$

$$\dfrac{\begin{matrix}\Pi & \\ A & \not{A}\end{matrix}}{\sharp}\uparrow \qquad \dfrac{\begin{matrix}[\not{A}]^{i} \\ \Pi \\ \sharp\end{matrix}}{A}\downarrow^{i}$$

$$\dfrac{X \succ \boxed{A}, Y}{X \succ \boxed{}\ A, Y} \qquad \dfrac{X \succ \boxed{}\ A, Y}{X \succ \boxed{A}, Y}$$

Add these rules for alternatives.

Keep the connective rules fixed.

(*Employing whichever discharge/binding discipline you prefer.*)

Now you have a classical version of your logic.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\cancel{p}]^2}{\sharp}\,\uparrow}{q}\,\downarrow}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [\cancel{p}]^2}{\sharp}\,\uparrow}{p}\,\downarrow^2}{((p \to q) \to p) \to p}\,{\to}I^3
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[{\color{red}p}]^1 \qquad [\not{p}]^2}{\sharp}\,\uparrow}{q}\,\downarrow}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [\not{p}]^2}{\sharp}\,\uparrow}{p}\,\downarrow^2}{}}{((p \to q) \to p) \to p}\,{\to}I^3
$$

$$
p \succ \boxed{p}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(p\to q)\to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\not p]^2}{\sharp}\uparrow}{q}\downarrow}{p\to q}\to I^1}{p}\to E \qquad [\not p]^2}{\sharp}\uparrow}{p}\downarrow^2}{((p\to q)\to p)\to p}\to I^3$$

$$p \succ \square p$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\cancel{p}]^2}{\sharp}\uparrow}{q}\downarrow}{p \to q}\to I^1}{p}\to E \qquad [\cancel{p}]^2}{\sharp}\uparrow}{p}\downarrow^2}{((p \to q) \to p) \to p}\to I^3$$

$$p \succ \boxed{q}, p$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\cancel{p}]^2}{\sharp}\,\uparrow}{q}\,\downarrow}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [\cancel{p}]^2}{\sharp}\,\uparrow}{p}\,\downarrow^2}{}}{((p \to q) \to p) \to p}\,{\to}I^3
$$

$$\succ \boxed{p \to q}, p$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\not{p}]^2}{\sharp}\,\uparrow}{q}\,\downarrow}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [\not{p}]^2}{\sharp}\,\uparrow}{p}\,\downarrow^2}{((p \to q) \to p) \to p}\,{\to}I^3$$

$$(p \to q) \to p \succ \boxed{p}, p$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\not p]^2}{\sharp}\uparrow}{q}\downarrow}{p \to q}{\to}I^1}{p}{\to}E \qquad [\not p]^2}{\sharp}\uparrow}{p}\downarrow^2}{((p \to q) \to p) \to p}{\to}I^3
$$

$$
(p \to q) \to p \succ \square\, p, p
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\cancel{p}]^2}{\sharp}\uparrow}{q}\downarrow}{p \to q}\to I^1}{p}\to E \qquad [\cancel{p}]^2}{\sharp}\uparrow}{p}\downarrow^2}{((p \to q) \to p) \to p}\to I^3$$

$$(p \to q) \to p \succ \boxed{p}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}\,[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\not p]^2}{\sharp}\uparrow}{q}\downarrow}{p \to q}\to I^1}{p}\to E \qquad [\not p]^2}{\sharp}\uparrow}{p}\downarrow^2}{((p \to q) \to p) \to p}\to I^3
\qquad \succ \; \boxed{((p \to q) \to p) \to p}
$$

$$x : A \qquad \dfrac{\begin{array}{c}[x : A]\\ \vdots \\ M : B\end{array}}{\lambda x M : A \rightarrow B}\rightarrow I^{x} \qquad \dfrac{\begin{array}{c}\vdots \\ M : A \rightarrow B\end{array} \quad \begin{array}{c}\vdots \\ N : A\end{array}}{(M N) : B}\rightarrow E$$

Terms

$$x : A \qquad \dfrac{\begin{matrix}[x : A] \\ \vdots \\ M : B\end{matrix}}{\lambda x M : A \to B} \to I^x \qquad \dfrac{\begin{matrix}\vdots \\ M : A \to B\end{matrix} \quad \begin{matrix}\vdots \\ N : A\end{matrix}}{(MN) : B} \to E$$

$$\dfrac{\begin{matrix}\vdots \\ M : A\end{matrix} \qquad \alpha : \cancel{A}}{\langle M | \alpha \rangle : \sharp} \uparrow$$

Terms

$$x : A \qquad \dfrac{\begin{matrix}[x : A] \\ \vdots \\ M : B\end{matrix}}{\lambda x M : A \to B} \to I^{x} \qquad \dfrac{\begin{matrix}\vdots \\ M : A \to B\end{matrix} \quad \begin{matrix}\vdots \\ N : A\end{matrix}}{(M N) : B} \to E$$

$$\dfrac{\begin{matrix}\vdots \\ M : A\end{matrix} \qquad \alpha : \not{A}}{\langle M | \alpha \rangle : \sharp} \uparrow$$

Terms Labels

$$x : A \qquad \dfrac{\begin{array}{c}[x : A] \\ \vdots \\ M : B\end{array}}{\lambda x M : A \to B}\to I^{x} \qquad \dfrac{\begin{array}{c}\vdots \\ M : A \to B\end{array} \quad \begin{array}{c}\vdots \\ N : A\end{array}}{(MN) : B}\to E$$

$$\dfrac{\begin{array}{c}\vdots \\ M : A\end{array} \qquad \alpha : \not{A}}{\langle M | \alpha \rangle : \sharp}\uparrow$$

Terms   Labels   Packages

$$x : A \qquad \dfrac{\begin{array}{c}[x : A]\\ \vdots\\ M : B\end{array}}{\lambda x M : A \to B}\to I^{x} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ M : A \to B & N : A\end{array}}{(MN) : B}\to E$$

$$\dfrac{\begin{array}{cc}\vdots & \\ M : A & \alpha : \cancel{A}\end{array}}{\langle M | \alpha \rangle : \sharp}\uparrow \qquad \dfrac{\begin{array}{c}[\alpha : \cancel{A}]\\ \vdots\\ P : \sharp\end{array}}{\mu\alpha P : A}\downarrow^{\alpha}$$

Terms Labels Packages

$$
\dfrac{\dfrac{\dfrac{\dfrac{[x : (p \to q) \to p] \qquad \dfrac{\dfrac{\dfrac{[y : p] \qquad [\alpha : \cancel{p}]}{\langle y | \alpha \rangle : \sharp}\uparrow}{\mu\beta\langle y | \alpha \rangle : q}\downarrow^{\beta}}{\lambda y\, \mu\beta\langle y | \alpha \rangle : p \to q}\to I^{y}}{(x \lambda y\, \mu\beta\langle y | \alpha \rangle) : p}\to E \qquad [\alpha : \cancel{p}]}{\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle : \sharp}\uparrow}{\mu\alpha\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle : p}\downarrow^{\alpha}}{\lambda x\, \mu\alpha\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle : ((p \to q) \to p) \to p}\to I^{x}
$$

$$(\lambda x M\; N) \quad \triangleright \quad M\{{}^{N}\!/_{x}\}$$

$$\dfrac{\dfrac{\begin{array}{c}[x : A]\\ \vdots \\ M : B\end{array}}{\lambda x M : A \to B}\,{\to}I^{x} \qquad \begin{array}{c}\vdots\\ N : A\end{array}}{(\lambda x M\; N) : B}\,{\to}E \qquad \triangleright \qquad \begin{array}{c}\vdots\\ N : A\\ \vdots\\ M\{{}^{N}\!/_{x}\} : B\end{array}$$

$$(\lambda x M\ N) \quad \triangleright \quad M\{^{N}/_{x}\}$$

$$\langle \mu\alpha P | \beta \rangle \quad \triangleright \quad P\{^{\beta}/_{\alpha}\}$$

$$\dfrac{\dfrac{\begin{array}{c}[\alpha : \cancel{A}]\\ \vdots \\ P : \sharp\end{array}}{\mu\alpha P : A}\downarrow^{\alpha} \qquad \beta : \cancel{A}}{\langle \mu\alpha P | \beta \rangle : \sharp}\uparrow \quad \triangleright \quad \begin{array}{c}[\beta : \cancel{A}]\\ \vdots \\ P\{^{\beta}/_{\alpha}\} : \sharp\end{array}$$

$$(\lambda x M\, N) \triangleright M\{{}^{N}/_{x}\}$$

$$\langle \mu\alpha P | \beta \rangle \triangleright P\{{}^{\beta}/_{\alpha}\}$$

$$(\mu\alpha P\, N) \triangleright ???$$

$$(\mu\alpha P\; N) \;\triangleright\; ???$$

$$\dfrac{\dfrac{\begin{array}{c}[\alpha : \cancel{A \to B}] \\ \vdots \\ P : \sharp\end{array}}{\mu\alpha P : A \to B}\downarrow^{\alpha} \qquad \begin{array}{c}\vdots \\ N : A\end{array}}{(\mu\alpha P\; N) : B}\to E$$

$$(\mu\alpha P\ N) \ \triangleright\ ???$$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ * : A \to B\end{array} \qquad [\alpha : \cancel{A \to B}]}{\langle * | \alpha \rangle : \sharp}\uparrow \\ \vdots \\ P : \sharp\end{array}}{\mu\alpha P : A \to B}\downarrow^{\alpha} \qquad \begin{array}{c}\vdots\\ N : A\end{array}}{(\mu\alpha P\ N) : B}\to E$$

$$(\mu\alpha P\, N) \;\triangleright\; ???$$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ * : A \to B\end{array} \qquad [\alpha : \not{A \to B}]}{\langle * | \alpha \rangle : \sharp}\uparrow \\ \vdots \\ P : \sharp\end{array}}{\mu\alpha P : A \to B}\downarrow^{\alpha} \qquad \begin{array}{c}\vdots\\ N : A\end{array}}{(\mu\alpha P\, N) : B}\to E
\quad\triangleright\quad
\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\\ * : A \to B\end{array} \qquad \begin{array}{c}\vdots\\ N : A\end{array}}{(*\, N) : B}\to E \qquad [\beta : \not{B}]}{\langle (*\, N) | \beta \rangle : \sharp}\uparrow \\ \vdots \\ P\{\langle (*\, N)|\beta\rangle / \langle * | \alpha\rangle\} : \sharp\end{array}}{\mu\beta P\{\langle (*\, N)|\beta\rangle / \langle * | \alpha\rangle\} : B}\downarrow^{\beta}$$

$$(\mu\alpha P\, N) \;\triangleright\; \mu\beta P\{\langle(*\, N)|\beta\rangle / \langle *|\alpha\rangle\}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{*\,:\,A \to B} \qquad [\alpha : \cancel{A \to B}]}{\langle *|\alpha\rangle : \sharp}\,\uparrow}{\dfrac{\vdots}{P : \sharp}}}{\mu\alpha P : A \to B}\,\downarrow^{\alpha} \qquad \dfrac{\vdots}{N : A}$$

$$\dfrac{\mu\alpha P : A \to B \qquad N : A}{(\mu\alpha P\, N) : B}\,\to_E$$

$$\triangleright$$

$$\dfrac{\dfrac{\dfrac{\vdots}{*\,:\,A \to B} \qquad \dfrac{\vdots}{N : A}}{(*\, N) : B}\,\to_E \qquad [\beta : \cancel{B}]}{\langle(*\, N)|\beta\rangle : \sharp}\,\uparrow$$

$$\dfrac{\dfrac{\vdots}{P\{\langle(*\, N)|\beta\rangle / \langle *|\alpha\rangle\} : \sharp}}{\mu\beta P\{\langle(*\, N)|\beta\rangle / \langle *|\alpha\rangle\} : B}\,\downarrow^{\beta}$$

$$
\begin{aligned}
(\lambda x M\, N) &\;\triangleright\; M\{^{N}/_{x}\} \\
\langle \mu\alpha P \,|\, \beta \rangle &\;\triangleright\; P\{^{\beta}/_{\alpha}\} \\
(\mu\alpha P\, N) &\;\triangleright\; \mu\beta P\{^{\langle (*\, N) \,|\, \beta \rangle}/_{\langle * \,|\, \alpha \rangle}\}
\end{aligned}
$$

$$\dfrac{[y : (p \to r)] \quad [\alpha : \cancel{p \to r}]}{\langle y | \alpha \rangle : \sharp}\uparrow$$

$$\dfrac{\langle y | \alpha \rangle : \sharp}{\mu\beta\langle y | \alpha \rangle : q}\downarrow^{\beta}$$

$$\dfrac{\mu\beta\langle y | \alpha \rangle : q}{\lambda y\, \mu\beta\langle y | \alpha \rangle : (p \to r) \to q}\to I^{y}$$

$$\dfrac{[x : ((p \to r) \to q) \to (p \to r)] \quad \lambda y\, \mu\beta\langle y | \alpha \rangle : (p \to r) \to q}{(x \lambda y\, \mu\beta\langle y | \alpha \rangle) : (p \to r)}\to E$$

$$\dfrac{(x \lambda y\, \mu\beta\langle y | \alpha \rangle) : (p \to r) \quad [\alpha : \cancel{p \to r}]}{\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle : \sharp}\uparrow$$

$$\dfrac{\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle : \sharp}{\mu\alpha\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle : p \to r}\downarrow^{\alpha}$$

$$\dfrac{\mu\alpha\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle : p \to r \quad [z : p]}{(\mu\alpha\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle z) : r}\to E$$

$$\dfrac{(\mu\alpha\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle z) : r}{\lambda z(\mu\alpha\langle (x \lambda y\, \mu\beta\langle y | \alpha \rangle) | \alpha \rangle z) : p \to r}\to I$$

$$\lambda z(\boxed{\mu\alpha}\langle(x\lambda y\,\mu\beta\langle y\,|\,\boxed{\alpha}\rangle)\,|\,\boxed{\alpha}\rangle z)$$

$$\lambda z(\boxed{\mu\alpha}\langle(x\lambda y\,\mu\beta\langle y|\boxed{\alpha}\rangle)|\boxed{\alpha}\rangle z)$$

$$= \lambda z(\boxed{\mu\alpha\langle(x\lambda y\,\mu\beta\langle y|\alpha\rangle)|\alpha\rangle}\,\boxed{z})$$

$$\lambda z(\boxed{\mu\alpha}\langle(x\lambda y\,\mu\beta\langle y|\boxed{\alpha}\rangle)|\boxed{\alpha}\rangle z)$$

$$= \lambda z(\boxed{\mu\alpha\langle(x\lambda y\,\mu\beta\langle y|\alpha\rangle)|\alpha\rangle}\boxed{z})$$

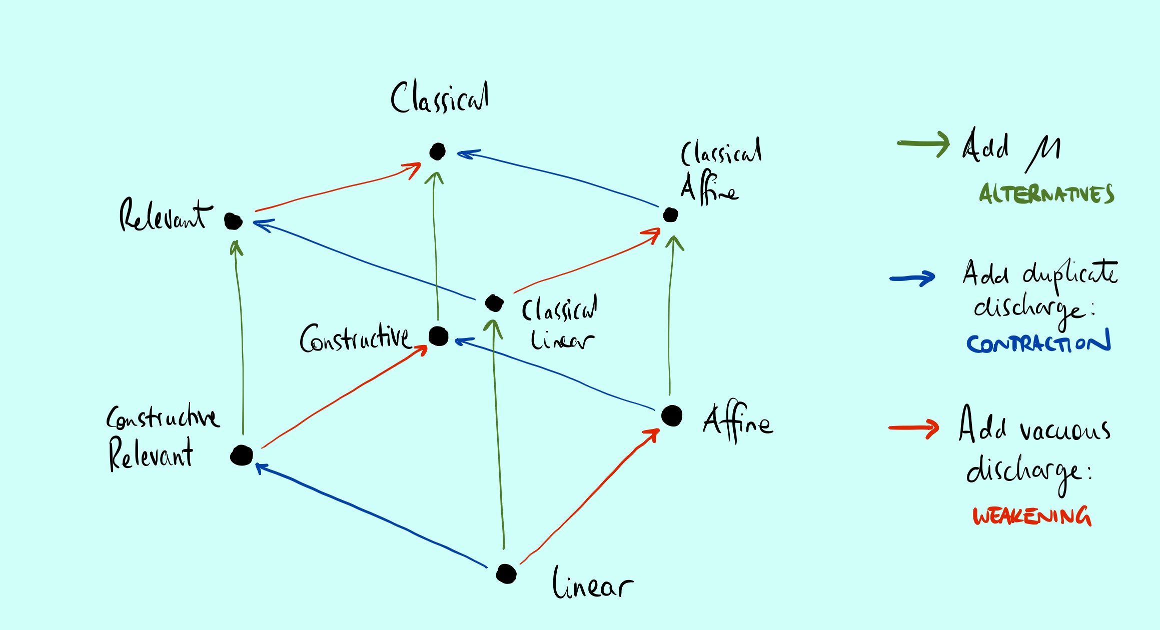
$$\rhd \lambda z\mu\gamma\langle((x\lambda y\,\mu\beta\langle(yz)|\gamma\rangle)z)|\gamma\rangle$$

$$\lambda z(\boxed{\mu\alpha}\langle(x\lambda y\,\mu\beta\langle y|\boxed{\alpha}\rangle)|\boxed{\alpha}\rangle z)$$

$$= \lambda z(\boxed{\mu\alpha\langle(x\lambda y\,\mu\beta\langle y|\alpha\rangle)|\alpha\rangle}\boxed{z})$$

$$\triangleright \lambda z\mu\gamma\langle((x\lambda y\,\mu\beta\langle(y z)|\gamma\rangle)z)|\gamma\rangle$$

$$= \boxed{\lambda z}\mu\gamma\langle((x\lambda y\,\mu\beta\langle(y\boxed{z})|\gamma\rangle)\boxed{z})|\gamma\rangle$$


Classical
Classical Affine
Relevant
Classical Linear
Constructive
Constructive Relevant
Affine
Linear
Add μ
ALTERNATIVES
Add duplicate discharge:
CONTRACTION
Add vacuous discharge:
WEAKENING

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

3 · TRANSLATION / NORMALISATION

It's well known that classical logic
can be found inside intuitionistic logic
using *double negation* translations.

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Classical logic is found inside intuitionistic logic
not only at the level of *provability*, but also
at the level of *proofs*, and even at the level
of proof dynamics—*normalisation*.

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Classical logic is found inside intuitionistic logic
not only at the level of *provability*, but also
at the level of *proofs*, and even at the level
of proof dynamics—*normalisation*.

These results are *robust*. They extend to all four
structural settings, linear, relevant, affine, and full.

We can translate a *classical* logic
(either linear, relevant, affine or full)
inside its constructive counterpart.

FIRST: For formulas, $\sharp$ and ~~slashed~~ formulas, add to the *constructive* language a fresh atom q, and set:

$$\overline{\sharp} = q \qquad \overline{\cancel{p}} = p \to q \qquad \overline{\cancel{A \to B}} = (\overline{A} \to \overline{B}) \to q$$

$$\overline{f} = q \qquad \overline{p} = \overline{\cancel{p}} \to q \qquad \overline{A \to B} = (\overline{\cancel{A \to B}}) \to q$$

We can translate a *classical* logic
(either linear, relevant, affine or full)
inside its constructive counterpart.

FIRST: For formulas, $\sharp$ and ~~slashed~~ formulas, add to the *constructive* language a fresh atom q, and set:

$$\overline{\sharp} = q \qquad \overline{\cancel{p}} = \neg_q p \qquad \overline{\cancel{A \to B}} = \neg_q(\overline{A} \to \overline{B})$$

$$\overline{f} = q \qquad \overline{p} = \neg_q \overline{\cancel{p}} \qquad \overline{A \to B} = \neg_q(\overline{\cancel{A \to B}})$$

SECOND: for terms variables and labels,

whenever $x : A$, we choose a unique variable $\overline{x} : \overline{A}$,

and whenever $\alpha : \not{A}$, we choose a unique variable $\overline{\alpha} : \overline{\not{A}}$.

We extend this translation to all terms and packages as follows . . .

$$
\begin{aligned}
\overline{\lambda x N} &= \lambda y(y \lambda \overline{x} \overline{N}) \\
\overline{(M\,N)} &= \lambda z(\overline{M}\,\lambda y((y\,\overline{N})z)) \\
\overline{\langle M|\alpha\rangle} &= (\overline{M}\,\overline{\alpha}) \\
\overline{\mu\alpha P} &= \lambda\overline{\alpha}\overline{P} \\
\overline{\mu P} &= \overline{P} \\
\overline{\langle N\rangle} &= \overline{N}
\end{aligned}
$$

$$\begin{aligned}
\overline{\lambda x N} &= \lambda y(y\lambda\overline{x}\overline{N}) \\
\overline{(M\,N)} &= \lambda z(\overline{M}\,\lambda y((y\,\overline{N})z)) \\
\overline{\langle M|\alpha\rangle} &= (\overline{M}\,\overline{\alpha}) \\
\overline{\mu\alpha P} &= \lambda\overline{\alpha}\overline{P} \\
\overline{\mu P} &= \overline{P} \\
\overline{\langle N\rangle} &= \overline{N}
\end{aligned}$$

$$\dfrac{[y : \neg_q(\overline{A} \to \overline{B})] \qquad \dfrac{\begin{matrix}[\overline{x} : \overline{A}] \\ \vdots \\ \overline{N} : \overline{B}\end{matrix}}{\lambda\overline{x}\overline{N} : \overline{A} \to \overline{B}}}{\dfrac{(y\lambda\overline{x}\overline{N}) : q}{\lambda y(y\lambda\overline{x}\overline{N}) : \neg_q\neg_q(\overline{A} \to \overline{B})}}$$

$$\overline{\lambda x N} = \lambda y (y \lambda \overline{x} \overline{N})$$

$$\overline{(M\, N)} = \lambda z (\overline{M}\, \lambda y ((y \overline{N}) z))$$

$$\overline{\langle M | \alpha \rangle} = (\overline{M}\, \overline{\alpha})$$

$$\overline{\mu \alpha P} = \lambda \overline{\alpha} \overline{P}$$

$$\overline{\mu P} = \overline{P}$$

$$\overline{\langle N \rangle} = \overline{N}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \overline{M} : \neg_q \neg_q (\overline{A} \to \overline{B})\end{matrix} \qquad \dfrac{\dfrac{\dfrac{[y : \overline{A} \to \overline{B}] \qquad \begin{matrix}\vdots \\ \overline{N} : \overline{A}\end{matrix}}{(y \overline{N}) : \overline{B}} \qquad [z : \cancel{\overline{B}}]}{((y \overline{N}) z) : q}}{\lambda y ((y \overline{N}) z) : \neg_q (\overline{A} \to \overline{B})}}{(\overline{M}\, \lambda y ((y \overline{N}) z)) : q}}{\lambda z (\overline{M}\, \lambda y ((y \overline{N}) z)) : \overline{B}}$$

$$\overline{\lambda x N} = \lambda y(y\lambda\overline{x}\,\overline{N})$$
$$\overline{(M\,N)} = \lambda z(\overline{M}\,\lambda y((y\,\overline{N})z))$$
$$\overline{\langle M|\alpha\rangle} = (\overline{M}\,\overline{\alpha})$$
$$\overline{\mu\alpha P} = \lambda\overline{\alpha}\,\overline{P}$$
$$\overline{\mu P} = \overline{P}$$
$$\overline{\langle N\rangle} = \overline{N}$$

$$\dfrac{\begin{matrix}\vdots & \\ \overline{M} : \overline{A} & \quad \overline{\alpha} : \overline{\cancel{A}}\end{matrix}}{(\overline{M}\,\overline{\alpha}) : q}$$

$$
\begin{aligned}
\overline{\lambda x N} &= \lambda y(y\lambda\overline{x}\overline{N}) \\
\overline{(M\,N)} &= \lambda z(\overline{M}\,\lambda y((y\overline{N})z)) \\
\overline{\langle M|\alpha\rangle} &= (\overline{M}\,\overline{\alpha}) \\
\overline{\mu\alpha P} &= \lambda\overline{\alpha}\overline{P} \\
\overline{\mu P} &= \overline{P} \\
\overline{\langle N\rangle} &= \overline{N}
\end{aligned}
$$

$$
\frac{\begin{array}{c}\vdots \\ \overline{M} : \overline{A}\end{array} \qquad \overline{\alpha} : \cancel{\overline{A}}}{(\overline{M}\,\overline{\alpha}) : q}
\qquad
\frac{\begin{array}{c}[\overline{\alpha} : \cancel{\overline{A}}] \\ \vdots \\ \overline{P} : q\end{array}}{\lambda\overline{\alpha}\overline{P} : \neg_q \cancel{\overline{A}}}
$$

$$\overline{\lambda x N} = \lambda y (y \lambda \overline{x}\, \overline{N})$$

$$\overline{(M\, N)} = \lambda z (\overline{M}\, \lambda y ((y\, \overline{N}) z))$$

$$\overline{\langle M | \alpha \rangle} = (\overline{M}\, \overline{\alpha})$$

$$\overline{\mu \alpha P} = \lambda \overline{\alpha}\, \overline{P}$$

$$\overline{\mu P} = \overline{P}$$

$$\overline{\langle N \rangle} = \overline{N}$$

$$\frac{\begin{array}{c}\vdots\\ \overline{M} : \overline{A}\end{array} \qquad \overline{\alpha} : \overline{\cancel{A}}}{(\overline{M}\, \overline{\alpha}) : q}
\qquad
\frac{\begin{array}{c}[\overline{\alpha} : \overline{\cancel{A}}]\\ \vdots\\ \overline{P} : q\end{array}}{\lambda \overline{\alpha}\, \overline{P} : \neg_q \overline{\cancel{A}}}$$

$$\frac{\begin{array}{c}\vdots\\ \overline{P} : q\end{array}}{\overline{\mu P} : q}
\qquad
\frac{\begin{array}{c}\vdots\\ \overline{N} : q\end{array}}{\overline{\langle N \rangle} : q}$$

$$
\begin{aligned}
\overline{\lambda x N} &= \lambda y(y\lambda\overline{x}\overline{N}) \\
\overline{(M\,N)} &= \lambda z(\overline{M}\,\lambda y((y\overline{N})z)) \\
\overline{\langle M|\alpha\rangle} &= (\overline{M}\,\overline{\alpha}) \\
\overline{\mu\alpha P} &= \lambda\overline{\alpha}\overline{P} \\
\overline{\mu P} &= \overline{P} \\
\overline{\langle N\rangle} &= \overline{N}
\end{aligned}
$$

$$
\dfrac{\begin{array}{c}\vdots \\ \overline{M} : \overline{A}\end{array} \qquad \overline{\alpha} : \overline{\cancel{A}}}{(\overline{M}\,\overline{\alpha}) : q}
\qquad
\dfrac{\begin{array}{c}[\overline{\alpha} : \overline{\cancel{A}}] \\ \vdots \\ \overline{P} : q\end{array}}{\lambda\overline{\alpha}\overline{P} : \neg_q \overline{\cancel{A}}}
$$

$$
\dfrac{\begin{array}{c}\vdots \\ \overline{P} : q\end{array}}{\overline{\mu P} : q}
\qquad
\dfrac{\begin{array}{c}\vdots \\ \overline{N} : q\end{array}}{\overline{\langle N\rangle} : q}
$$

This translation sends a classical proof of A to a constructive proof of $\overline{A}$.

NOTICE: If source terms are linear (or relevant, or affine), so are their translations.

$$(\lambda x M\, N) \triangleright M\{^N/_x\}$$

$$\begin{aligned}
\overline{(\lambda x M\, N)} &= \lambda z(\overline{\lambda x M}\, \lambda w((w\, \overline{N})z)) \\
&= \lambda z(\boxed{\lambda v(v\, \lambda \overline{x}\overline{M})}\ \boxed{\lambda w((w\, \overline{N})z)}) \\
&\triangleright \lambda z(\boxed{\lambda w((w\, \overline{N})z)}\ \boxed{\lambda \overline{x}\overline{M}}) \\
&\triangleright \boxed{\lambda z((\lambda \overline{x}\overline{M}\, \overline{N})z)} \\
&\triangleright_\eta (\boxed{\lambda \overline{x}\overline{M}}\ \boxed{\overline{N}}) \\
&\triangleright \overline{M}\{^{\overline{N}}/_{\overline{x}}\} \\
&= \overline{M\{^N/_x\}}
\end{aligned}$$

$$\langle \mu\alpha P \mid \beta \rangle \triangleright P\{^{\beta}/_{\alpha}\}$$

$$\begin{aligned}
\overline{\langle \mu\alpha P \mid \beta \rangle} &= (\overline{\mu\alpha P}\ \overline{\beta}) \\
&= (\boxed{\lambda\overline{\alpha}\overline{P}}\ \boxed{\overline{\beta}}) \\
&\triangleright \overline{P}\{\overline{\beta}/\overline{\alpha}\} \\
&= \overline{P\{\beta/\alpha\}}
\end{aligned}$$

$$\langle \mu P \rangle \triangleright P$$

$$\overline{\langle \mu P \rangle} = \overline{P}$$

$$(\mu\alpha P\, N) \triangleright \mu\beta P\{\langle(* N)\,|\,\beta\rangle / \langle * \,|\, \alpha\rangle\}$$

$$\begin{aligned}
\overline{(\mu\alpha P\, N)} &= \lambda y(\overline{\mu\alpha P}\,\lambda x((x\,\overline{N})y)) \\
&= \lambda y(\lambda\overline{\alpha}\overline{P}\,\lambda x((x\,\overline{N})y)) \\
&\triangleright \lambda y\overline{P}\{\lambda x((x\,\overline{N})y)/\overline{\alpha}\} \\
&\triangleright \lambda y\overline{P}\{((* N)y)/(*\,\overline{\alpha})\} \\
&= \lambda\overline{\beta}\,\overline{P}\{\overline{\langle(* N)\,|\,\beta\rangle}/\overline{\langle * \,|\, \alpha\rangle}\} \\
&= \overline{\mu\beta P\{\langle(* N)\,|\,\beta\rangle/\langle * \,|\, \alpha\rangle\}}
\end{aligned}$$

All the behaviour of *classical* proof
lives inside *constructive* proof,
for formulas of the form $\overline{A}$.

1 · TWO RULES / FOUR LOGICS

2 · ALTERNATIVES

3 · TRANSLATION / NORMALISATION

4 · MEANINGS

What does this mean for the relationship between classical and constructive reasoning?

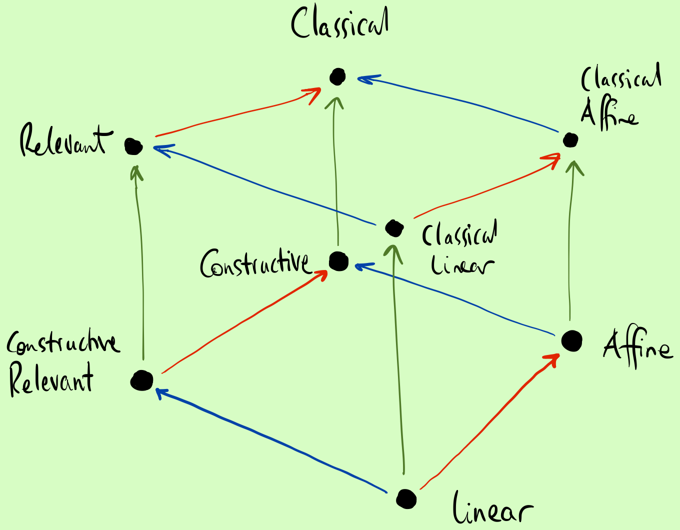
To see how some of the most basic results of classical analysis lack computational meaning, take the assertion that every bounded nonvoid set A of real numbers has a least upper bound. (The real number b is the *least upper bound* of A if $a \leqq b$ for all a in A, and if there exist elements of A that are arbitrarily close to b.) To avoid unnecessary complications, we actually consider the somewhat less general assertion that every bounded sequence (x_k) of rational numbers has a least upper bound b (in the set of real numbers). If this assertion were constructively valid, we could compute b, in the sense of computing a rational number approximating b to within any desired accuracy; in fact, we could program a digital computer to compute the approximations for us. For instance, the computer could be programmed to produce, one by one, a sequence $((b_k, m_k))$ of ordered pairs, where each b_k is a rational number and each m_k is a positive integer, such that $x_j \leqq b_k + k^{-1}$ for all positive integers j and k, and $x_{m_k} \geqq b_k - k^{-1}$ for all positive integers k. Unless there exists a general method M that produces such a computer program corresponding to each bounded, constructively given sequence (x_k) of rational numbers, we are not justified, by constructive standards, in asserting that each of the se-

Errett Bishop and Douglas Bridges, *Constructive Analysis* (1985)

PERSPECTIVE #1:

Classical reasoning *extends* constructive reasoning.

There are statements which can be *proved* classically that *cannot* be proved constructively.


Classical
Classical Affine
Relevant
Classical Linear
Constructive
Constructive Relevant
Affine
Linear

The law of the excluded middle provides a prime example. Constructively, this principle is not universally valid, as we have seen in Exercise **12.1**. Classically, however, it is valid, because every proposition is either false or not false, and being not false is the same as being true. Nevertheless, classical logic is consistent with constructive logic in that constructive logic does not refute classical logic. As we have seen, constructive logic proves that the law of the excluded middle is positively not refuted (its double negation is constructively true). Consequently, constructive logic is stronger (more expressive) than classical logic, because it can express more distinctions (namely, between affirmation and irrefutability), and because it is consistent with classical logic.

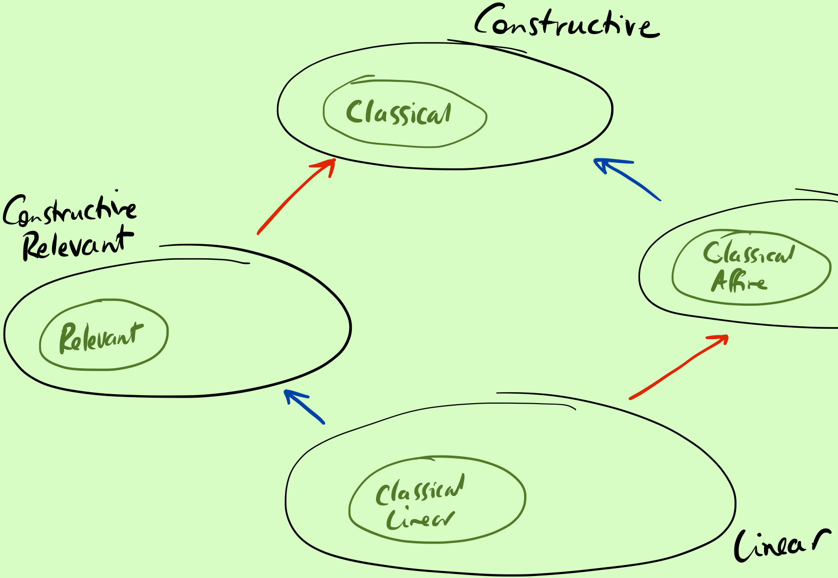
Proofs in constructive logic have computational content: they can be executed as programs, and their behavior is described by their type. Proofs in classical logic also have computational content, but in a weaker sense than in constructive logic. Rather than positively affirm a proposition, a proof in classical logic is a computation that cannot be refuted. Computationally, a refutation consists of a continuation, or control stack, that takes a proof of a proposition and derives a contradiction from it. So a proof of a proposition in classical logic is a computation that, when given a refutation of that proposition derives a contradiction, witnessing the impossibility of refuting it. In this sense, the law of the excluded middle has a proof, precisely because it is irrefutable.

Robert Harper, *Practical Foundations for Programming Languages* (2016)

PERSPECTIVE #2:

Constructive language *extends* classical language.

There are things we can *state* constructively
that *cannot* be stated classically.


Constructive
Classical
Constructive
Relevant
Relevant
Affine
Classical
Affine
Classical
Linear
Linear

Which of these pictures is *correct*?

Which of these pictures is *correct*?

It depends on what you *mean*.

Which of these pictures is *correct*?

It depends on what you *mean*.

That is, it depends on how you individuate the claims we make in our reasoning—the things that have meaning.

We usually take PERSPECTIVE #1 as *given*:
we have one field of statements, and
classical and constructive mathematicians argue
about which statements in that field are correct.

"Take ***the assertion*** *that every bounded non-void set* A *of real numbers has a least upper bound* . . ."

This fits the picture of classical logic as an extension
of constructive logic, allowing for more proofs.

If you take it that propositional content is determined
by what *norms* govern it, then the usual picture is not the *only* one.

Constructive justification is *stricter* than classical justification.

Since there are fewer ways to give constructive justification,
you can do more with such a justification when you have one.

CLASSICALLY: to state something is to rule something out, in that if you and I *rule out* the same things, we have *said* the same thing.

CONSTRUCTIVELY: p and ¬¬p *rule out* the same things, but they might (constructively) entail *different* things, so to say p and to say ¬¬p is to undertake different commitments.

PERSPECTIVE #2A: The constructive distinction between p and ¬¬p is a meaningful difference in what is *said*.

The classical logician erases or ignores differences that are present in propositional content.

PERSPECTIVE #2B: The constructive distinction between p and ¬¬p is not a difference in propositional content.

If we allow only constructive justification, we are in a wider field of *pre*-propositions, only some of which are governed by all the norms that determine propositional content, properly understood.

Our *formal* results are consistent with
PERSPECTIVES #1, #2A and #2B.

I think it is useful to *recognise* these different perspectives,
and to *learn* what is involved in taking up each stance.

Thank You!