# DAY 3

## POSITIONS, MODELS
## SPEECH ACTS & MORE

# TODAY'S PLAN

POSITIONS & LIMIT POSITIONS

COMPLETENESS PROOFS & LIMIT POSITIONS

SPEECH ACTS & BRIDGE PRINCIPLES

ASSERTION & DENIAL / WEAK & STRONG

RULES AS DEFINITIONS

EXTRA TOPICS

# POSITIONS, ASSERTION & DENIAL

What is the import of a proof from $A$ to $B$?

for ASSERTION & DENIAL, don't assert $A$ & deny $B$.

A position in which $A$ is asserted & $B$ is denied is OUT OF BOUNDS.

POSITIONS: $[X ; Y]$

ASSERTED

DENIED

$X$ & $Y$ are sets

# BRIDGE PRINCIPLES

If $X \succ Y$ is derivable, then don't ASSERT $X$ & DENY $Y$.

This is NEGATIVE

Is there a positive bridge principle indicating what you COULD do or SHOULD do with a valid sequent or with a proof?

# THIS DEPENDS on WHAT SPEECH ACTS are in PLAY....

A derivation of $X \succ A, Y$ shows how to infer $A$ in a position $[X : Y]$.

- What is it to infer $A$ in this sense?
- It's to comprehensively answer a justification request for $A$, (in a context where $[X : Y]$ is taken as given.)

(More on this later.)

THERE IS A CONNECTION between these two ACCOUNTS.

$X \succ A, Y$: In $[X:Y]$, denying $A$ is out of bounds—ie, relative to $[X:Y]$, $A$ is UNDENIABLE.

$X \succ A, Y$: We show that $A$ against a context $[X:Y]$.

To do this is to show that $A$ is undeniable, & if we show that $A$ is undeniable, we have (!) proved $A$.

(More on this, later)

# NORMS FOR BOUNDS

* $[A:A]$ is out of bounds

* If $[X:Y]$ is out of bounds,
  so are $[X,A:Y]$ and $[X:A,Y]$.

* If $[X:A,Y]$ & $[X,A:Y]$ are out of bounds,
  then so is $[X:Y]$

* If $[X:Y]$ is out of bounds then for some finite
  subsets $X' \subseteq X$; $Y' \subseteq Y$, $[X':Y']$ is out of bounds.

# NORMS FOR BOUNDS

* $[A : A]$ is out of bounds

$$A \succ A$$

* If $[X : Y]$ is out of bounds, so are $[X, A : Y]$ and $[X : A, Y]$.

$$\frac{X \succ Y}{X, A \succ Y} \qquad \frac{X \succ Y}{X \succ A, Y}$$

* If $[X : A, Y]$ & $[X, A : Y]$ are out of bounds, then so is $[X : Y]$

$$\frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}$$

* If $[X : Y]$ is out of bounds then for some finite subsets $X' \subseteq X$, $Y' \subseteq Y$, $[X' : Y']$ is out of bounds.

COMPACTNESS!

# AVAILABLE POSITIONS

- Let's call a position [X:Y] AVAILABLE when it is not out of bounds.

- So, if [X:Y] is available so is either [X,A:Y] or [X:A,Y].

# POSITIONS & SEQUENTS

$[X:Y]$ is out of bounds iff $X' \succ Y'$ is derivable for some finite $X' \subseteq X$, $Y' \subseteq Y$.

We write $X \rhd Y$ to say that $[X:Y]$ is out of bounds

[for now, we are no longer presuming that identity sequents are the only axioms — we allow other primitively analytically valid sequents — eg $Fa, a=b \succ Fb$; $Fa, b \geq_F a \succ Fb$; $0=1 \succ$ ]

# POSITION EXTENSION

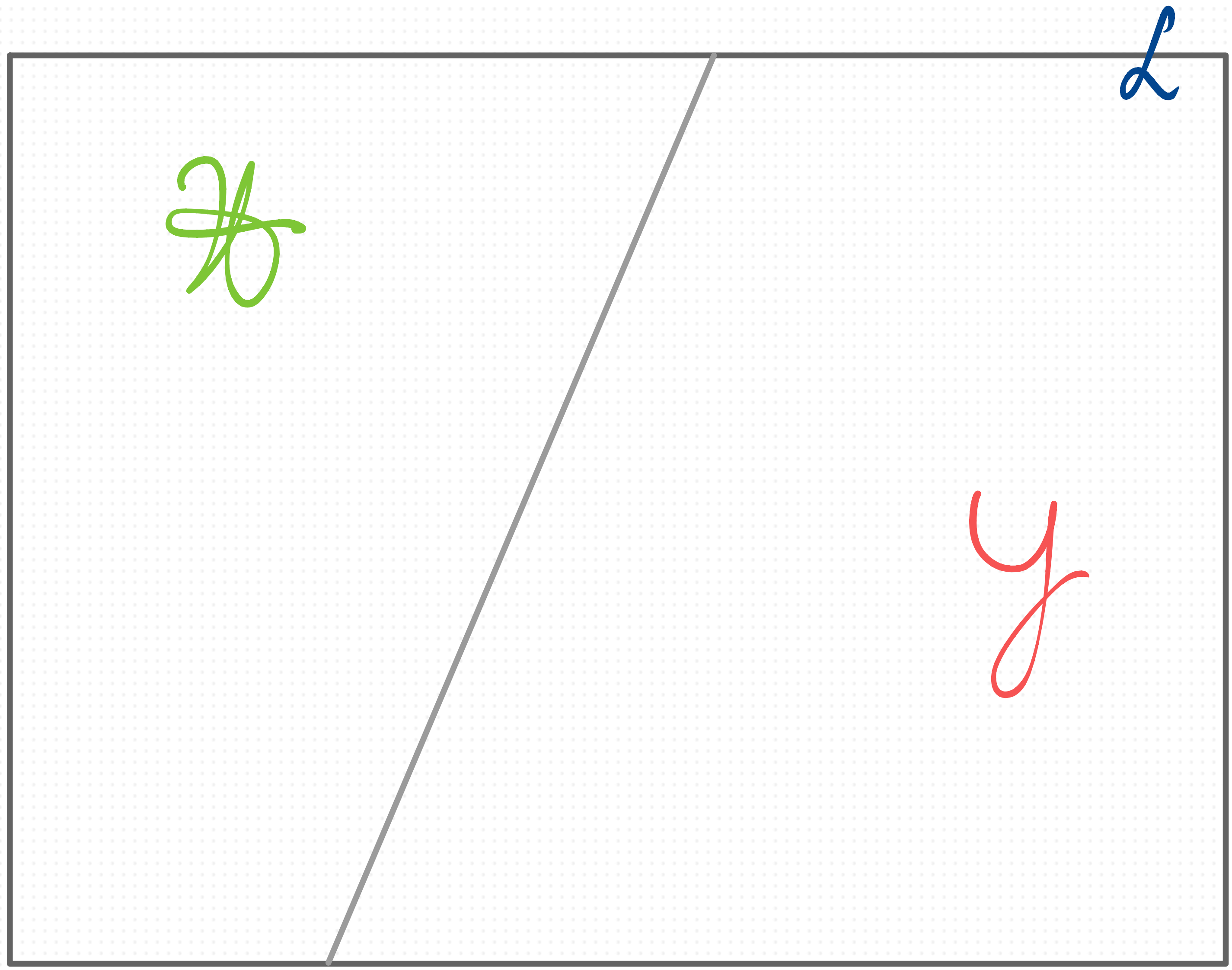
$$[X : Y] \leqslant [X' : Y']$$

iff $X \subseteq X'$ & $Y \subseteq Y'$!

# LIMIT POSITIONS

Given a language $\mathcal{L}$, a LIMIT POSITION $[\mathcal{X} : \mathcal{Y}]$ is a pair where

- $[\mathcal{X} : \mathcal{Y}]$ is a partition of $\mathcal{L}$
  - ie $\mathcal{X} \cup \mathcal{Y} = \mathcal{L}$;
    $\mathcal{X} \cap \mathcal{Y} = \emptyset$.
- $[\mathcal{X} : \mathcal{Y}]$ is AVAILABLE



$\mathcal{L}$
$\mathcal{H}$
$\mathcal{Y}$

# LIMIT POSITION FACT

For any language $\mathcal{L}$, any available position $[X:Y]$ is extended by some limit position $[\mathcal{X}:Y]$.

(We use Zorn's lemma, on the ordered set of available positions extending $[X:Y]$.
You can go without Zorn's lemma in the case of a countable language.)

# TRUTH & FALSITY IN POSITIONS

$A$ is TRUE IN $[X:Y]$ iff $[X:A,Y]$ is out of bounds.
(ie $X \rhd A, Y$)

$A$ is FALSE IN $[X:Y]$ iff $[X,A:Y]$ is out of bounds.
(ie $X, A \rhd Y$)

FACTS:
- ✱ If $A$ is both true & false in $[X:Y]$ then $[X:Y]$ is out of bounds.
- ✱ Every member of $X$ is true in $[X:Y]$.
- ✱ Every member of $Y$ is false in $[X:Y]$.

# POSITION EQUIVALENCE

$[X : Y]$ is equivalent to $[U : V]$ iff

$A$ is true in $[X : Y] \iff A$ is true in $[U : V]$

$A$ is false in $[X : Y] \iff A$ is false in $[U : V]$

EG. $[p, q : r]$ is equivalent to $[(p \wedge q) \wedge \neg r : \ ]$

# TRUTH / FALSITY FACTS

$A \wedge B$ is true in $[x:y]$ iff both $A$ & $B$ are true in $[x:y]$

$A \wedge B$ is false in $[x:y]$ if either $A$ or $B$ are false in $[x:y]$

$A \vee B$ is true in $[x:y]$ if either $A$ or $B$ are true in $[x:y]$

$A \vee B$ is false in $[x:y]$ iff both $A$ & $B$ are false in $[x:y]$

$\neg A$ is true in $[x:y]$ iff $A$ is false in $[x:y]$

$\neg A$ is false in $[x:y]$ iff $A$ is true in $[x:y]$

$A \wedge B$ is true in $[X:Y]$ iff both $A$ & $B$ are true in $[X:Y]$

$$\dfrac{X \rhd A \wedge B, Y \qquad \dfrac{A \succ A}{A \wedge B \succ A}\,\wedge L}{X \rhd A, Y}\,\text{cut} \qquad \dfrac{X \rhd A \wedge B, Y \qquad \dfrac{B \succ B}{A \wedge B \succ B}\,\wedge L}{X \rhd B, Y}\,\text{cut}$$

$$\dfrac{X \rhd A, Y \qquad X \rhd B, Y}{X \rhd A \wedge B, Y}\,\wedge R$$

$A \wedge B$ is false in $[X:Y]$ if either $A$ or $B$ are false in $[X:Y]$

$$\dfrac{X, A \rhd Y}{X, A \wedge B \rhd Y}\,\wedge L \qquad \dfrac{X, B \rhd Y}{X, A \wedge B \rhd Y}\,\wedge L$$

THAT if CANNOT, IN GENERAL BE STRENGTHENED to an iff.

$p \vee q$ is true in [$p \vee q$ : ], but we do not want either $p$ or $q$ true in [$p \vee q$ : ], in general, since we want to refute both

$$p \vee q \succ p \quad \& \quad p \vee q \succ q.$$

# HOWEVER, IN LIMIT POSITIONS....

If $[X:Y]$ is a limit position then

$A \wedge B$ is true in $[X:Y]$ iff both $A$ & $B$ are true in $[X:Y]$

$A \wedge B$ is false in $[X:Y]$ iff either $A$ or $B$ are false in $[X:Y]$

$A \vee B$ is true in $[X:Y]$ iff either $A$ or $B$ are true in $[X:Y]$

$A \vee B$ is false in $[X:Y]$ iff both $A$ & $B$ are false in $[X:Y]$

$\neg A$ is true in $[X:Y]$ iff $A$ is false in $[X:Y]$

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$A$ is true in $[X:Y]$ iff $A$ is not false in $[X:Y]$

$A$ is true in $[X:Y]$ iff $A$ is not false in $[X:Y]$ (where $[X:Y]$ is a limit position.)

If $X \rhd A, Y$ & $X, A \rhd Y$ then by Cut $X \rhd Y$,
& hence $[X:Y]$ is not available.

If $X \not\rhd A, Y$ & $X, A \not\rhd Y$ then $A \notin X$ & $A \notin Y$,
& hence $[X:Y]$ is not maximal.

$A \wedge B$ is false in $[X:Y]$ iff either $A$ or $B$ are false in $[X:Y]$

If $X, A \wedge B \rhd Y$, then $X \not\rhd A \wedge B, Y$ & so, either $X \not\rhd A, Y$ or $X \not\rhd B, Y$,
& hence, by maximality, either $X, A \rhd Y$ or $X, B \rhd Y$.

# So, LIMIT POSITIONS are BOOLEAN VALUATIONS

... and any Boolean valuation on $\mathcal{L}$ determines a limit position (setting $\mathcal{X} = \{A : v(A) = 1\}$, $\mathcal{Y} = \{B : v(B) = 0\}$) — provided that Identity sequents are the only axioms determining the bounds.

(More generally, we say that a valuation $v$ is a counterexample to $X \succ Y$ if $v(A) = 1$ for each $A \in X$ & $v(B) = 0$ for each $B \in Y$, and it respects $X \succ Y$ if it is not a counterexample to it. Then, any valuation that respects all axioms determines a limit position.)

# Completeness via Limit Positions

Suppose $[X:Y]$ is available (since $X \nvdash Y$.)

Then there is a limit position $[\mathcal{X}:\mathcal{Y}]$ extending $[X:Y]$.

This position determines a Boolean valuation $v$ which assigns each member of $X$ the value $1$ & each member of $Y$, the value $0$.

So, $X \nvDash Y$.

# THIS GENERALISES...

Intuitionistic logic: $[\mathcal{X} : \mathcal{Y}]$ is available iff for no $X \subseteq \mathcal{X}$ & $C \in \mathcal{Y}$ is $X \succ C$ derivable.

$[\neg\neg p : p]$ is available, & so, is extended by a limit position.

At any such position, $\neg\neg p$ is true & $p$ is false.

We do <u>not</u> have $\neg A$ true at a position iff $A$ is false there.

But, we have something that may be familiar...

$\neg A$ is true in $[X:Y]$ iff $X \rhd \neg A, Y$,

which, if $[X:Y]$ is available, means $X \rhd \neg A$,

& this holds iff $X, A \rhd$

Let's say $[X':Y']$ <u>extends</u> $[X:Y]$ iff $X \subseteq X'$.

Then $\neg A$ is true in $[X:Y]$ iff

$A$ is false in any available $[X':Y']$

that extends $[X:Y]$.

(& Similarly for the conditional: $A \to B$ is true in $[X:Y]$

iff $B$ is true in any available $[X':Y']$ extending $[X:Y]$

at which $A$ is true. We use $X \rhd A \to B$ iff $X, A \rhd B$.)

# THIS GENERALISES...

... and also to modal logics,

as we will see tomorrow.

# BUT WHAT ABOUT ASSERTION & DENIAL?

Assertion & Denial are opposed

( $[A : A]$ is out of bounds )

... but how, exactly?

What *is* denial?

# DENIALS: STRONG & WEAK

Abelard: Labour will win the Westminster election.

Eloise: No. The Lib Dems will win the Westminster election. (!)

This is a strong denial.

She rejects Abelard's claim as false.

Abelard: Labour will win the Westminster election.

Eloise: No. Labour or the Lib Dems will win the Westminster election.

This is a weak denial

She rejects Abelard's claim as unwarranted

# ASSERTION, DENIAL & THE COMMON GROUND

Represent the COMMON GROUND (what we, together have ruled in & what we have ruled out) as a position [X : Y].

X : positive common ground    Y : negative common ground.

STRONGLY DENY A — bid to add A to the negative c.g.

WEAKLY DENY A — block the addition of A to the positive c.g.

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STRONGLY DENY A — bid to add A to the negative c.g.

WEAKLY DENY A — block the addition of A to the positive c.g.

STRONGLY ASSERT A — bid to add A to the positive c.g.

WEAKLY ASSERT A — block the addition of A to the negative c.g.

# Isolating Strong Assertion & Denial

Abelard: Will Labour win?

Eloise: No, the Lib Dems will win.

✓

# ISOLATING STRONG ASSERTION & DENIAL

Abelard: Will Labour win?

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Abelard: Will Labour win?

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If Eloise's 'no' is appropriate as an answer to Abelard's question, then the follow-up is a strange way of saying that the Lib Dems will win!

# Isolating Strong Assertion & Denial

Abelard: Will Labour win?

Eloise: No, the Lib Dems will win.

✓

This cannot be a weak denial, because the question didn't place the claim into the cg, so there is nothing here to block.

Abelard: Will Labour win?

Eloise: No, either Labour or the Lib Dems will win.

???

If Eloise's 'no' is appropriate as an answer to Abelard's question, then the follow-up is a strange way of saying that the Lib Dems will win!

# BACK TO RULES FOR CONNECTIVES...

$$\frac{X, A \succ B, Y}{X \succ A \to B, Y} \to R \qquad \frac{X \succ A, Y \qquad U, B \succ V}{X, U, A \to B \succ Y, V} \to L$$

Why are these in harmony? How are they together a definition?

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$$\frac{X, A \succ B, Y}{X \succ A \to B, Y} \to R \qquad \frac{X \succ A, Y \quad U, B \succ V}{X, U, A \to B \succ Y, V} \to L$$

Why are these in harmony? How are they together a definition?

$$\frac{X, A \succ B, Y}{X \succ A \to B, Y} \to Df$$

This is a two-way rule

This is more obviously a definition.

# From ${\to}Df$ to ${\to}L/{\to}R$

$$\begin{array}{c} X, A \succ B, Y \\ \hline\hline X \succ A \to B, Y \end{array}\;{\to}Df$$

The $\downarrow$ direction is ${\to}R$

The $\uparrow$ direction justifies ${\to}L$, using Cut & Identity (1 Id + 2 Cuts)

$$\dfrac{\dfrac{X \succ A, Y \qquad \dfrac{\dfrac{}{A\to B \succ A\to B}\,\text{Identity}}{A\to B, A \succ B}\,{\to}Df{\uparrow}}{X, A\to B \succ B, Y}\,\text{Cut} \qquad U, B \succ V}{X, U, A\to B \succ Y, V}\,\text{Cut}$$

$$\dfrac{X \succ A, Y \qquad U, B \succ V}{X, U, A\to B \succ Y, V}\,{\to}L$$

# From $\to L$ / $\to R$ back to $\to Df$

$\to R$ just is $\to Df\downarrow$

$\to L$ justifies $\to Df\uparrow$, using Cut & Identity
(2 Ids + 1 Cut)

$$\dfrac{X \succ A\to B, Y \qquad \dfrac{\dfrac{}{A\succ A}\,\mathrm{Id} \qquad \dfrac{}{B\succ B}\,\mathrm{Id}}{A\to B, A \succ B}\,{\to}L}{X, A \succ B, Y}\,\mathrm{Cut}$$

$$\dfrac{X \succ A\to B, Y}{X, A \succ B, Y}\,{\to}Df\uparrow$$

- This generalizes to the other connectives
- No Contraction or Weakening is ever used.

# QUANTIFIERS?

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\ \forall Df$$

This works, as a definition of the universal quantifier, but to recover the $\forall L$ Rule, we need to do some work.

Obvious side condition on $n$ in force. — $n$ is absent from the bottom sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x A(x) \succ \forall x A(x)}\ Id}{\forall x A(x) \succ A(n)}\ \forall Df\uparrow}{\forall x A(x) \succ A(t)}\ Spec \qquad X, A(t) \succ Y}{X, \forall x A(x) \succ Y}\ Cut$$

The Specialise rule is required in the system with the Df Rules — it makes the eigenvariables inferentially general...