Proof Theory

NCS 2024

Greg Restall

University of St Andrews

COURSE PLAN

TODAY	NATURAL DEDUCTION
TUES	SEQUENT SYSTEMS
WED	POSITIONS / MODELS & MORE
THU	MODAL LOGIC

CLASS NOTES & READINGS

& SLIDES, after each day

https://consequently.org/class/2024/nls-proof-theory/

DAY 1

NATURAL DEDUCTION

TODAY'S PLAN

NATURAL DEDUCTION PROOFS FOR $\to$

NORMAL PROOFS

CONSEQUENCES OF NORMALITY

NEGATION & FALSITY

ALTERNATIVES & CLASSICAL LOGIC

EXTRA TOPICS (if time)

What is a proof?

A proof from P_1, P_2, P_3 to C shows that C, in a context in which P_1, P_2 & P_3 are taken as given.

NATURAL DEDUCTION RULES for a conditional

A

$$\frac{A \to B \qquad A}{B} \to E \qquad\qquad \frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B} \to I$$

Let's prove $(s\to q)\to((s\to p)\to(s\to r))$ from $p\to(q\to r)$

$p\to(q\to r)$

$s\to q$

$s\to p$

s

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p\to(q\to r)\qquad \dfrac{[s\to p]^2\quad [s]^1}{p}\to E}{q\to r}\to E\qquad \dfrac{[s\to q]^3\quad [s]^1}{q}\to E}{r}\to E}{s\to r}\to I^1}{(s\to p)\to(s\to r)}\to I^2}{(s\to q)\to((s\to p)\to(s\to r))}\to I^3
$$

STRUCTURAL RULES & DISCHARGE

what is going on here??

$$\dfrac{\begin{array}{c}[A]^i \\ \vdots \\ B\end{array}}{A \to B} \to I^i$$

$$p \vdash q \to p$$

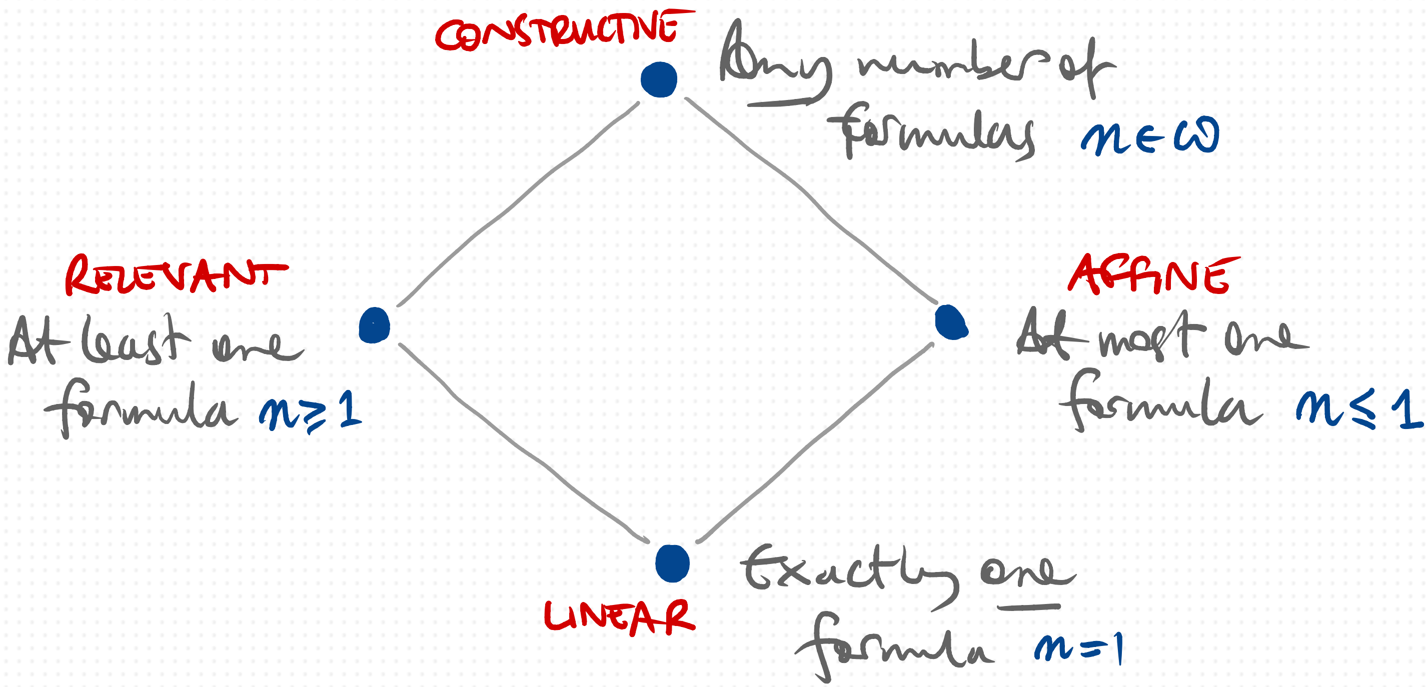
$$\dfrac{p}{q \to p} \to I$$

$$p \to (p \to q) \vdash p \to q$$

$$\dfrac{\dfrac{\dfrac{p \to (p \to q) \qquad (p)^1}{p \to q} \qquad (p)^1}{q}}{p \to q}$$

FOUR DIFFERENT LOGICS

from four discharge policies



The Introduction/Elimination Detour

$$\dfrac{\dfrac{\begin{array}{c}[A]\\ \Pi_1\\ B\end{array}}{A\to B}\,{\scriptstyle \to I} \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{B}\,{\scriptstyle \to E} \qquad \rhd \qquad \begin{array}{c}\Pi_2\\ A\\ \Pi_1\\ B\end{array}$$

That maximum is gone.

local maximum in complexity.

HARMONY

What you can do with $A \to B$ ....

$$\frac{A \to B \qquad A}{B} \to E$$

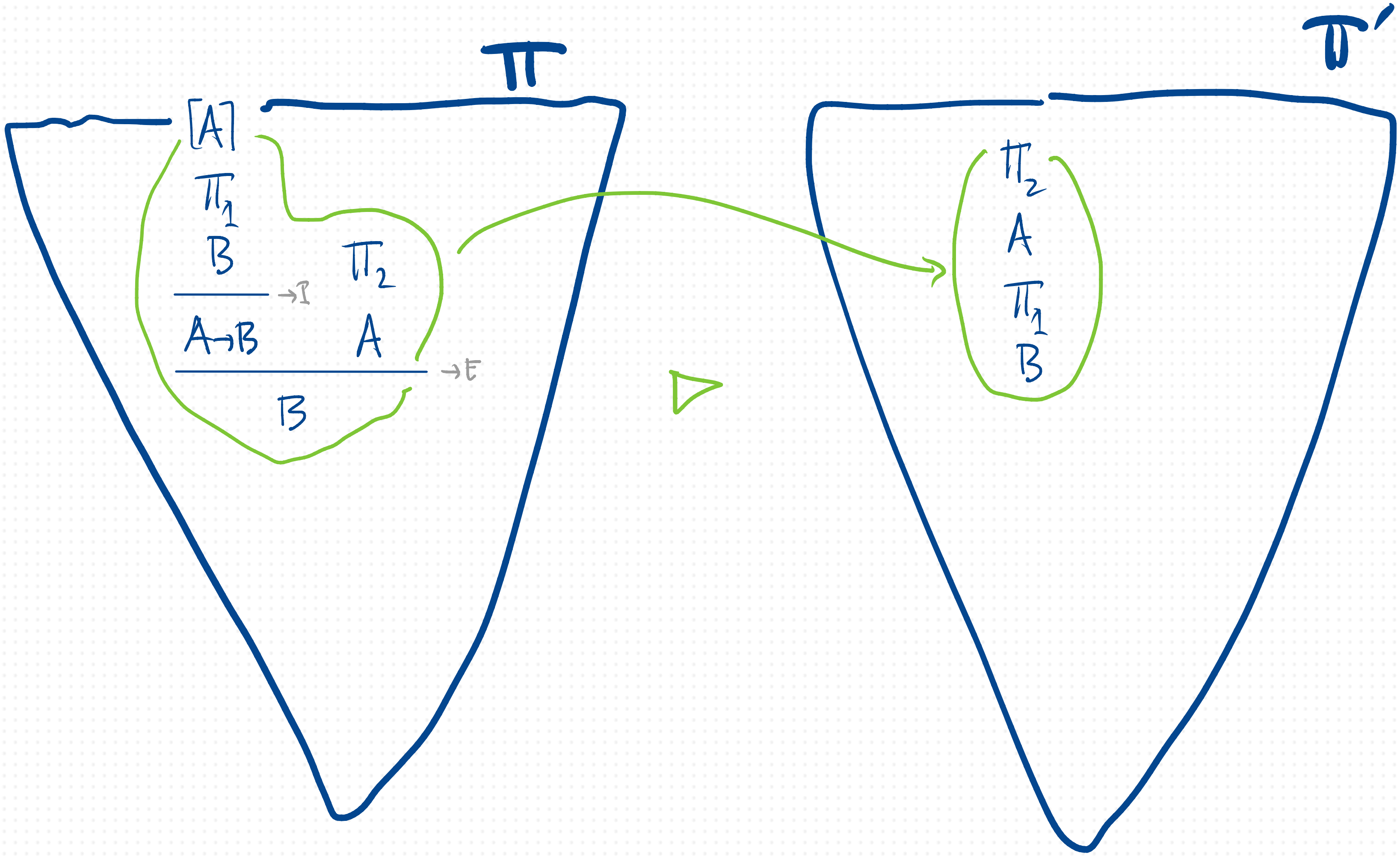
Infer from A to B

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B} \to I$$

Is what it takes to get $A \to B$.

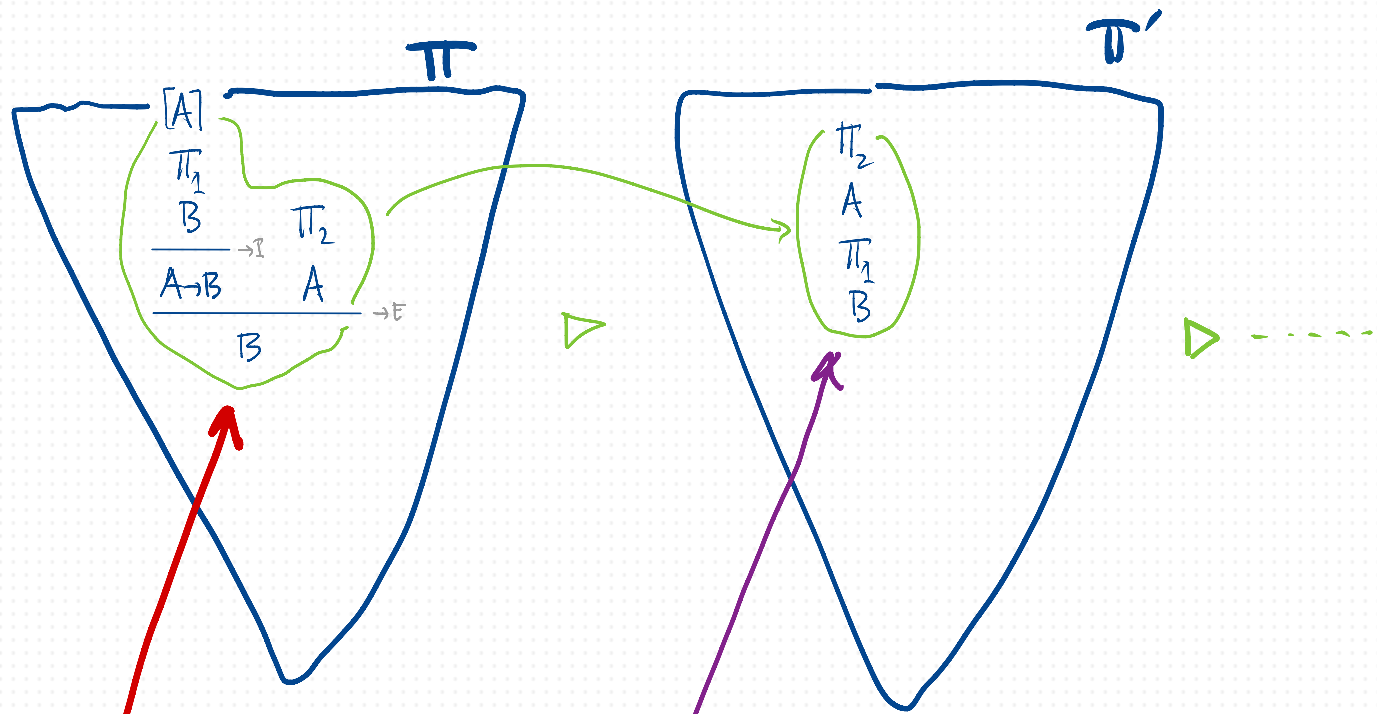
A NORMAL PROOF IS ONE WITH NO DETOURS

NORMALIZATION



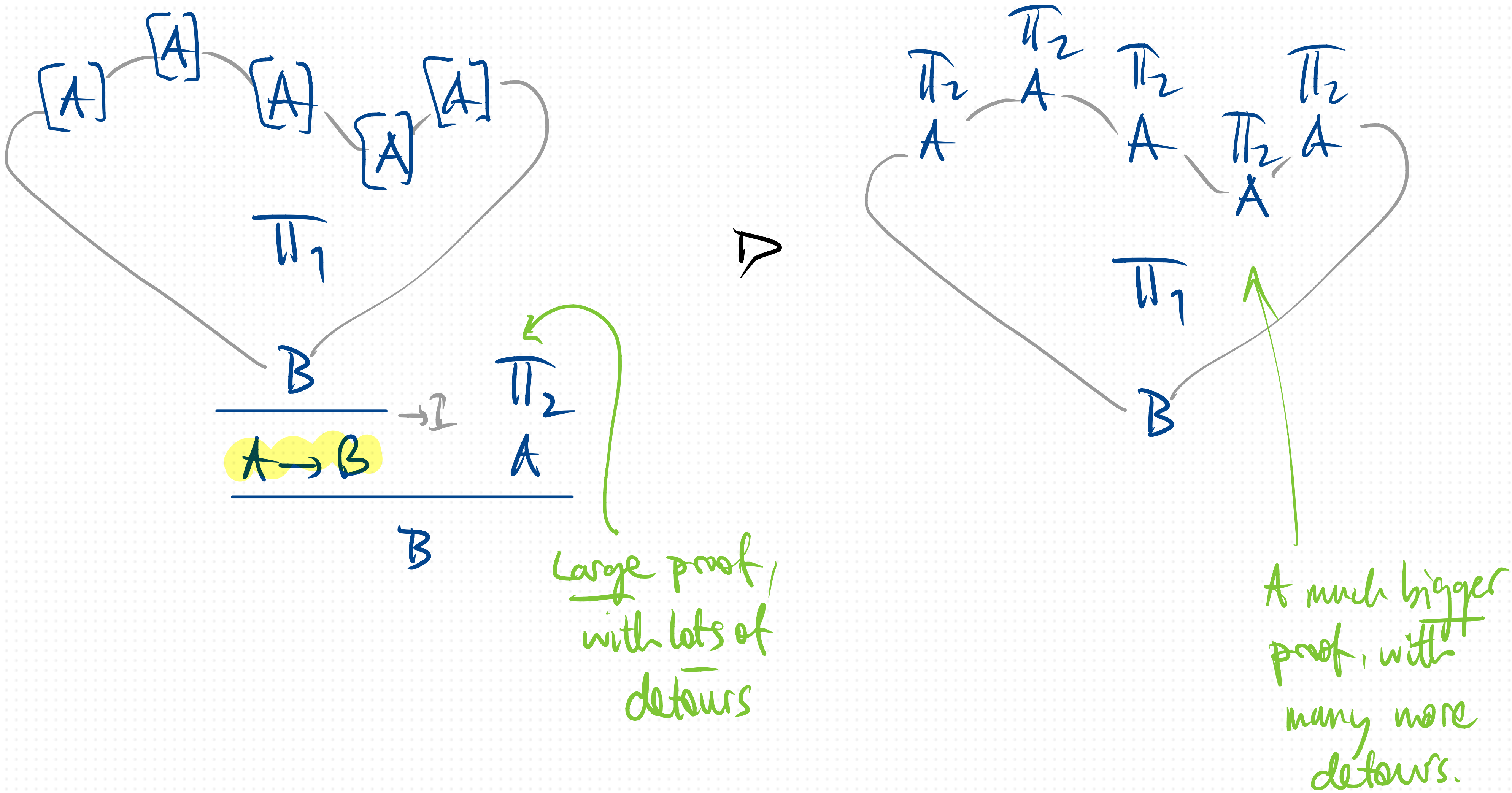
Keep doing this until
the result is normal

LINEAR/AFFINE NORMALISATION

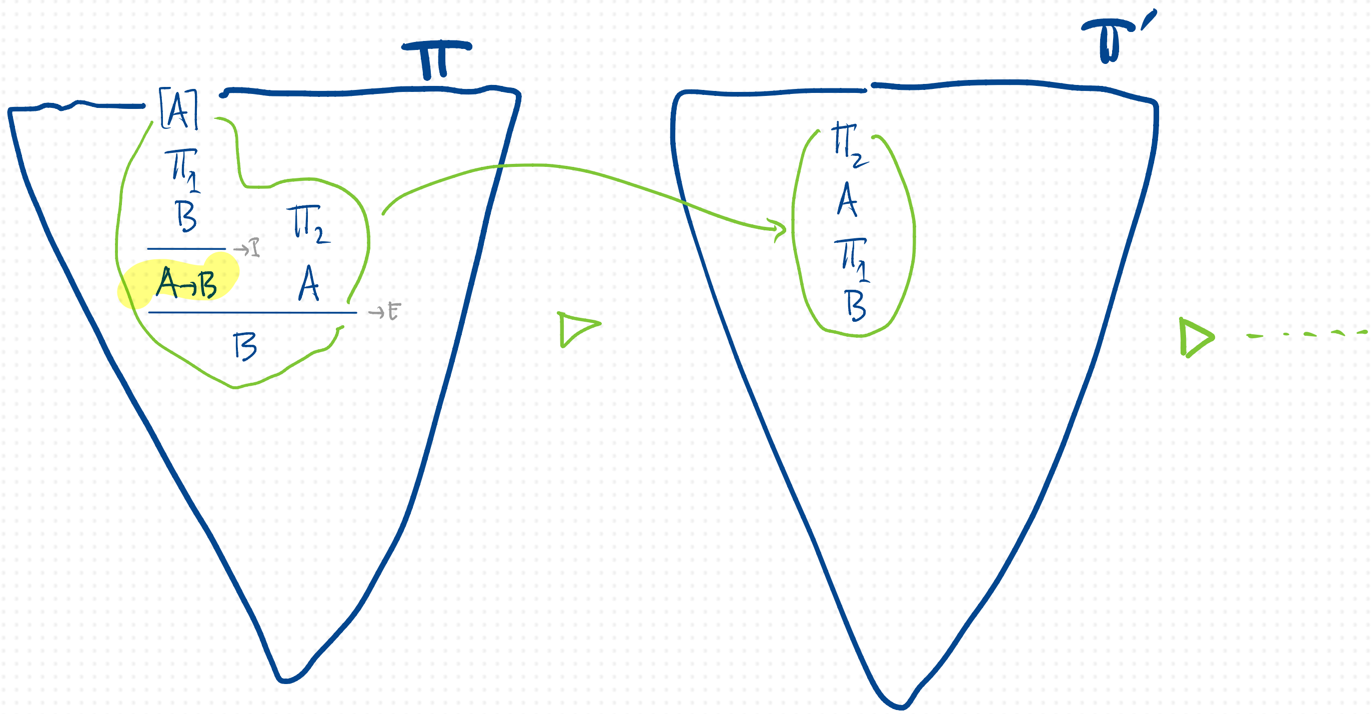


This is larger than this, so the process of normalising must come to an end, since the proofs get smaller.

DUPLICATE DISCHARGE IS A PROBLEM



USE A SMART REDUCTION STRATEGY



- Pick a most complex detour formula,
  - with no detour formulas of that size any higher (ie in Π_1 or in Π_2)
- Π' has fewer detour formulas of that complexity than Π.
- The detour measure $\langle d_1, d_2, \ldots, d_n \rangle$ always decreases.
  (d_i = # detour formulas of complexity i, n complexity of largest d.f.)

... REDUCTION IS CONFLUENT

$$
\begin{tikzcd}
 & \Pi_a \arrow[dr, phantom, "\rhd^*"] & \\
\Pi \arrow[ur, phantom, "\rhd"] \arrow[dr, phantom, "\rhd"] & & \Pi_c \\
 & \Pi_b \arrow[ur, phantom, "\rhd^*"] &
\end{tikzcd}
$$

... AND STRONGLY NORMALIZING.

NORMAL PROOFS ARE ANALYTIC.

In any normal proof Π from X to A, every formula in Π is a subformula of the formulas in $X \cup \{A\}$.

Why?

An induction on the construction of Π.

▷ Π an assumption proof of A from A? Simply obvious.

$$\dfrac{\begin{matrix} X,[A] \\ \Pi' \\ B \end{matrix}}{A \to B}\,{\to}I$$

▷ Π made from a normal proof Π' by $\to I$?

- Everything in Π' is inside X, A, B, so everything in Π is inside $X, A \to B$!

$$\dfrac{\begin{matrix} X \\ \Pi_1 \\ C \to A \end{matrix} \qquad \begin{matrix} Y \\ \Pi_2 \\ C \end{matrix}}{A}\,{\to}E$$

▷ Π is made from normal proofs of $\overset{\Pi_1}{C \to A}$ & $\overset{\Pi_2}{C}$, by $\to E$ and Π_1 does not end in $\to I$…

WHAT IS SO SPECIAL ABOUT ANALYTIC PROOFS?

- SEPARABILITY OF RULES
- PROOF SEARCH
- MEANING?

WHAT ABOUT NEGATION?

A false proposition?

$$\bot$$

$$\neg A =_{df} A \to \bot$$

PROOFS & REFUTATIONS

$$\frac{A \qquad B}{C} \qquad\qquad \frac{A \qquad B}{\#}$$

ASSERTION & DENIAL & CLASSICAL PROOF

$$\frac{A \qquad \cancel{A}}{\#}\uparrow \qquad\qquad \frac{\begin{array}{c}[\cancel{A}]^i\\ \Pi\\ \#\end{array}}{A}\downarrow^i$$

LET'S PROVE $((p \to q) \to p) \to p$

$$
\begin{prooftree}
\AxiomC{$((p \to q) \to p)^3$}
\AxiomC{$[p]^1$}
\AxiomC{$(\cancel{p})^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\#$}
\UnaryInfC{q}
\RightLabel{$\to I^1$}
\UnaryInfC{$p \to q$}
\RightLabel{$\to E$}
\BinaryInfC{p}
\AxiomC{$(\cancel{p})^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\#$}
\RightLabel{$\downarrow$}
\UnaryInfC{p}
\UnaryInfC{$((p \to q) \to p) \to p$}
\end{prooftree}
$$

NEGATION: TWO OPTIONS

①

$$\frac{\#}{f}\,fI \qquad\qquad \frac{f}{\#}\,fE$$

$$\neg A =_{df} A \to f$$

②

$$\frac{\neg A \quad A}{\#}\,\neg E \qquad\qquad \begin{array}{c}[A]^{i}\\ \vdots \\ \dfrac{\#}{\neg A}\,\neg I\end{array}$$

LETS PROVE $\neg\neg p \rightarrow p$

$$\dfrac{\dfrac{\neg\neg p \qquad \dfrac{\dfrac{[p]^1 \quad [\cancel{p}]^2}{\#}\;\uparrow}{\neg p}\;\neg I^1}{\#}\;\neg E}{p}\;\downarrow^2$$

NORMAL PROOFS WITH ALTERNATIVES

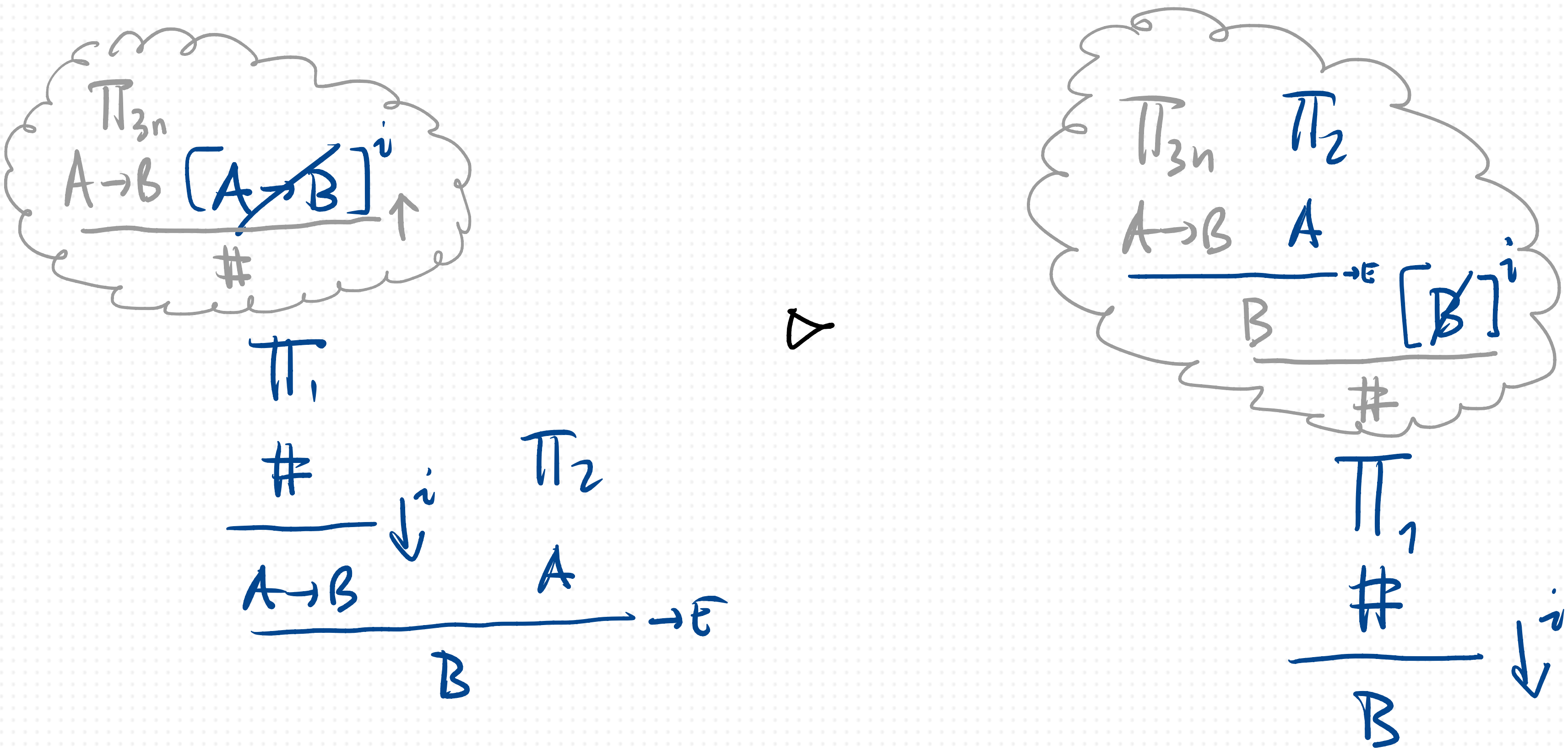
$\downarrow\ \uparrow$ detour

$$\dfrac{\dfrac{\begin{array}{c}[\cancel{A}]^i\\ \Pi\\ \#\end{array}}{A}\downarrow^i \qquad \cancel{A}}{\#}\uparrow \quad \rhd \quad \begin{array}{c}\cancel{A}\\ \Pi\\ \#\end{array}$$

$\downarrow$ $\to E$ detour

$$\dfrac{\dfrac{\begin{array}{c}[\cancel{A \to B}]^i\\ \Pi_1\\ \#\end{array}}{A \to B}\downarrow^i \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{B}\to E \quad \rhd \quad \cdots$$

ZOOMING IN TO WHERE $A \to B$ IS STORED



(In the resulting proof, if $A \to B$ is introduced in Π_{3n}, it now can be normalised away)

NORMALISATION WITH ALTERNATIVES

$\to I/\to E$ $\quad \downarrow/\uparrow$ $\quad \downarrow/\to E$ $\quad$ (& $\mathrm{f}I/\mathrm{f}E$ or $\neg I/\neg E$) detours can all be normalised away, resulting in a unique normal form.

① DIRECT PROOF

② PROOF BY TRANSLATION into proofs <u>without</u> alternatives by way of a double negation translation

OTHER CONNECTIVES

Define $A \otimes B$ as $\neg(A \to \neg B)$

$$\frac{A \qquad B}{A \otimes B}\,\otimes I \qquad \text{ie} \qquad \frac{\dfrac{\dfrac{[A \to \neg B]^{i} \qquad A}{\neg B}\,\to E \qquad B}{\#}\,\neg E}{\neg(A \to \neg B)}\,\neg I^{i}$$

$$\frac{A \otimes B}{A}\,\otimes E \qquad \frac{A \otimes B}{B}\,\otimes E \qquad ???$$

ONLY IN THE PRESENCE OF WEAKENING

(VACUOUS DISCHARGE)

$$
\dfrac{\neg(A \to \neg B) \qquad \dfrac{\dfrac{\dfrac{\cancel{[A]}^2 \quad [A]^1}{\#}\uparrow}{\neg B}\downarrow^{*}}{A \to \neg B}\to I^1}{\dfrac{\#}{A}\downarrow^2}\to E
$$

$$
\dfrac{\neg(A \to \neg B) \qquad \dfrac{\dfrac{\dfrac{\cancel{[B]}^2 \quad (B)^1}{\#}\uparrow}{\neg B}\neg I^1}{A \to \neg B}\to I^{*}}{\dfrac{\#}{B}\downarrow^2}\neg E
$$

THIS IS NOT SURPRISING!!

$$
\dfrac{\dfrac{A \qquad B}{A \otimes B}}{A}
\qquad\qquad
\dfrac{\dfrac{A \qquad B}{A \otimes B}}{B}
$$

THE 'REAL' $\otimes E$ RULE

this makes sense in the presence/absence of contraction & weakening.

$$
\dfrac{A \otimes B \qquad \begin{array}{c}[A]^i, [B]^j \\ \Pi \\ C\end{array}}{C}\otimes E^{i,j}
$$

ie

$$
\dfrac{\neg(A \to \neg B) \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}[A]^i\,[B]^j \\ \Pi \\ C\end{array} \qquad [\cancel{C}]^k}{\#}\uparrow}{\neg B}\neg I^j}{A \to \neg B}\to I^i}{\dfrac{\#}{C}\downarrow^k}\neg E
$$

DISJUNCTION?

Define $A \oplus B$ as $\neg A \to B$

$$\dfrac{A \oplus B \qquad \begin{array}{c}[A]^i\\ \Pi_1\\ \#\end{array} \qquad \begin{array}{c}[B]^j\\ \Pi_2\\ \#\end{array}}{\#}\ \oplus E^{i,j}$$

ie

$$\begin{array}{c} \dfrac{\neg A \to B \qquad \dfrac{\begin{array}{c}[A]^i\\ \Pi_1\\ \#\end{array}}{\neg A}\ \neg I^i}{B}\ \to E \\ \Pi_2 \\ \# \end{array}$$

$$\dfrac{\begin{array}{c}[\cancel{A}]^i\\ \Pi\\ B\end{array}}{A\oplus B}\ \oplus I^i$$

ie

$$\dfrac{\dfrac{\dfrac{[\neg A]^k \qquad \dfrac{\dfrac{\begin{array}{c}[\cancel{A}]^i\\ \Pi\\ B\end{array} \quad [\cancel{B}]^j}{\#}}{A}\ \downarrow^i}{\#}\ \neg E}{B}\ \downarrow^j}{\neg A \to B}\ \to I^k$$

WHAT ABOUT QUANTIFIERS?

SUBSTITUTION

$$\frac{\forall x\, A(x)}{A(t)}\ \forall E$$

$$\frac{\begin{array}{c} \cancel{x} \\ \vdots \\ A(n) \end{array}}{\forall x\, A(x)}\ \forall I$$

SIDE CONDITION

... AND IDENTITY?

SUBSTITUTION

$$\frac{s=t \qquad A(s)}{A(t)}\,{=}E \qquad\qquad \frac{s=t \qquad A(t)}{A(s)}\,{=}E$$

SIDE CONDITION

$$\frac{\begin{array}{cc}[Fs] & [Ft] \\ \vdots & \vdots \\ Ft & Fs\end{array}}{s=t}\,{=}I \qquad\qquad \frac{}{s=s}\,\mathrm{Refl}$$