Proofs, and what they're good for

Greg Restall

THE UNIVERSITY OF
MELBOURNE

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My Aim

To explain the nature of proof, from the perspective of a normative pragmatic account of meaning, using the formal tools of proof theory.
Outline

Motivation

Background

What Proofs Are

How Proofs Work
MOTIVATION
Example Proof 1

Every drink (in our fridge) is either a beer or a lemonade
Every *drink* (in our fridge) is either a *beer* or a *lemonade* 

\( (\forall x)(Dx \supset (Bx \lor Lx)) \).
Every drink (in our fridge) is either a beer or a lemonade
\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every drink is a beer or some drink is a lemonade.
Every *drink* (in our fridge) is either a *beer* or a *lemonade* 
\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every *drink* is a *beer* or some *drink* is a *lemonade* 
\((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\).
Every drink (in our fridge) is either a beer or a lemonade

\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every drink is a beer or some drink is a lemonade \((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why?
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\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every drink is a beer or some drink is a lemonade \((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary drink.
Every drink (in our fridge) is either a beer or a lemonade \((\forall x)(Dx \supset (Bx \lor Lx))\). So either every drink is a beer or some drink is a lemonade \((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary drink. It’s either a beer or a lemonade.
Every drink (in our fridge) is either a beer or a lemonade \((\forall x)(Dx \supset (Bx \lor Lx))\). So either every drink is a beer or some drink is a lemonade \((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary drink. It’s either a beer or a lemonade. If it’s a lemonade, we have the conclusion that some drink is a lemonade.
Every *drink* (in our fridge) is either a *beer* or a *lemonade* 

\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every *drink* is a *beer* or some *drink* is a *lemonade* 

\((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary *drink*. It’s either a *beer* or a *lemonade*. If it’s a *lemonade*, we have the conclusion that some *drink* is a *lemonade*. If we don’t have that conclusion, then that arbitrary *drink* is a *beer*
Every drink (in our fridge) is either a beer or a lemonade
\((\forall x)(Dx \supset (Bx \lor Lx))\). So either every drink is a beer or some drink is a lemonade \((\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)\). Why? Take an arbitrary drink. It’s either a beer or a lemonade. If it’s a lemonade, we have the conclusion that some drink is a lemonade. If we don’t have that conclusion, then that arbitrary drink is a beer, and so, all the drinks are beers
Every *drink* (in our fridge) is either a *beer* or a *lemonade* $(\forall x)(Dx \supset (Bx \lor Lx))$. So either every *drink* is a *beer* or some *drink* is a *lemonade* $(\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)$. Why? Take an arbitrary *drink*. It’s either a *beer* or a *lemonade*. If it’s a *lemonade*, we have the conclusion that some *drink* is a *lemonade*. If we don’t have that conclusion, then that arbitrary *drink* is a *beer*, and so, all the *drinks* are *beers*, and so, we also have our conclusion.
Example Proof 1 (the formal structure)

\[
\begin{align*}
&\quad \text{Ba} \rightarrow \text{Ba} \quad \text{La} \rightarrow \text{La} \\
\hline
\text{Da} \rightarrow \text{Da} &\Rightarrow \text{Ba} \lor \text{La} \rightarrow \text{Ba}, \text{La} \\
\hline
\text{Da} \supset (\text{Ba} \lor \text{La}), \text{Da} \rightarrow \text{Ba}, \text{La} \\
\hline
(\forall x)(\text{D}x \supset (\text{B}x \lor \text{L}x)), \text{Da} \rightarrow \text{Ba}, \text{La} \\
\hline
(\forall x)(\text{D}x \supset (\text{B}x \lor \text{L}x)), \text{Da} \rightarrow \text{Ba}, \text{Da} \land \text{La} \\
\hline
(\forall x)(\text{D}x \supset (\text{B}x \lor \text{L}x)), \text{Da} \rightarrow \text{Ba}, (\exists x)(\text{D}x \land \text{L}x) \\
\hline
(\forall x)(\text{D}x \supset (\text{B}x \lor \text{L}x)) \rightarrow \text{Da} \supset \text{Ba}, (\exists x)(\text{D}x \land \text{L}x) \\
\hline
(\forall x)(\text{D}x \supset (\text{B}x \lor \text{L}x)) \rightarrow (\forall x)(\text{D}x \supset \text{B}x), (\exists x)(\text{D}x \land \text{L}x) \\
\hline
(\forall x)(\text{D}x \supset (\text{B}x \lor \text{L}x)) \rightarrow (\forall x)(\text{D}x \supset \text{B}x) \lor (\exists x)(\text{D}x \land \text{L}x)
\end{align*}
\]
Example Proof 2 (The Bridges of Königsberg)

It’s not possible to walk a circuit through Königsberg, *crossing each bridge exactly once.*
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It’s not possible to walk a circuit through Königsberg, crossing each bridge exactly once. Why? Any bridge takes you from one landmass (A, B, C, D) to another.
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It’s not possible to walk a circuit through Königsberg, *crossing each bridge exactly once*. Why? Any bridge takes you from one landmass \((A, B, C, D)\) to another. In any *circuit*, you must *leave* a landmass as many times as you *arrive*. 
Example Proof 2 (The Bridges of Königsberg)

It’s not possible to walk a circuit through Königsberg, crossing each bridge exactly once. Why? Any bridge takes you from one landmass (A, B, C, D) to another. In any circuit, you must leave a landmass as many times as you arrive. So, if you are use every bridge exactly once, each landmass must have an even number of bridges entering and exiting it.
Example Proof 2 (The Bridges of Königsberg)

It’s not possible to walk a circuit through Königsberg, *crossing each bridge exactly once*. Why? Any bridge takes you from one landmass (A, B, C, D) to another. In any *circuit*, you must *leave* a landmass as many times as you *arrive*. So, if you are use every bridge exactly once, each landmass must have an even number of bridges entering and exiting it. Here, each landmass has an *odd* number of bridges, so a circuit is impossible.
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But what I say here can be extended to proof relying on other concepts.
How can proofs expand our knowledge, when the conclusion is (in some sense) already present in the premises?
Puzzles about proof

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- How can we be ignorant of a conclusion which logically follows from what we already know?
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Puzzles about proof

- How can proofs expand our knowledge, when the conclusion is (in some sense) already present in the premises?
- How can we be ignorant of a conclusion which logically follows from what we already know?
- What grounds the necessity in the connection between the premises and the conclusion?
- (Notice that these are important questions for proofs in first order predicate logic, as much as for proof more generally.)
BACKGROUND
Assertions and Denials

\[ X : Y \]
Assertions and denials are moves in a practice.
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I can deny what you assert.
Assertions and denials are moves in a practice.

I can *deny* what you *assert*.

I can *retract* an assertion or a denial.
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I can ‘try on’ assertion or denial hypothetically.
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 I can *deny* what you *assert*.

 I can *retract* an assertion or a denial.

 I can ‘try on’ assertion or denial hypothetically.

 They are connected to other speech acts, too, like imperatives, interrogatives, recognitives, observatives, *etc.*
Assertions and denials take a *stand* (*pro* or *con*) on something.

DENIAL clashes with assertion.
ASSERTION clashes with denial.
These norms give rise to **BOUNDS** for positions.
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IDENTITY: $[A : A]$ is out of bounds.
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**Identity:** $[A : A]$ is out of bounds.

**Weakening:** If $[X : Y]$ is out of bounds, then $[X, A : Y]$ and $[X : A, Y]$ are also out of bounds.
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**Cut:** If \([X, A : Y]\) and \([X : A, Y]\) are out of bounds, then so is \([X : Y]\).

A position that is out of bounds doesn’t succeed in taking a stand.
Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.
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\[ x \text{ is a square} \quad =_{df} \quad x \text{ is a rectangle } \land \quad \text{all sides of } x \text{ are equal in length.} \]
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Concepts defined explicitly are *sharply delimited* (contingent on the definition).
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Concepts defined explicitly are *sharply delimited* (contingent on the definition).

Logical concepts are similarly sharply delimited, but they cannot all be given explicit definitions.
Definition through a rule for use

\[ [X, A \land B : Y] \text{ is out of bounds} \]
if and only if

\[ [X, A, B : Y] \text{ is out of bounds} \]
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\[
\frac{X, A, B \vdash Y}{X, A \land B \vdash Y} \quad \land \text{Df}
\]
What about when to deny a conjunction?

\[
\begin{align*}
X \vdash A ; Y \\
X \vdash B ; Y \\
A \land B \vdash A \land B \\
\text{Id} \\
X \vdash B , Y \\
A , B \vdash A \land B \\
\wedge \text{Df} \\
X \vdash A , Y \\
X , A \vdash A \land B , Y \\
\text{Cut} \tag{1} \\
X \vdash A \land B , Y \\
\text{Cut} 
\end{align*}
\]
What about when to deny a conjunction?

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\begin{align*}
\text{Id} & \quad A \land B \vdash A \land B \\
\land \text{Df} & \quad X \vdash B, Y \quad A, B \vdash A \land B \\
\text{Cut} & \quad X \vdash A, Y \quad X, A \vdash A \land B, Y \\
\text{Cut} & \quad X \vdash A \land B, Y \\
\end{align*}
\]

So, we have

\[
\begin{align*}
\text{Id} & \quad A \land B \vdash A \land B \\
\land \text{Df} & \quad X \vdash B, Y \\
\text{Cut} & \quad X, A \vdash A \land B, Y \\
\land \text{R} & \quad X \vdash A \land B, Y \\
\end{align*}
\]
Definitions for other logical concepts

\[
\begin{align*}
X &\vdash A, Y \\
\frac{X, \neg A \vdash Y}{\neg Df}
\end{align*}
\]

\[
\begin{align*}
X &\vdash A \supset B, Y \\
\frac{X, A \vdash B, Y}{\supset Df}
\end{align*}
\]

\[
\begin{align*}
X &\vdash A, B, Y \\
\frac{X \vdash A \lor B, Y}{\lor Df}
\end{align*}
\]
Definitions for other logical concepts

\[ X \vdash A, Y \]
\[ X, \neg A \vdash Y \quad \neg \text{Df} \]
\[ X, A \vdash B, Y \]
\[ X \vdash A \supset B, Y \quad \supset \text{Df} \]
\[ X \vdash A \lor B, Y \quad \lor \text{Df} \]

\[ X \vdash Fa, Y \]
\[ X \vdash (\forall x)Fx, Y \quad \forall \text{Df} \]
\[ X, Fa \vdash Y \]
\[ X, (\exists x)Fx \vdash Y \quad \forall \text{Df} \]
\[ X, Gb \vdash Gc, Y \]
\[ X \vdash b = c, Y \quad \forall \text{Df} \]

(Where \( a \) and \( G \) are not present in \( X \) and \( Y \).)
Concepts defined in this way...

- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)

- Are *conservatively extending*. (Adding a logical concept to your vocabulary in this way doesn’t constrain the bounds in the original language.)

- Play useful dialogical roles. (You can do things with these concepts that you cannot do without. Denying a conjunction does something different to denying the conjuncts.)

- Are *subject matter neutral*. (They work wherever you assert and deny—and have singular terms and predicates.)

- In Brandom’s terms, they *make explicit* some of what was implicit in the practice of assertion and denial.
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WHAT PROOFS ARE
A Tiny Proof

If it’s Friday, I’m in Melbourne.

It’s Friday.

Therefore, I’m in Melbourne.
A Tiny Proof

If it’s Friday, I’m in Melbourne.

It’s Friday.

Therefore, I’m in Melbourne.

\[
\frac{A \supset B}{\therefore A \supset B, A \vdash B} \quad \supset \text{Df}
\]

\[
\frac{A \supset B \vdash A \supset B \quad \text{Id}}{A \supset B, A \vdash B}
\]
A Tiny Proof

If it’s Friday, I’m in Melbourne.

It’s Friday.

Therefore, I’m in Melbourne.

\[
\begin{align*}
A \supset B \vdash A \supset B & \quad \text{Id} \\
A \supset B, A \vdash B & \quad \supset \text{Df}
\end{align*}
\]

[It’s Friday $\supset$ I’m in Melbourne, It’s Friday : I’m in Melbourne]

(This is out of bounds.)
Take a context in which I’ve asserted

\( \text{it's Friday} \supset \text{I'm in Melbourne} \)

and I’ve asserted \( \text{it's Friday} \),
then \( \text{I'm in Melbourne} \) is undeniable.
Take a context in which I’ve asserted

\[ \text{it’s Friday} \supset \text{I’m in Melbourne} \]

and I’ve asserted \( \text{it’s Friday} \),

then \( \text{I’m in Melbourne} \) is undeniable.

Adding the assertion makes explicit
what was implicit before that assertion.
The Undeniable

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\[ \text{it’s Friday} \supset \text{I’m in Melbourne} \]

and I’ve asserted \text{it’s Friday},
then \text{I’m in Melbourne} is undeniable.

Adding the assertion makes explicit what was \textit{implicit} before that assertion.

The \textit{stance} (pro or con)
on \text{I’m in Melbourne} was already made.
A proof of $X \vdash Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.
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In this sense, proofs are *analytic*. They apply, given the definitions, independently of the positions taken by those giving the proof.
A proof of $A, B \vdash C, D$ can be seen as a proof of $C$ from $[A, B : D]$, and a refutation of $A$ from $[B : C, D]$, and more.
HOW PROOFS WORK
Observation 0: Proofs are *analytic*

These proofs are grounded in the *rules* defining the concepts used in them.
Observation 1: *Specification outstrips Recognition*

Our ability to *specify* concepts and consequence far outstrips our ability to *recognise* that consequence.
SUCCESSOR AXIOMS:

PA1: \( \forall x \forall y (x' = y' \supset x = y) \);

PA2: \( \forall x (0 \neq x') \).

ADDITION AXIOMS:

PA3: \( \forall x (x + 0 = x) \);

PA4: \( \forall x (x + y' = (x + y)') \).

MULTIPLICATION AXIOMS:

PA5: \( \forall x (x \times 0 = 0) \);

PA6: \( \forall x \forall y (x \times y' = (x \times y) + x) \).

INDUCTION SCHEME:

PA7: \( (\phi(0) \land \forall x (\phi(x) \supset \phi(x'))) \supset \forall x \phi(x) \).

GOLDBACH’S CONJECTURE:

GC: \( \forall x \exists y \exists z (\text{Prime } y \land \text{Prime } z \land 0'' \times x = y + z) \)
Observation 1: *Specification* outstrips *Recognition*

Is \([\text{PA} : \text{GC}]\) out of bounds?
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Is \([PA : GC]\) out of bounds?

We have no idea.
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This is not a *bug*. It’s a *feature*.
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Our concepts are rich and expressive. We can say things whose significance we continue to work out.
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Verifying a putative proof is straightforward. Checking that something *has* a proof is not so easy.
Are we logically omniscient?

Suppose that $\text{PA} \vdash \text{GC}$ (but we don’t possess that proof) and that we *know* $\text{PA}$.

Do we know $\text{GC}$?
In a weak sense of ‘know’, yes, we do know $\text{GC}$

- It’s a logical consequence of what we know.
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- It’s a logical consequence of what we know.
- It is implicitly present in what we already know.
In a weak sense of ‘know’, yes, we do know GC

- It’s a logical consequence of what we know.
- It is implicitly present in what we already know.
- There is no epistemic possibility (no circumstance consistent with our knowledge) that leaves GC out.
In a not-so-weak sense, we don't know GC

- Do we believe GC?
In a not-so-weak sense, we don't know GC

- Do we *believe* GC?
- If we believed it, do we believe it *in the right way*?
In a not-so-weak sense, we don't know GC

- Do we believe GC?
- If we believed it, do we believe it *in the right way*?
- There is evidence for GC (its proof from PA, for example), but if that evidence plays no role in our belief...
Observation 2: Proofs Preserve Truth

- The account of consequence does not use the concept of *truth*. 
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- The account of consequence does not use the concept of truth.

- However, given plausible (minimal) assumptions concerning T, we can show that (for example) if $A, B \vdash C$ then $T\langle A \rangle, T\langle B \rangle \vdash T\langle C \rangle$. 
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- However, given plausible (minimal) assumptions concerning $T$, we can show that (for example) if $A, B \vdash C$ then $T\langle A \rangle, T\langle B \rangle \vdash T\langle C \rangle$.

- This follows from the concepts of consequence and truth.
Observation 3: Proofs Can Preserve Warrant

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- However, given plausible (less minimal) assumptions concerning warrant, we can show that (for example) if \( p \) is a proof for \( A, B \vdash C \) then \( x : A, y : B \vdash p(x, y) : C \).

Here, \( p \) transforms warrants for the premises into warrant for the conclusion. This works only for categorical, conclusive warrants (grounds), not for defeasible warrants.
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A Caveat on Defeasible Warrants

Consider the “Lottery Paradox.”
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\[
\begin{align*}
(\exists x)(T_x \land W_x),
\end{align*}
\]

\[
(\forall x)(T_x \equiv (x = t_1 \lor x = t_2 \lor \cdots \lor x = t_{1,000,000}))
\]

\[
: W_{t_1}, W_{t_2}, \ldots, W_{t_{1,000,000}}
\]
Consider the “Lottery Paradox.”

\[
\begin{align*}
(\exists x)(Tx \land Wx), \\
(\forall x)(Tx \equiv (x = t_1 \lor x = t_2 \lor \cdots \lor x = t_{1,000,000}))
\end{align*}
\]

\[ : Wt_1, Wt_2, \ldots, Wt_{1,000,000} \]

We have a very high degree of confidence in each part. Each component is highly probable. But the whole position is out of bounds.
“Well, now, let’s take a little bit of the argument in that First Proposition—just two steps, and the conclusion drawn from them. Kindly enter them in your note-book. And in order to refer to them conveniently, let’s call them $A$, $B$, and $Z$:

(A) Things that are equal to the same are equal to each other.

(B) The two sides of this Triangle are things that are equal to the same.

(Z) The two sides of this Triangle are equal to each other.

Readers of Euclid will grant, I suppose, that $Z$ follows logically from $A$ and $B$, so that any one who accepts $A$ and $B$ as true, must accept $Z$ as true?”

“Undoubtedly! The youngest child in a High School—as soon as High Schools are invented, which will not be till some two thousand years later—will grant that.”

“And if some reader had not yet accepted $A$ and $B$ as true, he might still accept the sequence as a valid one, I suppose?”
“No doubt such a reader might exist. He might say ‘I accept as true the Hypothetical Proposition that, if $A$ and $B$ be true, $Z$ must be true; but, I don’t accept $A$ and $B$ as true.’ Such a reader would do wisely in abandoning Euclid, and taking to football.”

“And might there not also be some reader who would say ‘I accept $A$ and $B$ as true, but I don’t accept the Hypothetical’?”

“Certainly there might. He, also, had better take to football.”

“And neither of these readers,” the Tortoise continued, “is as yet under any logical necessity to accept $Z$ as true?”

“Quite so,” Achilles assented.

“Well, now, I want you to consider me as a reader of the second kind, and to force me, logically, to accept $Z$ as true.”

“A tortoise playing football would be—” Achilles was beginning

“—an anomaly, of course,” the Tortoise hastily interrupted. “Don’t wander from the point. Let’s have $Z$ first, and football afterwards!”

“I’m to force you to accept $Z$, am I?” Achilles said musingly. “And your present position is that you accept $A$ and $B$, but you don’t accept the Hypothetical—”

“Let’s call it $C$, ” said the Tortoise.

“—but you don’t accept

($C$) If $A$ and $B$ are true, $Z$ must be true.”

“That is my present position,” said the Tortoise.

“Then I must ask you to accept $C$.”
Our Analysis

\[ A, B \vdash Z \]

This doesn't mean when I accept \( A \) and I accept \( A \), I ought to also accept \( Z \). However, if I assert \( A \) and \( A \), then \( Z \) is undeniable.
Our Analysis

\[ A, B \vdash Z \]

or

\[ A, A \models Z \vdash Z \]

This doesn't mean when I accept \( A \) and I accept \( A \), I ought to also accept \( Z \). However, if I assert \( A \) and \( A \), then \( Z \) is undeniable.
A, B ⊢ Z

or

A, A ⊨ Z ⊢ Z

This *doesn’t* mean when I accept A and I accept A ⊨ Z, I ought to also accept Z.
A, B \vdash Z

or

A, A \models Z \vdash Z

This *doesn’t* mean when I accept A
and I accept A \models Z,
I ought to also accept Z.

However, if I assert A and A \models Z then Z is *undeniable*.
If I assert $A$ and if $A$ then $Z$ and deny $Z$, then I am using ‘if ... then’ in a way that deviates from the defining rule for $\supset$, or I am explicitly contradicting myself.
If I assert $A$ and if $A$ then $Z$ and deny $Z$, then I am using ‘if ... then’ in a way that deviates from the defining rule for $\supset$, or I am explicitly contradicting myself.

$$
\frac{A \supset B \vdash A \supset B}{A \supset B, A \vdash B} \supset Df
$$
An account of the logical concepts given in terms of defining rules governing assertions and denials helps explain how (first order predicate logic) proof works, how possessing a proof can expand our knowledge, while proofs make explicit what is implicit in what we know.
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THANK YOU!


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